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ON HARDY'S INEQUALITY AND ITS GENERALIZATION

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1. G. H. Hardy, [1], p. 239 has proved the following

THEOREM A. *If* $p>1$, $a_n \geqq 0$, $(n=1,2,\cdots)$ *and* $A_n = a_1 + a_2 + \cdots + a_n$, *then*

$$\sum_{n=1}^{\infty}(A_n/n)^p < \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p \tag{1}$$

unless all the a_n *vanish. The constant* $(p/(p-1))^p$ *is best possible.*

In [2], pp. 273-275, he proved that the arithmetic mean of (a_n) in (1) can be replaced by a more general mean which contains the Euler mean, Cesàro mean and Hölder mean as particular cases. Another more general case has been studied in [3] and [4] where the following has been proved.

THEOREM B. *Let* $C=(c_{m,k})$ *be a positive triangular matrix (i.e.* $c_{m,k}=0$ *for* $k>m$, *and* $c_{m,k}>0$ *for* $k \leqq m$ $(m=1,2,\cdots)$), *satisfying*

$$0 < \frac{c_{n,k}}{c_{m,k}} \leqq A_1 \qquad \text{for all } k \leqq m \leqq n\,, \tag{2}$$

and there exists a sequence $f(k)$, $f(k)\nearrow\infty$ *such that*

$$0 < f(n)c_{n,k}/f(m)c_{m,k} \leqq A_2\,, \qquad \text{for all } k \leqq m \leqq n \tag{3}$$

and

$$\sum_{n=k}^{\infty}[f(n)]^{-p} \leqq A_3[f(k)]^{1-p}\,, \qquad p>1\,, \tag{4}$$

where A_1, A_2 *and* A_3 *are constants.*

If $a_n \geqq 0$, and if $p>1$

$$\sum_{m=1}^{\infty}\left(\sum_{k=1}^{m} c_{m,k}\,a_k\right)^p \leqq A \sum_{m=1}^{\infty} (a_m f(m)\, c_{m,m})^p . \tag{5}$$

The case $c_{m,k} = 1/m$, $k \leqq m$, $c_{m,k} = 0$, $k > m$, and $f(k) = k$ satisfies the conditions of Theorem B and (5) reduces to (1).

2. In this paper we shall investigate various generalizations of these two theorems, the letters A, A_1, A_2 etc. will denote various constants independent of the terms under the summation sign.

THEOREM 1. *Let $p > 1$, $a_m \geqq 0$ $(m = 1, 2, \cdots)$, $g(m) > 0$ $(m = 1, 2, \cdots)$ and $C = (c_{m,k})$ be a positive triangular matrix. If*

$$\sum_{m=1}^{\infty} c_{m,m} < \infty \tag{6}$$

and

$$\sum_{m=n}^{\infty} g(m)\, c_{m,n}^p \leqq A_1\, g(n)\, c_{n,n}^{p-1} \qquad \textit{for all } n \geqq 1, \tag{7}$$

then

$$\sum_{m=1}^{\infty} g(m)\left(\sum_{n=1}^{m} c_{m,n}\, a_n\right)^p \leqq A_2 \sum_{m=1}^{\infty} g(m)\, a_m^p . \tag{8}$$

PROOF. Denote by U and V the left hand side and right hand side sums, respectively, in (8). Using Minkowski's inequality and (7)

$$U^{1/p} \leqq \sum_{n=1}^{\infty} a_n \left(\sum_{m=n}^{\infty} g(m)\, c_{m,n}^p\right)^{1/p} \leqq A_1^{1/p} \sum_{n=1}^{\infty} a_n [g(n)]^{1/p}\, c_{n,n}^{(p-1)/p} .$$

By Hölder's inequality and (6)

$$U^{1/p} \leqq A_2 \left(\sum_{n=1}^{\infty} a_n^p\, g(n)\right)^{1/p} \left(\sum_{n=1}^{\infty} c_{n,n}\right)^{1/q} \leqq A\, V^{1/p}$$

where $1/p + 1/q = 1$. This completes the proof of the Theorem 1.

