Tôhoku Math. Journ.
22(1970), 200-209.

HIGHER ORDER TANGENT BUNDLES OF PROJECTIVE SPACES AND LENS SPACES

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(Received Oct. 22, 1969)

Introduction. In [6], [7] and [8] H. Suzuki considered higher order non-immersions of projective spaces in real affine spaces or projective spaces by means of characteristic classes, γ-operations and spin operations. In [9] C. Yoshioka obtained complete formulas of Stiefel-Whitney classes of higher order tangent bundles of complex projective spaces and Dold manifolds and he applied his results to higher order non-immersions of these spaces. In this paper, we shall study higher order tangent bundles of complex projective spaces, quaternion projective spaces and lens spaces and compute characteristic classes of them and apply these results to higher order non-immersions of quaternion projective spaces and lens spaces.

I am grateful to Prof. H. Suzuki for his valuable suggestions.

1. Preliminaries. Let G be a compact connected Lie group, F be R or C, the real or complex number field. Let V be a finite dimensional G-vector space over F and $[V]$ be a G-isomorphism class of V, then the dimension of V is said to be the degree of $[V]$.

We denote k-fold symmetric product over F of V by O^kV, then G acts on O^kV as follows:

$$g(v_1 \bigcirc v_2 \bigcirc \cdots \bigcirc v_k) = gv_1 \bigcirc gv_2 \bigcirc \cdots \bigcirc gv_k \quad \text{for} \quad g \in G,$$

where $v_1 \bigcirc v_2 \bigcirc \cdots \bigcirc v_k$ is the image of $v_1 \otimes v_2 \otimes \cdots \otimes v_k$ by the symmetrization operator from k-fold tensor product $\otimes^k V$ to O^kV. Thus O^kV is a G-vector space. We have the following lemma:

LEMMA 1.1. *Let V be one dimensional G-vector space, then O^kV is isomorphic to $\otimes^k V$.*

Let $M_F(G)$ be a semiring which consists of all G-isomorphism classes of finite dimensional G-vector spaces over F. The sum and product in $M_F(G)$ are induced by direct sum and tensor product of finite dimensional G-vector spaces over F. We define $O^k[V]$ for $[V] \in M_F(G)$ as follows

$$O^k[V] = [O^k V],$$

then O^k induces an operation of $M_F(G)$ having following properties:

i) $O^0(x) = 1, O^1(x) = x$ for $x \in M_F(G)$,

ii) $O^k(x+y) = \sum_{i+j=k} O^i(x)O^j(y)$ for $x, y \in M_F(G)$,

iii) $O^k(x) = x^k$ for $x \in M_F(G)$; of degree 1.

Let $R_F(G)$ be the ring completion of $M_F(G)$ and $\theta: M_F(G) \longrightarrow R_F(G)$ be the natural inclusion map.

Then the above O^k can be extended to $R_F(G)$ and the properties i) ,ii) are preserved in $R_F(G)$ too, but the properties iii) is satisfied in $\mathrm{Im}\theta$ only.

Now, let r, c, ψ_C^{-1} be the following operations

$$r: R_C(G) \longrightarrow R_R(G) \text{ realification,}$$

$$c: R_R(G) \longrightarrow R_C(G) \text{ complexification,}$$

$$\psi_C^{-1}: R_C(G) \longrightarrow R_C(G) \text{ complex conjugation.}$$

Then we have the following lemma (see [1]).

LEMMA 1.2. i) *r is a group homomorphism and c and ψ_C^{-1} are ring homomorphisms.*

ii) $rc = 2$, $cr = 1 + \psi_C^{-1}$.

iii) *c is injective.*

iv) $cO^k = O^k c$.

Next, let t be an indeterminate and let $1 + R_F(G)[[t]]^+$ be the multiplicative group which consists of all units in the ring $R_F(G)[[t]]$.

We define a map

$$O_t: R_F(G) \longrightarrow 1 + R_F(G)[[t]]^+$$

by

$$O_t(x) = \sum_{k=0}^{\infty} O^k(x)t^k \quad \text{for} \quad x \in R_F(G).$$

Then, by Lemma 1.2, we have

$$O_t(cx) = \sum_{k=0}^{\infty} (cO^k(x))t^k \quad \text{for} \quad x \in R_R(G).$$

The following theorem is described in [2].

