Tôhoku Math. Journ.
22 (1970), 1-13.

AHLFORS'S CONJECTURE ON THE SINGULAR SET OF SOME KLEINIAN GROUP

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(Received March 10, 1969)

Introduction. In the theory of automorphic functions it is important to investigate the properties of the singular sets of the properly discontinuous groups. Recently I proved the existence of Kleinian groups with fundamental domains bounded by mutually disjoint circles whose singular sets have positive (3/2)-dimensional measure ([3]). Now in the natural way the following problem arises: *To what extent does the dimension of the singular sets of Kleinian groups climb up, when the number N of the boundary circles increase*? It is well known that the 2-dimensional measure of the singular set is always zero in the case that the fundamental domains are bounded by mutually disjoint circles ([9]).

Let G be a Kleinian group which acts both on the unit ball B in the 3-dimensional space and on its boundary S. Its action on B is always discontinuous, but its action on S will not in general be so. There is an exceptional set E of singular points. Ahlfors [2] proved that, if the isometric polyhedron of G in B has only a finite number of sides, then either E is all of S or the 2-dimensional measure of E in S is zero. But it seems still open that the 2-dimensional measure of the singular set of all finitely generated Kleinian groups is always zero (Ahlfors's conjecture).

The purpose of this paper is to solve this problem in the special case.

1. Let G be a group of linear transformations

$$S(z)=\frac{az+b}{cz+d}, \qquad ad-bc=1$$

of the extended complex plane. A point z_0 is called a singular point of G if it is an accumulation point of $S(z_1)$, $S\in G$, for some z_1. The set of singular points will be called the singular set of G and denoted by E.

E is a closed set, and invariant under G; we denote its complement by D.

This work was supported in part by a research contract AF 49 (638)-1591 (1967-68).

As soon as D is not empty we say that G is discontinuous, and D is its set of discontinuity. It is easy to classify all groups for which E is void, consists of a single point, or of two points. All other discontinuous groups will be called Kleinian groups, that is, a Kleinian group is a discontinuous group with more than two singular points.

A Kleinian group which leaves a circle invariant is said to be Fuchsian. The singular set lies on the invariant circle. If E is the whole circle, the Fuchsian group is of the first kind. If not, E is nowhere dense on the invariant circle, and the group is of the second kind.

2. Let π be the quotient space D/G. It has a natural complex structure such that the projection map $P: D \rightarrow \pi$ is holomorphic. Thus, the components π_i of π are Riemann surfaces. We shall write $D_i = P^{-1}\pi_i$. In general, the D_i are not connected, and we denote the components of D_i by D_{ij}.

With any Kleinian group G we have thus associated the decompositions $D = \cup D_i = \cup D_{ij}$ and $\pi = \cup \pi_i$. Each D_i is invariant under the full group G, and the boundary of D_i is all of E. The components D_{ij} are ramified covering surfaces of D_i whose branch points are elliptic fixed points.

Now add the hypothesis that one or several D_i are connected, and hence invariant under the full group G. Such groups and domains D_i have been called function groups and invariant regions. The fact that all invariant regions D_i have the same boundary E does not by itself preclude the existence of any number of such regions. However, the existence of non-elliptic fixed points does impose a severe restriction. R. Accola [1] proved the following important theorem : there are at most two invariant regions D_i, and if there are two, they are simply connected.

3. Let G be a finitely generated Kleinian group and further add the assumption of function group. For brevity we call it a finitely generated simple Kleinian group and write it F. S. K-group followed by Maskit [6]. Then from No. 2 there are at most two invariant regions D_i $(i = 1, 2)$. If there are two, the common boundary is a Jordan curve. Without loss of generality we may assume that the infinity is an ordinary point.

Now we consider a fundamental domain B_i $(i = 1, 2)$ for G with respect to D_i $(i = 1, 2)$ respectively, where D_1 is a component which does not contain the infinity. We can take a domain as a fundamental domain such that each B_i is bounded by open arcs of circles, called sides, where the endpoints of these arcs are called vertices and every vertex is an endpoint of precisely two sides. If the number of sides is finite, B_i is called finitely sided.