As an immediate consequence we have

COROLLARY 1.1. *If* (6) *and*

$$g(m)\, c_{m,n}^{p} \leqq A\, g(n)\, c_{n,n}^{p-1} c_{m,m}\,. \tag{9}$$

for all $m \geqq n \geqq 1$, *hold, then* (8) *must follow.*

Let $g(m) = 1$ $(m=1, 2, \cdots)$, in Theorem 1 to obtain

COROLLARY 1.2. *If* (6) *and*

$$\sum_{m=n}^{\infty} c_{m,n}^{p} \leqq A\, c_{n,n}^{p-1} \qquad \text{for all } n \geqq 1\,, \tag{10}$$

then

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{m} c_{m,n}\, a_n\right)^p \leqq A \sum_{m=1}^{\infty} a_m^p\,.$$

Condition (6) is quite restrictive and we shall now attempt to find a more satisfactory alternative.

THEOREM 2. *Let* $a_m \geqq 0$ $(m = 1, 2, \cdots)$, $g(m) > 0$ $(m = 1, 2, \cdots)$ *and* $C = (c_{m,k})$ *be a positive triangular matrix which satisfies the conditions*

$$\sum_{m=\nu}^{\infty} g(m)\, c_{m,\mu} c_{m,\nu} \leqq A_1\, g(\nu)\, c_{\nu,\mu} \qquad \text{for all } \mu \leqq \nu \tag{11}$$

then

$$\sum_{m=1}^{\infty} g(m)\left(\sum_{\nu=1}^{m} c_{m,\nu}\, a_\nu\right)^2 \leqq A_2 \sum_{m=1}^{\infty} g(m)\, a_m^2\,. \tag{12}$$

PROOF. From the left hand side of (12)

$$\begin{aligned}
\sum_{m=1}^{\infty} g(m)\left(\sum_{\nu=1}^{m} c_{m,\nu}\, a_\nu\right)^2 &= \sum_{m=1}^{\infty} g(m)\left(\sum_{\nu=1}^{m} c_{m,\nu}\, a_\nu\right)\left(\sum_{\mu=1}^{m} c_{m,\mu}\, a_\mu\right)\\
&= \sum_{\nu=1}^{\infty} a_\nu\left(\sum_{m=\nu}^{\infty} c_{m,\nu}\, g(m)\left(\sum_{\mu=1}^{m} c_{m,\mu}\, a_\mu\right)\right)\\
&\leqq \sum_{\nu=1}^{\infty} a_\nu \sum_{m=\nu}^{\infty} c_{m,\nu}\, g(m)\left(\sum_{\mu=1}^{\nu} c_{m,\mu}\, a_\mu + \sum_{\mu=\nu}^{m} c_{m,\mu}\, a_\mu\right)\\
&= \sum_{\nu=1}^{\infty} a_\nu \sum_{\mu=1}^{\nu} a_\mu \sum_{m=\nu}^{\infty} c_{m,\mu} c_{m,\nu}\, g(m) + \sum_{\nu=1}^{\infty} a_\nu \sum_{\mu=\nu}^{\infty} a_\mu \sum_{m=\mu}^{\infty} c_{m,\mu} c_{m,\nu}\, g(m)\\
&= S + T\,.
\end{aligned}$$

From condition (11) and Schwarz's inequality, it follows that

$$S \leqq A_3 \sum_{\nu=1}^{\infty} a_\nu \sum_{\mu=1}^{\nu} a_\mu g(\nu) c_{\nu,\mu} = A_3 \sum_{\nu=1}^{\infty} a_\nu (g(\nu))^{\frac{1}{2}} (g(\nu))^{\frac{1}{2}} \sum_{\mu=1}^{\nu} a_\mu c_{\nu,\mu}$$

$$\leqq A_3 \left(\sum_{\nu=1}^{\infty} g(\nu) a_\nu^2 \right)^{1/2} \left(\sum_{\nu=1}^{\infty} g(\nu) \left(\sum_{\mu=1}^{\nu} c_{\nu,\mu} a_\mu \right)^2 \right)^{1/2} = A_3 S' \tag{13}$$

and

$$T \leqq A_4 \sum_{\nu=1}^{\infty} a_\nu \sum_{\mu=\nu}^{\infty} a_\mu g(\mu) c_{\mu,\nu} = A_4 \sum_{\mu=1}^{\infty} a_\mu g(\mu) \sum_{\nu=1}^{\mu} c_{\mu,\nu} a_\nu \leqq A_4 S' \tag{14}$$