THEOREM 1.3. *$R_C(U(1))$ equals $Z[Z, Z^{-1}]$, where z is a $U(1)$-isomorphism class of degree 1 such that $U(1)$ acts on one dimensional vector space C (the field of complex numbers) as follows: $(e^{i\theta}, w) \longmapsto e^{i} \cdot w$ for $e^{i\theta} \in U(1)$, $w \in C$. And $z^{-1} = \psi_C^{-1} z$.*

Let $\eta = rz - 2 \in R_R(U(1))$, then we have the following lemma:

LEMMA 1.4. i) $\psi_R^k(\eta) = rz^k - 2$, $\psi_R^0(\eta) = 0$, $\psi_R^{-k}(\eta) = \psi_R^k(\eta)$,

ii) $\eta^j = \sum_{i=1}^{j} (-1)^{j-i} \binom{2j}{j-i} \psi_R^i(\eta)$, iii) $\psi_R^k(\eta) = \sum_{j=1}^{k} A_j^k \eta^j$,

iv) $\psi_R^i(\eta)\psi_R^j(\eta) = \psi_R^{i+j}(\eta) + \psi_R^{j-i}(\eta) - 2(\psi_R^i(\eta) + \psi_R^j(\eta))$,

where ψ_R^k is the real Adams operation (see [1]) and

$$A_j^k = \frac{2}{(2j)!} \prod_{i=0}^{j-1} (k^2 - i^2) = \frac{k}{j} \binom{k+j-1}{2j-1}.$$

Since proofs require only tedious calculations, we omit them.

2. Calculations and applications. In the first place, we prove the following key lemma:

LEMMA 2.1.

$$O^k((n+1)rz) = \sum_{j=0}^{[\frac{k-1}{2}]} \binom{n+j}{j}\binom{n+k-j}{k-j} \psi_R^{k-2j}(\eta) + \binom{2n+k+1}{k}.$$

PROOF. Since z and z^{-1} are of degree 1

$$\begin{aligned} O_t((n+1)crz) &= (O_t(z + z^{-1}))^{n+1} = (1 - zt)^{-(n+1)}(1 - z^{-1}t)^{-(n+1)} \\ &= 1 + \sum_{k=1}^{\infty} \sum_{i+j=k} \binom{n+i}{i}\binom{n+j}{j} z^{j-i} t^k. \end{aligned}$$

Hence

$$cO^k((n+1)rz) = \sum_{i+j=k} \binom{n+i}{i}\binom{n+j}{j} z^{j-i}.$$

By Lemma 1.2 and 1.4, the proof is completed.

Let h_C be the canonical complex line bundle over an n-dimensional complex projective space CP^n.

By Suzuki's theorem (1.1) of [7], the k-th order tangent bundle of an n-dimensional complex projective space CP^n is given by

$$\tau_k(CP^n) = O^k((n+1)rh_C) - O^{k-1}((n+1)rh_C) - 1.$$

Hence by Lemma 2.1 we have

THEOREM 2.2. *Let $y = rh_C - 2$, then*

$$\tau_k(CP^n) + 1 = \sum_{j=0}^{[\frac{k-1}{2}]} \binom{n+j}{j}\left\{\binom{n+k-j}{k-j}\psi_R^{k-2j}(y) - \binom{n+k-j-1}{k-j-1}\psi_R^{k-2j-1}(y)\right\} + \binom{2n+k}{k}.$$

For example, by Lemma 1.4 we have

$$\tau_2(CP^n) + 1 = \binom{n+2}{2}y^2 + \binom{2n+3}{2}y + \binom{2n+2}{2},$$

$$\tau_3(CP^n) + 1 = \binom{n+3}{3}y^3 + (2n+5)\binom{n+2}{2}y^2 + 6\binom{2n+4}{3}y + \binom{2n+3}{3},$$

$$\tau_4(CP^n) + 1 = \binom{n+4}{4}y^4 + (2n+7)\binom{n+3}{3}y^3 + 3(2n+5)\binom{n+3}{3}y^2 + 2\binom{2n+5}{4}y + \binom{2n+4}{4}.$$

Next we calculate k-th order Pontrjagin class $P(\tau_k(CP^n))$ of CP^n.