Further we impose a restrictive condition to B_i. Let C_j $(j = 1, \cdots, 2p)$ be the boundary circles which form the part of the boundary of B_i $(i = 1, 2)$. Let D_j $(j = 1, \cdots, 2p)$ be a closed disc bounded by C_j $(j = 1, \cdots, 2p)$. Then the set

$\{D_j\}$ $(j=1, \cdots, 2p)$ are called a finite closed chain, if they satisfy the following two conditions : (i) any two of the D_j have no interior point in common, and (ii) D_j has a common boundary point with D_{j+1} for $j=1,\cdots, 2p$ (where $C_{2p+1}=C_1$). We call such a group an F. S. K-group with finite closed chain.

Starting from an F. S. K-group, we reached the fundamental domain of it. Conversely let us form any finite closed chain. If we can define the discontinuous group whose fundamental domain is one of two components, which are the complementary sets of this chain, it is known that the singular set of this group is a Jordan curve and has a complicated shape ([5], [7]).

4. Now let G be an F. S. K-group with finite closed chain. The purpose of this paper is to prove that the 2-dimensional measure of the singular set E of G is zero.

Let B_1 be a fundamental domain bounded by $2p$ circular arcs $\{\overline{C}_i\}(i=1,\cdots,2p)$ which does not contain the infinity, where $\overline{C}_i$ is a closed circular arc on the circle C_i. Without loss of generality we can assume that the origin 0 is contained in B_1. The elements of G, which map some side $\overline{C}_{i_1}$ of B_1 onto some other side $\overline{C}'_{i_1}$ of B_1, generate G.

Let S_{i_1} denote a generating transformation which transforms the outside of C_{i_1} onto the inside of C'_{i_1}, that is, maps the side $\overline{C}_{i_1}$ onto the side $\overline{C}'_{i_1}$ of B_1, where C_{i_1} and C'_{i_1} are some circles of $\{C_i\}$ $(i=1,\cdots,2p)$. Obviously $S_{i_1}^{-1}$ transforms the outside of C'_{i_1} onto the inside of C_{i_1}.

In general, we denote by ST the transformation obtained by composition of transformations S and T, that is, $ST(z)=S(T(z))$. Consider the totality G of all linear transformations in the form

$$(1) \qquad S=S_{i_k}^{\lambda_k} S_{i_{k-1}}^{\lambda_{k-1}} \cdots S_{i_1}^{\lambda_1}, \text{ viz, } S(z)=S_{i_k}^{\lambda_k}(S_{i_{k-1}}^{\lambda_{k-1}}(\cdots(S_{i_1}^{\lambda_1}(z))\cdots))$$

together with the identical transformation, where λ_j $(1\leqq j\leqq k)$ are integers and $i_j\neq i_{j-1}$ for $j=2,\cdots,k$.

We call the sum

$$m=\sum_{j=1}^{k}|\lambda_j|$$

the grade of S. The image $S(B_1)$ of the fundamental domain B_1 of G by S $(\in G)$ with grade m $(\neq 0)$ is bounded by $2p$ mutually tangent circular arcs $\overline{C}_i^{(m)}$ $(i=1,\cdots,2p)$, the one $\overline{C}_{i_0}^{(m)}$ of which is contained in the boundary of the image of B_1 under some T $(\in G)$ with grade $m-1$. For simplicity, we say that the $2p-1$ circular arcs of $S(B_1)$ different from $\overline{C}_{i_0}^{(m)}$ are circular arcs of grade m. Circular arcs $\{\overline{C}_i\}$ $(i=1,\cdots,2p)$, which bound B_1, are of grade zero. The number of circular arcs of grade m is obviously equal to $2p(2p-1)^m$.

The circumstance with respect to the fundamental domain B_2 containing the infinity is the same as B_1. If we denote the closure of the complementary circular arc of the circular arc $\overline{C}_i^{(m)}$ of grade m by $\overline{\overline{C}}_i^{(m)}$, which is also called a circular arc of grade m, the join $\overline{C}_i^{(m)} \cup \overline{\overline{C}}_i^{(m)}$ is a complete circle and is denoted by $C_i^{(m)}$, which is called a circle of grade m.

5. Given a set $\mathcal{E}$ of points in the z-plane and a positive number δ, we denote by $I(\delta, \mathcal{E})$ a family of a countable number of closed discs U of diameter $l_U \leqq \delta$ such that every point of $\mathcal{E}$ is an interior point of at least one U.