Combining (13) and (14), we have the desired inequality (12).

An immediate consequence of Theorem 2 is the following

COROLLARY 2.1. *Let* $a_m \geqq 0$ $(m = 1, 2, \cdots)$, $g(m) > 0$ $(m = 1, 2, \cdots)$, $C = (c_{m,k})$ *be a positive triangular matrix and* α *any real number. If*

$$\sum_{m=\nu}^{\infty} (c_{m,m}^{1+\alpha} / g(m)^\alpha) \leqq A c_{\nu,\nu}^\alpha / g(\nu)^\alpha \qquad \text{for all } \nu \geqq 1$$

and

$$(g(m))^{(1+\alpha)/2} c_{n,n}^{(1+\alpha)/2} c_{m,\mu} \leqq A_1 (g(n))^{(1+\alpha)/2} c_{m,m}^{(1+\alpha)/2} c_{n,\mu}$$

for all $m \geqq n \geqq \mu$, *then inequality* (12) *holds.*

The case $\alpha = 0$ gives

COROLLARY 2.2. *If* $\sum_{m=1}^{\infty} c_{m,m} < \infty$ *and*

$$g(m) c_{n,n} c_{m,\mu}^2 \leqq A\, g(n) c_{m,m} c_{n,\mu}^2$$

for all $m \geqq n \geqq \mu$, *then inequality* (12) *holds.*

Another corollary of Theorem 2 is

COROLLARY 2.3. *Let* $a_m \geqq 0$ $(m = 1, 2, \cdots)$, $g(m) > 0$ $(m = 1, 2, \cdots)$, $C = (c_{m,k})$ *be a positive triangular matrix and* β *any real number. If*

$$\sum_{m=\nu}^{\infty} (c_{m,\nu} c_{m,m}^{1+\beta}(g(m))^{-\beta}) \leqq A c_{\nu,\nu}^{1+\beta}(g(\nu))^{-\beta} \qquad \text{for all } \nu \geqq 1$$

and

$$g(m)^{1+\beta} c_{n,n}^{1+\beta} c_{m,\mu} \leqq A_1 g(n)^{1+\beta} c_{m,m}^{1+\beta} c_{n,\mu} \qquad \text{for all } m \geqq n \geqq \mu,$$

then inequality (12) *holds.*

For the case $\beta=0$, we have

COROLLARY 2.4. *If*

$$\sum_{m=\nu}^{\infty} c_{m,\nu} c_{m,m} < A c_{\nu,\nu} \qquad \text{for all } \nu \geqq 1,$$

and

$$g(m) c_{n,n} c_{m,\mu} \leqq A_1 g(n) c_{m,m} c_{n,\mu} \qquad \text{for all } m \geqq n \geqq \mu,$$

then inequality (12) *holds.*

In Theorem 2, let us consider the case $c_{m,k} = p_k/q_m$, ($p_k > 0$, $q_m > 0$) and $g(m)=1$. We have

COROLLARY 2.5. *If* $\sum_{m=\nu}^{\infty} q_m^{-2} \leqq A(p_\nu q_\nu)^{-1}$ *then*

$$\sum_{m=1}^{\infty} \left(\frac{1}{q_m} \sum_{\nu=1}^{m} p_\nu a_\nu \right)^2 \leqq A_1 \sum_{\nu=1}^{\infty} a_\nu^2 .$$