Let x be a generator of $H^2(CP^n; Z)$. Then we can see easily that the Pontrjagin class of rh_C^j is given by

$$P(\psi_R^j(y)) = P(rh_C^j) = 1 + j^2x^2 .$$

Since $H^*(CP^n; Z)$ has no 2-torsion, we have the following corollary:

COROLLARY 2.3.

$$P(\tau_k(CP^n)) = \sum_{j=0}^{[\frac{k-1}{2}]} \left\{ \frac{(1+(k-2j)^2x^2)^{\binom{n+k-j}{k-j}}}{(1+(k-2j-1)^2x^2)^{\binom{n+k-j-1}{k-j-1}}} \right\}^{\binom{n+j}{j}} .$$

Now we calculate higher order tangent bundles and higher order charactristic classes of the quaternion projective space HP^n.

Let $\pi: CP^{2n+1} \longrightarrow HP^n$ be the canonical S^2-bundle and let h_H be the canonical complex plane bundle over the quaternion projective space HP^n. We have following commutative diagram

$$\begin{array}{ccccc} KO(CP^{2n+1}) & \underset{r}{\overset{c}{\rightleftarrows}} & K(CP^{2n+1}) & \overset{ch}{\longrightarrow} & H^*(CP^{2n+1}; Q) \\ \uparrow \pi^! & & \uparrow \pi^! & & \uparrow \pi^* \\ KO(HP^n) & \underset{r}{\overset{c}{\rightleftarrows}} & K(HP^n) & \overset{ch}{\longrightarrow} & H^*(HP^n; Q) \end{array}$$

where all vertical arrows are ring monomorphisms (see [4]).

The following lemma is easily seen by means of results described in [4].

LEMMA 2.4. i) *Complex conjugation* $\psi_C^{-1}: K(HP^n) \to K(HP^n)$ *is identity.* ii) *First order tangent bundle* $\tau(HP^n) = (n+1)rh_H - \frac{(rh_H)^2}{4}$. iii) $\pi^! rh_H = 2rh_C$. iv) $\pi^! h_H = h_C + h_C^{-1}$. v) $crh_H = 2h_H$. vi) $\pi^* q = x^2$, *where q and x are the generators of* $H^4(HP^n; Q)$(*or* $H^4(HP^n; Z)$) *and* $H^2(CP^{2n+1}; Q)$(*or* $H^2(CP^{2n+1}; Z)$) *respectively.* vii) *Restriction of* $c: KO(CP^{2n+1}) \to K(CP^{2n+1})$ *to free part of* $KO(CP^{2n+1})$ *and* $c: KO(HP^n) \to K(HP^n)$ *are injective.*

By this lemma we have following fact

$$\pi^! c(\tau(HP^n) + 1) = 2(n+1)(h_C + h_C^{-1}) - h_C^2 - h_C^{-2} - 1 .$$

Thus

$$O_t(\pi^! c(\tau(HP^n) + 1)) = O_t(2(n+1)(h_C + h_C^{-1}))(1 - h_C^2 t)(1 - h_C^{-2} t)(1 - t)$$

Hence

$$\pi^! \tau_k(HP^n) + 1 = O^k - O^{k-3} - (rh_C^2 + 1)(O^{k-1} - O^{k-2}),$$

where $O^j = O^j(2(n+1)rh_C)$.