We call the quantity

$$m_\eta(\mathcal{E}) = \lim_{\delta \to 0} \left[\inf_{\{I(\delta,\varepsilon)\}} \sum_{U \in I(\delta,\varepsilon)} l_U^\eta \right] \tag{2}$$

the η-dimensional measure of $\mathcal{E}$.

6. Let E be the singular set of an F. S. K-group defined in No. 4. Let us give a covering of E by using the set of circles of grade m ($\geqq m_0$) in order to calculate the 2-dimensional measure of E.

Denote by $\mathcal{D}_m$ a finite closed chain, that is, a continuum consisting of the whole closed discs bounded by $2p(2p-1)^m$ circles of grade m, which are tangent externally in turns. Evidently $\{\mathcal{D}_m\}(m = 1, 2, \cdots)$ is a monotone decreasing sequence of continua. The set $\bigcap_{m=1}^{\infty} \mathcal{D}_m$ is consistent with E. We call $\mathcal{D}_m$ the m-th net of E. Since the points of contact during circles are also singular points, we can not give up them by the definition (2). Still more when the number of grade m increases, the set of points of contact will have regardful strength. Therefore we must form another covering of such points.

7. Let ξ_j ($j = 1, \cdots, 2p$) be the points of contact between C_j and C_{j+1}, where $C_{2p+1} = C_1$. These ξ_j ($j = 1, \cdots, 2p$) are divided into some systems, which are called parabolic cycles. Take a parabolic point, for example, ξ_{i_n} from a parabolic cycle $\{\xi_{i_1}, \xi_{i_2}, \cdots, \xi_{i_k}\}$, ($k \leqq 2p$). Suppose that ξ_{i_n} is the point of contact between C_{i_n} and C_{i_n+1}. Then, from the definition of a parabolic cycle, ξ_{i_n} is the fixed point of the parabolic transformation $S_{(k)}$ of grade k.

Let $O_n^{(q)}$ ($q = 1, 2, \cdots$) be the image circles, which go through ξ_n and are tangent internally with C_n at ξ_n, by $S_{(k)}^q$, ($q = 1, 2, \cdots$) and $O_{n+1}^{(q)}$ ($q = -1, -2, \cdots$) be the image circles, which go through ξ_n and are tangent internally with C_{n+1} at ξ_n, by $S_{(k)}^q$, ($q = -1, -2, \cdots$). Circles $O_n^{(q)}$ ($q = 1, 2, \cdots$) and $O_{n+1}^{(q)}$ ($q = -1, -2, \cdots$) are called oricycles of C_n and C_{n+1} at ξ_n, respectively. It is obvious that $O_n^{(q)}$ and $O_{n+1}^{(q)}$ are circles of grade $|q|k$. If $|q|$ tends to infinity, both oricycles converge to ξ_n from both sides.

8. Now describe the circles K_n $(n=1,\cdots,2p)$ with centers ξ_n $(n=1,\cdots,2p)$ and sufficiently small radii r_n $(n=1,\cdots,2p)$ and denote by d_n $(n=1,\cdots,2p)$ the circular arcs of K_n $(n=1,\cdots,2p)$ cut off by B_1, where ξ_n is a parabolic point belonging to some parabolic cycle.

Consider any transformation $S_{(\nu)}$, of grade ν for a sufficiently large integer ν. Then the number of images $S_{(\nu)}(d_n)$ of d_n $(n=1,\cdots,2p)$ by whole transformations $S_{(\nu)}$ of grade ν is obviously equal to $(2p)^2(2p-1)^{\nu-1}$. The images $S_{(\nu)}(d_n)$ $(n=1,\cdots,2p)$ are the $2p$ circular arcs which join the $2p$ boundary circular arcs of the image $S_{(\nu)}(B_1)$ in turn. If we take some $S_{(\nu)}(d_n)$, then $S_{(\nu)}(d_n)$ and the boundary circular subarcs of $S_{(\nu)}(B_1)$ joined by $S_{(\nu)}(d_n)$ form a circular triangle, one of whose vertices facing $S_{(\nu)}(d_n)$ is a parabolic point, that is, a singular point. In this triangle we denote the smaller one of two euclidian distances joining this parabolic point with the other two vertices by $r_n^{(\nu)}$ and describle a circle $K_n^{(\nu)}$ of radius $r_n^{(\nu)}$ with center at this parabolic point. There are $2p(2p-1)^\nu$ parabolic points of $S_{(\nu)}(z)$ in all. The set of these parabolic points $\{\xi_n^{(\nu)}\}$ $(n=1,2,\cdots,2p(2p-1)^\nu)$ consist of the points of contact of the finite closed chain $\mathfrak{D}_\nu$.