By the same substitution in Theorem 1 we obtain

COROLLARY 2.6. *If* $\sum_{m=1}^{\infty} (p_m/q_m) < \infty$ *and*

$$\sum_{m=\nu}^{\infty} q_m^{-p} \leqq A(p_\nu^{-1} q_\nu^{-p+1}) \qquad \text{for all } \nu \geqq 1$$

then

$$\sum_{m=1}^{\infty} \left(\frac{1}{q_m} \sum_{\nu=1}^{m} p_\nu a_\nu \right)^p \leqq A_1 \sum_{m=1}^{\infty} a_m^p .$$

3. By an M matrix we shall denote a positive triangular matrix $C=(c_{m,k})$ which satisfies

$$0 \leqq \frac{c_{m,k}}{c_{n,k}} \leqq K \quad (0 \leqq k \leqq n \leqq m) \tag{15}$$

and for which there exists an $f(m) \nearrow \infty$ such that

$$\frac{c_{m,k}}{c_{n,k}} \leqq K_1 \frac{f(n)}{f(m)} \qquad (0 \leqq k \leqq n \leqq m) \tag{16}$$

where K and K_1 are constants.

THEOREM 3. *Let $p>1$. If $C=(c_{m,k})$ is an M matrix and $a_k \geqq 0$ $(k=1,2,\cdots)$, and if*

$$\sum_{m=1}^{\infty} \frac{c_{m,m}}{(f(m))^{p-1}} \quad \textit{converges} \tag{17}$$

and

$$\sum_{m=\nu}^{\infty} \frac{c_{m,m}}{(f(m))^{p-1}} \leqq \frac{A}{(f(\nu))^{p-1}} \tag{18}$$

it follow that

$$\sum_{m=1}^{\infty} c_{m,m} f(m) \left\{ \sum_{\nu=1}^{m} c_{m,\nu} a_\nu \right\}^p \leqq A_1 \sum_{m=1}^{\infty} c_{m,m} f(m) a_m^p . \tag{19}$$

PROOF. We first observe that from Lemma 1 in [4] we have

$$\begin{aligned}\left(\sum_{\nu=1}^{m} c_{m,\nu} a_\nu\right)^p &\leqq A_2 \sum_{\nu=1}^{m} c_{m,\nu} a_\nu \left(\sum_{r=1}^{\nu} c_{m,r} a_r\right)^{p-1} \\ &\leqq A_3 \sum_{\nu=1}^{m} c_{m,\nu} a_\nu \left(\frac{f(\nu)}{f(m)}\right)^{p-1} \left(\sum_{r=1}^{\nu} c_{\nu,r} a_r\right)^{p-1} .\end{aligned}$$

Hence

$$\begin{aligned}S_N &= \sum_{m=1}^{N} c_{m,m} f(m) \left\{ \sum_{\nu=1}^{m} c_{m,\nu} a_\nu \right\}^p \\ &\leqq A_3 \sum_{m=1}^{N} \left[\sum_{\nu=1}^{m} c_{m,m} f(m) c_{m,\nu} a_\nu \left(\frac{f(\nu)}{f(m)}\right)^{p-1} \left(\sum_{r=1}^{\nu} c_{\nu,r} a_r\right)^{p-1} \right]\end{aligned}$$

and it follows that

$$S_N \leqq A_3 \sum_{\nu=1}^{N} \left(\sum_{r=1}^{\nu} c_{\nu,r} a_r \right)^{p-1} \left[\sum_{m=\nu}^{N} c_{m,m} f(m)\, c_{m,\nu}\, a_\nu \left(\frac{f(\nu)}{f(m)} \right)^{p-1} \right].$$