The set $\{\xi_n^{(\nu)}\}$ $(n=1,2,\cdots,2p(2p-1)^\nu)$ are divided into two kinds in the following.

(1) If $\xi_n^{(\nu)}$is a parabolic point of some $S_{(\nu)}(z)$, but not a parabolic point of any $S_{(\nu-1)}(z)$, that is, there is no circular arcs of grade $\nu-1$ which go through $\xi_n^{(\nu)}$, we let a circle $K_n^{(\nu)}$ of radius $r_n^{(\nu)}$ with center at $\xi_n^{(\nu)}$ correspond to $\xi_n^{(\nu)}$.

(2) If $\xi_n^{(\nu)}$ is a parabolic point of some $S_{(\nu)}(z)$ and still also of some $S_{(\nu-1)}(z)$, that is, there is two externally tangent circular arcs of grade $\nu-1$ which go through $\xi_n^{(\nu)}$, we get two circular triangles with common vertex $\xi_n^{(\nu)}$, each of which is formed by $S_{(\nu)}(d_n)$ and two boundary circular subarcs of $S_{(\nu)}(B_1)$ joined by $S_{(\nu)}(d_n)$. In these two triangles we denote the minimum of four euclidean distances joining this parabolic point $\xi_n^{(\nu)}$ with the other four vertices by $r_n^{(\nu)}$ and let a circle $K_n^{(\nu)}$ of radius $r_n^{(\nu)}$ with center at $\xi_n^{(\nu)}$ correspond to $\xi_n^{(\nu)}$.

For $2p(2p-1)^\nu$ parabolic points $\{\xi_n^{(\nu)}\}$ $(n=1,2,\cdots,2p(2p-1)^\nu)$ we form closed discs $\Delta_n^{(\nu)}$ $(n=1,2,\cdots,2p(2p-1)^\nu)$ bounded by $K_n^{(\nu)}$ in the above methods (1) and (2). It is obvious that the set $\{\Delta_n^{(\nu)}\}$ $(n=1,2,\cdots,2p(2p-1)^\nu)$ form a covering of parabolic points that are singular points. Considering the set $\mathfrak{D}_\nu \cup \left(\bigcup_{n=1}^{2p(2p-1)^\nu} \Delta_n^{(\nu)}\right)$, we get a covering of E and call this the complete ν-th net of E.

9. In the above we got a covering of E in order to investigate the 2-dimensional measure of E. It is important to calculate the radii of discs in this complete ν-th net of E.

Let $c^{(\nu)}$ be a coefficient of z in the denominator of any transformation $S_{(\nu)}$ $(\in G)$ of grade ν such that

$$S_{(\nu)}(z)=\frac{a^{(\nu)}z+b^{(\nu)}}{c^{(\nu)}z+d^{(\nu)}},\qquad a^{(\nu)}d^{(\nu)}-b^{(\nu)}c^{(\nu)}=1. \tag{3}$$

We state the well known lemma, which needs later, without proof.

LEMMA 1. (Ford [4]) *If the point at infinity is an ordinary point of G, the series $\Sigma|c^{(\nu)}|^{-2m}$ converges for $m\geqq 2$, where in the summation the finite number of terms for which $c^{(\nu)}=0$ are omitted.*

10. Now let us solve the Ahlfors's conjecture in this special case by using the complete ν-th net of E formed in No. 8. Denote by $R_i^{(\nu)}$ $(i=1,\cdots,2p(2p-1)^{\nu})$ the radii of circles of $\mathcal{D}_\nu$ and by $r_n^{(\nu)}$ $(n=1,\cdots,2p(2p-1)^{\nu})$ the radii of circles $K_n^{(\nu)}$ to the complete ν-th net of E, respectively. In order to prove that the 2-dimensional measure $m_2(E)$ of the singular set E of an F. S. K-group G with finite closed chain is zero, it is enough to show that

$$I\equiv 4\left\{\sum_{i=1}^{2p(2p-1)^{\nu}}(R_i^{(\nu)})^2+\sum_{n=1}^{2p(2p-1)^{\nu}}(r_n^{(\nu)})^2\right\} \tag{4}$$

is less than any given small ε.

At first we shall give the estimate of $R_i^{(\nu)}$ from the above. Let $R_i^{(\nu)}$ be the radius of $C_i^{(\nu)}$. Then $C_i^{(\nu)}$ is a complete circle consisting of two circular arcs which are images of boundary circular arcs of B_1 and B_2 by the transformation $S_{(\nu)}$ of grade ν in the form (3). We describe two circles K_j and K_{j+1} with small radii and centers at the points of contact ξ_j and ξ_{j+1} of the boundary circle C_j of B_1, respectively.

Denote by $\widetilde{C}_j$ the join of two circular arcs of C_j outside these K_j and K_{j+1}. Taking sufficiently small radii of K_j and K_{j+1}, we can always assume that the length of the image of $\widetilde{C}_j$ by $S_{(\nu)}$ on $C_i^{(\nu)}$ is larger than the half of circumference of the circle $C_i^{(\nu)}$. Therefore we have

$$R_i^{(\nu)}=\frac{1}{2\pi}\int_{C_j}\left|\frac{dS_{(\nu)}(z)}{dz}\right||dz|<\frac{1}{\pi}\int_{\widetilde{C}_j}\left|\frac{dS_{(\nu)}(z)}{dz}\right||dz| \tag{5}$$

and

$$\frac{1}{\pi}\int_{\widetilde{C}_j}\left|\frac{dS_{(\nu)}(z)}{dz}\right||dz|=\frac{1}{\pi|c^{(\nu)}|^2}\int_{\widetilde{C}_j}\frac{|dz|}{\left|z+\dfrac{d^{(\nu)}}{c^{(\nu)}}\right|^2} \tag{6}$$

$$<\frac{2R_j}{\{\Delta(j)\}^2|c^{(\nu)}|^2}=k(j)\frac{1}{|c^{(\nu)}|^2},\quad k(j)=\frac{2R_j}{(\Delta(j))^2},$$

where R_j is the radius of C_j and $\Delta(j)$ is a quantity smaller than the distances from $\widetilde{C}_j$ to the pole $-\frac{d^{(\nu)}}{c^{(\nu)}}$. Since $k(j)$ is a constant depending only on K_j and K_{j+1} and there are $2p$ such constants in all with respect to the boundary circles C_j $(j=1,\cdots,2p)$, we can take the maximum σ of these constants and have from (5) and (6)

$$(7)\qquad R_j^{(\nu)}<\sigma\frac{1}{|c^{(\nu)}|^2}.$$

11. Now we shall estimate $r_n^{(\nu)}$ in (4) from the above. Let us return to the notations of No. 8. Let $l_n^{\pm}$ be the circular arcs, which form the two sides of a triangle, different from d_n and $(l_n^{\pm})^{(\nu)}$ and $\xi_n^{(\nu)}$ be the images of $l_n^{\pm}$ and ξ_n by the transformation $S_{(\nu)}$ of grade ν, respectively. The minimum of two euclidian distances from this parabolic point $\xi_n^{(\nu)}$ to the endpoints of these $(l_n^{\pm})^{(\nu)}$ is the radius $r_n^{(\nu)}$ of $K_n^{(\nu)}$ and $\{\Delta_n^{(\nu)}\}(n=1,\cdots,2p(2p-1)^\nu)$ bounded by $K_n^{(\nu)}$ $(n=1,\cdots,2p(2p-1)^\nu)$ form a covering of parabolic points of the transformations of grade ν.