Also, we have by applying (16) and then (17),

$$\sum_{m=\nu}^{N} c_{m,m} f(m)\, c_{m,\nu}\, a_\nu \left(\frac{f(\nu)}{f(m)} \right)^{p-1} = a_\nu (f(\nu))^{p-1} \sum_{m=\nu}^{N} c_{m,m} c_{m,\nu} (f(m))^{2-p}$$

$$= a_\nu (f(\nu))^{p-1} c_{\nu,\nu} \sum_{m=\nu}^{N} c_{m,m} \frac{c_{m,\nu}}{c_{\nu,\nu}} (f(m))^{2-p}$$

$$\leqq a_\nu (f(\nu))^{p-1} c_{\nu,\nu} K_1 \sum_{m=\nu}^{N} c_{m,m} f(\nu) (f(m))^{1-p}$$

$$\leqq a_\nu (f(\nu))^{p} c_{\nu,\nu} K_1 \sum_{m=\nu}^{N} c_{m,m} (f(m))^{1-p} \leqq a_\nu f(\nu) c_{\nu,\nu} K_1 A \leqq A_4 a_\nu f(\nu) c_{\nu,\nu} \,.$$

Substituting this relation in the previous expression and using Hölder's inequality

$$\sum_{m=1}^{N} c_{m,m} f(m) \left(\sum_{\nu=1}^{m} c_{m,\nu} a_\nu \right)^{p} \leqq A_3 A_4 \sum_{m=1}^{N} \left(\sum_{\nu=1}^{m} c_{m,\nu} a_\nu \right)^{p-1} a_m f(m) c_{m,m}$$

$$\leqq A_5 \left\{ \sum_{m=1}^{N} c_{m,m} f(m) a_m^p \right\}^{1/p} \left\{ \sum_{m=1}^{N} c_{m,m} f(m) \left(\sum_{\nu=1}^{m} c_{m,\nu} a_\nu \right)^{p} \right\}^{1/q},$$

where $1/p + 1/q = 1$.

Dividing both sides of the inequality by the last factor on the right and raising to the pth power,

$$\sum_{m=1}^{N} c_{m,m} f(m) \left(\sum_{\nu=1}^{m} c_{m,\nu} a_\nu \right)^{p} \leqq A_1 \sum_{m=1}^{\infty} c_{m,m} f(m) a_m^p \,.$$

By letting N tend to infinity, we prove our theorem.

Suppose an additional condition is satisfied by the matrix, namely

$$c_{m,m} f(m) \nearrow, \tag{20}$$

then let

$$a_n = (c_{n,n} f(n))^{-1/p} \mu_n$$

and substituting in (19), we have

$$\sum_{m=1}^{\infty}\left(\sum_{\nu=1}^{m} c_{m,\nu}\,\mu_\nu\right)^p \leqq A_1 \sum_{m=1}^{\infty} \mu_m^p . \tag{21}$$

A matrix satisfying (15), (16), (18) and (20) in the case $p=2$ is given by

$$c_{m,\nu} = \begin{cases} \nu^{1/2} m^{-3/2} & (\nu \leqq m) \\ 0 & (\nu > m). \end{cases}$$

For this matrix

$$\frac{c_{m,\nu}}{c_{n,\nu}} = \frac{\nu^{1/2}\, m^{-3/2}}{\nu^{1/2}\, n^{-3/2}} \leqq 1 \quad (0 \leqq \nu \leqq n \leqq m),$$

if $f(m) = m^{3/2}$,

$$\frac{c_{m,\nu}}{c_{n,\nu}} \leqq \frac{f(n)}{f(m)} \quad (0 \leqq \nu \leqq n \leqq m).$$

Furthermore,

$$\sum_{m=\nu}^{\infty} c_{m,m}[f(m)]^{-1} = \sum_{m=\nu}^{\infty} m^{-1} m^{-3/2} \leqq \int_{\nu-1}^{\infty} x^{-5/2}\, dx \leqq A\,\nu^{-3/2} = \frac{A}{f(\nu)},$$

and $c_{m,m} f(m) = m^{1/2} \nearrow$ so that the stated conditions are satisfied. From (21) this implies

$$\sum_{m=1}^{\infty}\left(\sum_{\nu=1}^{m} \nu^{1/2} m^{-3/2} \mu_\nu\right)^2 \leqq A_1 \sum_{m=1}^{\infty} \mu_m^2 .$$