We can easily obtain

$$(8)\qquad |r_n^{(\nu)}|\leqq\int_{l_n^{\pm}}\frac{|dz|}{|c^{(\nu)}z+d^{(\nu)}|^2}=\frac{1}{|c^{(\nu)}|^2}\int_{l_n^{\pm}}\frac{|dz|}{\left|z+\dfrac{d^{(\nu)}}{c^{(\nu)}}\right|^2}.$$

(i) First we consider the case in which $S_{(\nu)}^{-1}(\infty)=-\frac{d^{(\nu)}}{c^{(\nu)}}$, that is, the pole of $S_\nu(z)$, is not contained in the domain bounded by the oricycle $O_n^{(1)}$ or $O_{n+1}^{(-1)}$, which is the image of C_n by $S_{(k)}$ or C_{n+1} by $S_{(k)}^{-1}$, where C_n and C_{n+1} are two boundary circles of the finite closed chain which are tangent externally at ξ_n. When z is on $l_n^{\pm}$, there exists a constant $\mathcal{K}(n)$ depending only on $l_n^{\pm}$ such that it holds

$$(9)\qquad \left|z+\frac{d^{(\nu)}}{c^{(\nu)}}\right|>\mathcal{K}(n).$$

Denoting the minimum of $\mathcal{K}(n)$ $(n=1,\cdots,2p)$ by

$$\mathcal{K}=\min_{1\leqq n\leqq 2p}\mathcal{K}(n),$$

we obtain from (8) and (9)

$$|r_n^{(\nu)}| < \frac{\max|l_n^{\pm}|}{\mathcal{K}} \cdot \frac{1}{|c^{(\nu)}|^2}, \tag{10}$$

where $|l_n^{\pm}|$ denotes the euclidean length of $l_n^{\pm}$ and $\mathcal{K}$ is a constant bounded below depending only on G.

(ii) Secondly we consider the case in which $S_{(\nu)}^{-1}(\infty) = -\dfrac{d^{(\nu)}}{c^{(\nu)}}$ is contained in the domain bounded by the oricycle $O_n^{(1)}$ or $O_{n+1}^{(-1)}$.

(a) Suppose that $-\dfrac{d^{(\nu)}}{c^{(\nu)}}$ is sufficiently close to ξ_n and is contained in the domain bounded by two oricycles $O_n^{(q)}$ and $O_n^{(q+1)}$ ($q>0$; integer), which are the images of C_n by $S_{(k)}^q$ and $S_{(k)}^{q+1}$, respectively. We put

$$S_{(\nu)}(z) = S(S_{(k)}^{-q}(z)). \tag{11}$$

(b) In the case in which $-\dfrac{d^{(\nu)}}{c^{(\nu)}}$ is contained in the domain bounded by two oricycles $O_{n+1}^{(-q)}$ and $O_{n+1}^{(-(q+1))}$, which are the images of C_{n+1} by $S_{(k)}^{-q}$ and $S_{(k)}^{-(q+1)}$, respectively, we put

$$S_{(\nu)}(z) = S(S_{(k)}^{q}(z)).$$

Since the discussion about (b) is the same as (a), it is enough to consider the case (a) only. Then it is obvious that $S^{-1}(\infty)$ is in the domain $D_n^{(1,2)}$ bounded by two oricycles $O_n^{(1)}$ and $O_n^{(2)}$.

It is obvious that

$$\left|\frac{dS_{(\nu)}(z)}{dz}\right| = \left|\frac{dS(z_q)}{dz_q}\right|\left|\frac{dz_q}{dz}\right|, \tag{12}$$

where $z_q = S_{(k)}^{-q}(z)$ and z is any point on $l_n^{\pm}$.

Let us estimate the two factors of the right hand side of (12) from the above. Since $z_q = S_{(k)}^{-q}(z)$ is parabolic, it is represented by the following form:

$$\frac{1}{z_q - \xi_n} = \frac{1}{z - \xi_n} + bq, \tag{13}$$

where b is a constant independent of q. Then we easily obtain

$$\left|\frac{dz_q}{dz}\right| = \frac{1}{|bq|^2} \frac{1}{\left|z - \left(\xi_n - \dfrac{1}{bq}\right)\right|^2}. \tag{14}$$