4. We next turn our attention to

THEOREM 4. *If* $a_n \geqq 0$ $(n = 1, 2, \cdots)$, *and* $C = (c_{m,k})$ *is a positive triangular matrix, satisfying*

$$\sum_{m=n}^{\infty} (c_{m,k})^p \leqq A_1 n^{-p+1} \qquad \textit{for } 1 \leqq k \leqq n, \tag{22}$$

$(n=1, 2, \cdots)$ *for some* $p > 1$, *then*

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} c_{n,k}\, a_k\right)^p \leqq A_2 \sum_{n=1}^{\infty} a_n^p . \tag{23}$$

We shall also prove

THEOREM 5. *If* $p>1$ $(n=1,2,\cdots)$, $f(m)\geqq 0$ $(m=1,2,\cdots)$ *and* $C=(c_{m,k})$ *is a positive triangular matrix, satisfying the condition*

$$\sum_{n=k}^{\infty} n^{\varepsilon}(c_{n,k})^p \leqq A_1 k^{\varepsilon}(f(k)\, c_{k,k})^p \tag{24}$$

for a small $\varepsilon>0$ *and for all* $k\geqq 1$, *then*

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} c_{n,k}a_k\right)^p \leqq A_2 \sum_{n=1}^{\infty} n^{-1}(n\, a_n f(n)\, c_{n,n})^p\,. \tag{25}$$

PROOF OF THEOREM 4. Let $1/p+1/q=1$ and $0<r<1/q$. Then by Hölder's inequality,

$$\sum_{m=1}^{\infty}\left(\sum_{k=1}^{m} c_{m,k}a_k\right)^p = \sum_{m=1}^{\infty}\left(\sum_{k=1}^{m} k^r c_{m,k}a_k k^{-r}\right)^p \leqq \sum_{m=1}^{\infty}\left(\sum_{k=1}^{m} k^{rp}(c_{m,k}a_k)^p\right)\left(\sum_{k=1}^{m} k^{-rq}\right)^{p/q}$$

$$\leqq A_3\sum_{m=1}^{\infty} m^{p-rp-1}\sum_{k=1}^{m} k^{rp}(c_{m,k}a_k)^p \leqq A_4\sum_{k=1}^{\infty} k^{rp}\, a_k^p \sum_{m=k}^{\infty} m^{p-rp-1}(c_{m,k})^p\,. \tag{26}$$

Let $C_{n,k}=\sum_{m=n}^{\infty}(c_{m,k})^p$ $(k\leqq n)$ so that by (22)

$$C_{n,k}\leqq A_1 n^{-p+1} \qquad \text{for all } k\leqq n.$$

Hence, for the inner sum on the right hand side of (26),

$$\sum_{m=k}^{\infty} m^{p-rp-1}(c_{m,k})^p = \sum_{m=k}^{\infty} m^{p-rp-1}(C_{m,k}-C_{m+1,k})$$

$$= k^{p-rp-1}C_{k,k} + \sum_{m=k}^{\infty}[(m+1)^{p-rp-1}-m^{p-rp-1}]C_{m+1,k}$$

$$\leqq A_1 k^{-rp} + A_5\sum_{m=k}^{\infty}(m+1)^{p-rp-2}(m+1)^{-p+1} \leqq A_6 k^{-rp}\,. \tag{27}$$

Substituting (27) into (26) we complete the proof of Theorem 4.

PROOF OF THEOREM 5. As before, let $1/p+1/q=1$ and $0<r<1/q$. In (26) let r be chosen so that $\varepsilon=p-rp-1$, then (24) becomes

$$\sum_{m=k}^{\infty} m^{p-rp-1}(c_{m,k})^p \leqq A_1 k^{p-rp-1}(f(k)c_{k,k})^p$$

and then, by (26)

$$\sum_{m=1}^{\infty}\left(\sum_{k=1}^{m} c_{m,k}\, a_k\right)^p \leqq A_3 \sum_{k=1}^{\infty} k^{-1}(k\, a_k f(k) c_{k,k})^p$$

which is the required result.