We choose the orthogonal coordinate system such that the x-axis goes through the points $\xi_n - \frac{1}{bq}$, $(q=1,2,\cdots)$ and ξ_n is the origin. Then we can deform (14) in the following form :

$$\left|\frac{dz_q}{dz}\right| = \frac{1}{(|b|qx+1)^2+(|b|qy)^2}\,. \tag{15}$$

If we consider the right hand side of (15) as the function of q, we can easily find that this take the maximum value $1+\left(\frac{x}{y}\right)^2$ at $q=-\frac{x}{|b|(x^2+y^2)}$. Since z is on $l_n^\pm$, $\frac{x}{y}$ tends to 0 for $z\to\xi_n$ along $l_n^\pm$. Hence the maximum value in (15) is bounded and there exists some positive number μ such that it holds

$$\left|\frac{dz_q}{dz}\right| < \mu \qquad \text{for any } z\in l_n^\pm\,.$$

The discussion for each parabolic point ξ_n is the same as the above and therefore we see easily that it holds for ξ_n $(n=1,\cdots,2p)$

$$\left|\frac{dz_q}{dz}\right| < \mu \qquad \text{for any } z\in l_n^\pm\,. \tag{16}$$

12. By using the estimate that was given by Myrberg [8] in the proof of divergence of the (-2)-dimensional Poincaré theta series with respect to Fuchsian groups of the first kind whose fundamental domain is finitely sided, we obtain the following Lemma.

LEMMA 2. *Suppose that* $S_{(\nu)}(z)=S(S_{(k)}^{-q}(z))$ *has the property* (a) *in the case* (ii), *where* $S(z)$ *has the following form :*

$$S(z)=\frac{\alpha z+\beta}{\gamma z+\delta}\,, \quad \alpha\delta-\beta\gamma=1\,. \tag{17}$$

Then it holds

$$\left|\frac{dS_{(\nu)}(z)}{dz}\right| < s\frac{1}{|\gamma|^2}\,, \tag{18}$$

where s is a constant which is bounded above, PROOF. We assumed that the origin 0 is contained in B_1 (No. 4). Since $S^{-1}(\infty)$ is in the domain $D_n^{(1,2)}$ bounded by two oricycles $O_n^{(1)}$ and $O_n^{(2)}$, it is obvious that $S^{-1}(0)$ is also in $D_n^{(1,2)}$. Denote by K_0 and K_1 the circles with center at the origin whose radii r and R are sufficiently small and large, respectively, such that K_0 is contained in B_1 and K_1 contains the finite closed chain $\mathcal{D}_0$ and is contained in B_2. Since the point of B_1, which is sufficiently close to ξ_n, is also mapped into the point of $D_n^{(1,2)}$, which is sufficiently close to ξ_n, by the parabolic transformation $S_{(k)}(z)$, we can describe a circle τ_n of small radius ρ with center ξ_n such that $K_0 \cap \tau_n = \emptyset$ and the point $S^{-1}(0)$, which is equivalent to 0, is not contained in τ_n.

Then from the above we obtain

$$\left|\xi_n + \frac{\beta}{\alpha}\right| = |\xi_n - S^{-1}(0)| > \rho. \tag{19}$$

Since $S(\xi_n)$ is also a singular point from the invariance of the singular set by any element of this group, it holds

$$\left|\frac{\alpha\xi_n+\beta}{\gamma\xi_n+\delta}\right| = \left|\frac{\alpha}{\gamma}\right| \left|\frac{\xi_n + \frac{\beta}{\alpha}}{\xi_n + \frac{\delta}{\gamma}}\right| < R. \tag{20}$$

It is easily seen that

$$\left|\frac{\alpha}{\gamma}\right| = |S(\infty)| > r \tag{21}$$

from the assumption that the infinity is contained in B_2. Then we obtain, from (19), (20), and (21), the following inequality :

$$\left|\xi_n + \frac{\delta}{\gamma}\right| > \rho\frac{r}{R}. \tag{22}$$

We can easily find from (13) and (14)

$$\left|\frac{z_q - \xi_n}{z - \xi_n}\right| = \sqrt{\left|\frac{dz_q}{dz}\right|}. \tag{23}$$

Since first we took the arbitrarily small circle K_n of radius r_n with center ξ_n in No. 7, we can take r_n so small that K_n may be contained in τ_n, that is, $r_n < \varepsilon_1 < \rho$. Then it holds from (16) and (23)

(24) $$|z_q-\xi_n|<r_n\sqrt{\mu}<\varepsilon_1\sqrt{\mu}$$

for any point z on $l_n^{\pm}$. Hence it implies from (22) and (24)

(25) $$\left|z_q+\frac{\delta}{\gamma}\right|>\omega=\rho\frac{r}{R}-\varepsilon_1\sqrt{\mu}\ ,$$

where ω is a positive constant bounded below. Because r_n is arbitrarily small and hence ε_1 can be taken so small that the right hand side of (25) may be positive. Hence we obtain from (25) the following inequality :

(26) $$\left|\frac{dS(z_q)}{dz_q}\right|=\frac{1}{|\gamma z_q+\delta|^2}<\frac{1}{\omega^2}\cdot\frac{1}{|\gamma|^2}\ .$$

By using (16) and (26), we have the following inequality:

$$\left|\frac{dS_{(\nu)}(z)}{dz}\right|<s\cdot\frac{1}{|\gamma|^2}\qquad\text{for any } z\in l_n^{\pm},$$

where $s=\mu/\omega^2$ is a constant bounded above. q. e. d.

Thus we have from (18) of Lemma 2 the following inequality:

(27) $$|r_n^{(\nu)}|\leqq\int_{l_n^{\pm}}\left|\frac{dS_{(\nu)}(z)}{dz}\right|<\frac{s}{|\gamma|^2}\ \max\ |l_n^{\pm}|\ .$$

Put $\delta=\max\left(s,\dfrac{1}{\mathcal{K}^2}\right)$, where $\mathcal{K}$ is a constant given in (10). If we correspond $S(S_{(k)}^{-q}(z))$, $(q=0,1,2,\cdots)$ to $S_{(\nu)}$, we obtain, by arranging (10) and (27),

(28) $$|r_n^{(\nu)}|<\delta\ \max\ |l_n^{\pm}|\cdot\frac{1}{|\gamma|^2}\ .$$

13. Thus we get from (7) and (28) the following estimate of (4) from the above :

(29) $$I\leqq 4\left\{\sigma\sum_{j=1}^{2p(2p-1)^{\nu}}\frac{1}{|c_j^{(\nu)}|^4}+\delta^2\max_{1\leqq n\leqq 2p}|l_n^{\pm}|^2\left(\sum_S{}'\frac{1}{|\gamma|^4}\right)\right\},$$

where Σ' denotes the summation with respect to terms corresponding to (28).

Since it holds from Lemma 1 that $\sum_{S \in G} \frac{1}{|\gamma|^4} < \infty$ for all transformations except ones satisfying $\gamma = 0$, the first summation $\sum_{j=1}^{2p(2p-1)^\nu} \frac{1}{|c_j^{(\nu)}|^4}$ of (29) tends to zero for $\nu \to \infty$. We note that $\gamma = 0$ arises only in the case of the identical transformation, since the point at infinity is an ordinary point and G does not contain the elliptic transformation in this case. Hence we can determine a sufficiently large ν depending only on any given small number ε such that the firs term in the brace does not exceed $\frac{\varepsilon}{8}$. Since $\sum_S' \frac{1}{|\gamma|^4}$ is finite, we can make the radii of circles K_n $(n = 1, \cdots, 2p)$ with centers ξ_n $(n = 1, \cdots, 2p)$ sufficiently small for such fixed ν such that the second term in the brace is smaller than $\frac{\varepsilon}{8}$. Thus we obtain the following inequality:

$$4\left\{\sigma \sum_{j=1}^{2p(2p-1)^\nu} \frac{1}{|c_j^{(\nu)}|^4} + \delta^2 \max_{1 \leq n \leq 2p} |l_n^{\pm}|^2 \left(\sum_S{}' \frac{1}{|\gamma|^4}\right)\right\} < \varepsilon. \tag{30}$$

Since ε is an arbitrary small number, this leads to the conclusion that $m_2(E)$ is equal to zero in our special case.

Therefore we have the following

THEOREM. *Let G be an F. S. K-group whose fundamental domain is bounded by the boundary circular arcs of a finite closed chain. Then the 2-dimensional measure of the singular set of G, which is a Jordan curve and a common boundary of two invariant regions, is zero.*

14. REMARK. We treated the case of a finite closed chain consisting of $2p$ boundary circles thus far. Generally the case of the odd number of boundary circles may occur and in this case the group has at least one elliptic transformation with period 2 as a generator. But if we take a reflection with respect to this circle, we shall obtain a subgroup G' of G formed by the even number of generators. It is easily seen that the singular set of G' is consistent with one of G. Hence it is sufficient to consider the group G with $2p$ generators as the case we treated.

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