Consider the Nörlund mean

$$c_{n,k} = \frac{p_{n-k+1}}{P_n} \qquad \text{for } k \leqq n \text{ and } c_{n,k} = 0 \text{ for } k > n,$$

where $p_n \geqq 0$ and $P_n = p_1 + p_2 + \cdots + p_n > 0$. Then we have the following

COROLLARY 5.1. *If* $a_n \geqq 0$, $f(m) \geqq 0$, $p_n \geqq 0$ *and*

$$\sum_{n=k}^{\infty} n^{\varepsilon}(p_{n-k+1}/P_n)^p \leqq A_1 k^{\varepsilon}(f(k)/P_k)^p \tag{28}$$

for an $\varepsilon > 0$ *and all* $k \geqq 1$, *then we have*

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} p_{n-k} a_k / P_n\right)^p \leqq A_2 \sum_{n=1}^{\infty} n^{-1}(n\, a_n f(n)/P_n)^p .$$

For example, we take $p_n \cong 1/\log n$, then $P_n \cong n \log n$ and (28) is satisfied for $0 < \varepsilon < p - 1$ and $f(n) = n^{1/p}/\log(n+1)$. Thus we have

$$\sum_{n=1}^{\infty}\left(\frac{\log n}{n} \sum_{k=1}^{n} \frac{a_k}{\log(n-k+1)}\right)^p \leqq A_2 \sum_{n=1}^{\infty} a_n^p .$$

We shall next consider the case $p_n \cong n^{-\alpha}$ $(0 \leqq \alpha < 1)$ then $P_n \cong n^{1-\alpha}$ and (28) is satisfied for $0 < \varepsilon < p - 1$ and $[f(n)]^p = n^{1-\alpha p}$, $\log n$ or 1 according as $0 \leqq \alpha < 1/p$, $\alpha = 1/p$ or $\alpha > 1/p$. Then we have

$$\sum_{n=1}^{\infty}\left(n^{\alpha-1} \sum_{k=1}^{n} \frac{a_k}{(n-k+1)^{\alpha}}\right)^p \leqq A_2 \sum_{n=1}^{\infty} a_n^p\, g(n)$$

where $g(n)$ is 1, $\log n$ or $n^{\alpha p-1}$ according as $0 \leqq \alpha < 1/p$, $\alpha = 1/p$ or $1/p < \alpha < 1$.

Finally we consider the case $p_n = 1/n$, then $P_n \cong \log n$ and $f(n) = 1$. The inequality that results is

$$\sum_{n=1}^{\infty}\left(\frac{1}{\log(n+1)}\sum_{k=1}^{n}\frac{a_k}{n-k+1}\right)^p \leqq A_2\sum_{n=1}^{\infty}\frac{a_n^p n^{p-1}}{\log(n+1)}.$$

Another corollay is

COROLLARY 5.2. *Let* $p>1$, $f(k)\geqq 0$ $(k=1,2,\cdots)$, *and* $C=(c_{n,k})$ *a positive triangular matrix. If*

$$\sum_{m=k}^{\infty} m^{\varepsilon}(c_{m,k})^p \leqq A_1 k^{\varepsilon}[f(k)]^p$$

for an $\varepsilon>0$ *and for all* $k\geqq 1$, *then*

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{c_{n,k}}{k}\right)^p \leqq A_2\sum_{n=1}^{\infty}\frac{[f(n)]^p}{n}.$$

5. In the previous paragraph, the conditions were imposed on the column sums of the matrix. We shall exhibit a theorem involving row sums.

THEOREM 6. *Let* $C=(c_{n,k})$ *be a positive triangular matrix and* $a_n\geqq 0$ $(n=1,2,\cdots)$, $f(n)\geqq 0$ $(n=1,2,\cdots)$, $g(n)\geqq 0$ $(n=1,2,\cdots)$, *and* $p>2$. *If*

$$\sum_{k=1}^{n}(c_{n,k})^p \leqq A_1(f(n)c_{n,n})^p \qquad \text{for all } n\geqq 1 \tag{29}$$

and

$$\sum_{n=k}^{\infty} n^{\varepsilon}(f(n)c_{n,n})^p \leqq A_2 k^{\varepsilon}(f(k)g(k)c_{k,k})^p \tag{30}$$

for an $\varepsilon>0$ *and for all* $k\geqq 1$, *then we have*

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}c_{n,k}a_k\right)^p \leqq A_3\sum_{n=1}^{\infty}n^{-2}(n a_n f(n) g(n) c_{n,n})^p. \tag{31}$$

PROOF. Let $p>2$, $1/p+1/q=1$ and $b>0$. By Hölder's inequality

$$\sum_{m=1}^{\infty}\left(\sum_{k=1}^{m}c_{m,k}a_k\right)^p = \sum_{m=1}^{\infty}\left(\sum_{k=1}^{m}k^b a_k k^{-b}c_{m,k}\right)^p$$

$$\leqq \sum_{m=1}^{\infty}\left(\sum_{k=1}^{m} k^{bp} a_k^p\right)\left(\sum_{j=1}^{m} j^{-bq} c_{m,j}^q\right)^{p/q} = \sum_{k=1}^{\infty} k^{bp} a_k^p \sum_{m=k}^{\infty}\left(\sum_{j=1}^{m} j^{-bq} c_{m,j}^q\right)^{p/q}. \tag{32}$$

Using Hölder's inequality again,

$$S = \sum_{m=k}^{\infty}\left(\sum_{j=1}^{m} j^{-bp} c_{m,j}^q\right)^{p/q} = \sum_{m=k}^{\infty}\left(\sum_{j=1}^{m} c_{m\, j}^p\right)\left(\sum_{r=1}^{m} r^{-bp/p-2}\right)^{p-2}$$

$$\leqq A_4 \sum_{m=k}^{\infty} m^{p-bp-2} \sum_{j=1}^{m} c_{m,j}^p\,, \tag{33}$$

if $0 < b < (p-2)/p$. We take b close to $p-2/p$ so that $\varepsilon = p-bp-2$, then (29) and (30) imply that

$$S \leqq A k^{-bp-2}(kf(k)\, g(k)\, c_{k,k})^p$$

and

$$\sum_{m=1}^{\infty}\left(\sum_{k=1}^{m} c_{m,k}\, a_k\right)^p \leqq A_3 \sum_{k=1}^{\infty} k^{-2}(ka_k f(k)\, g(k)\, c_{k,k})^p\,,$$

which completes the proof.

In the particular case $c_{n,k} = p_{n-k+1}/P_n$ for $k \leqq n$ and $c_{n,k} = 0$ for $k > n$, $P_n = p_1 + p_2 + \cdots + p_n > 0$ $(n=1,2,\cdots)$, (29) is satisfied when

$$f(n) = \left(\sum_{k=0}^{n} p_k^p\right)^{1/p}$$

and condition (30) is satisfied when

$$(f(k)\, g(k)/P_k)^p = k^{-\varepsilon} \sum_{n=k}^{\infty} n^{\varepsilon}.f(n)^p\, P_n^{-p}\,.$$

The conclusion of Theorem 6 becomes

$$\sum_{n=1}^{\infty}\left(P_n^{-1} \sum_{k=0}^{n} p_{n-k} a_k\right)^p \leqq A \sum_{n=1}^{\infty} a_n^p n^{p-2-\varepsilon} \sum_{m=n}^{\infty} m^{\varepsilon}\left(P_m^{-p} \sum_{k=0}^{m} p_k^p.\right) \tag{34}$$

Thus we have

COROLLARY 6.1. *Let $\varepsilon > 0$ and $p > 1$. If (p_n) is any positive sequence and $P_n = p_1 + p_2 + \cdots + p_n$ $(n = 1, 2, \cdots)$, then* (34) *holds for any series $\sum a_k$ with positive terms.*

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