By Lemma 2.1 and iv) of Lemma 1.4, we have

THEOREM 2.5.

$$\pi^! \tau_k(HP^n) + 1 = \sum_{j=0}^{[\frac{k-1}{2}]} \binom{2n+1+j}{j} \binom{2n+1+k-j}{k-j} \psi_R^{k-2j}(y)$$

$$- \sum_{j=0}^{[\frac{k}{2}]-1} \binom{2n+1+j}{j} \binom{2n+1+k-1-j}{k-1-j} (\psi_R^{k+1-2j}(y) + \psi_R^{k-1-2j}(y) + \psi_R^{k-3-2j}(y))$$

$$+ \sum_{j=0}^{[\frac{k-1}{2}]-1} \binom{2n+1+j}{j} \binom{2n+1+k-2-j}{k-2-j} (\psi_R^{k-2j}(y) + \psi_R^{k-2-2j}(y) + \psi_R^{k-4-2j}(y))$$

$$- \sum_{j=0}^{[\frac{k}{2}]-2} \binom{2n+1+j}{j} \binom{2n+1+k-3-j}{k-3-j} \psi_R^{k-3-2j}(y)$$

$$+ (-1)^k \binom{2n+1+\left[\frac{k-1}{2}\right]}{\left[\frac{k-1}{2}\right]}^2 \psi_R^2(y) + \binom{4n+k}{k},$$

where $y = rh_C - 2$.

Next we calculate k-th order Stiefel-Whitney class $W(\tau_k(HP^n))$.

$$W(\psi_R^j(y)) = W(rh_C^j) = C(h_C^j) \bmod 2 = 1 + j\bar{x},$$

where $\bar{x} = x \bmod 2$, and $C(h_C^j)$ is Chern class of h_C^j.

Hence,

$$W(\psi_R^j(y)) = \begin{cases} 1 & \text{for even } j \\ 1+\bar{x} & \text{for odd } j\,. \end{cases}$$

In general, for an odd positive integer l

$$\sum_{j=0}^{\frac{l-1}{2}} \binom{2n+1+j}{j}\binom{2n+1+l-j}{l-j} = \frac{1}{2}\binom{4n+l+3}{l},$$

and

$$\binom{4n+l+3}{l} \equiv 0 \pmod 4 .$$

Thus, by Theorem 2.5,

$$\bar{\pi}^* W(\tau_k(HP^n)) = \begin{cases} (1+\bar{x})^{2N_0(n,k)} & \text{for odd } k \\ (1+\bar{x})^{-2N_e(n,k)} & \text{for even } k, \end{cases}$$

where $\bar{\pi}^* = \pi^* \bmod 2$ is injective and

$$N_0(n, k) = \frac{1}{4}\binom{4n+k+3}{k} + \frac{3}{4}\binom{4n+k+1}{k-2} \text{ for odd } k,$$

$$N_e(n, k) = \frac{3}{4}\binom{4n+k+2}{k-1} + \frac{1}{4}\binom{4n+k}{k-3} \text{ for even } k.$$

Now $(1+\bar{x})^2 = 1+\bar{x}^2 = 1+\bar{\pi}^*\bar{q} = \bar{\pi}^*(1+\bar{q})$, where $\bar{q} = q \bmod 2$. Therefore

THEOREM 2.6.

$$W(\tau_k(HP^n)) = \begin{cases} (1+\bar{q})^{N_0(n,k)} & (k: \text{odd}) \\ (1+\bar{q})^{-N_e(n,k)} & (k: \text{even}). \end{cases}$$

Let $\delta_0(n, k)$, $\sigma_0(n, k)$, $\delta_e(n, k)$, $\sigma_e(n, k)$ be following integers

$$\delta_0(n, k) = \max\left\{1 \leqq i \leqq n ; \binom{N_0(n, k)}{i} \not\equiv 0 \pmod 2\right\},$$

$$\sigma_0(n, k) = \max\left\{1 \leqq i \leqq n ; \binom{N_0(n, k)+i-1}{i} \not\equiv 0 \pmod 2\right\},$$

$$\delta_e(n, k) = \max\left\{1 \leqq i \leqq n ; \binom{N_e(n, k)}{i} \not\equiv 0 \pmod 2\right\},$$

$$\sigma_e(n, k) = \max\left\{1 \leqq i \leqq n ; \binom{N_e(n, k)+i-1}{i} \not\equiv 0 \pmod 2\right\}.$$

By this theorem and Theorem (1.1) of [6], we have the following corollary.

COROLLARY 2.7. *When k is odd, if m is an integer such that $-4\delta_0(n,k)<m<4\sigma_0(n,k)$ and when k is even, if m is an integer such that $-4\sigma_e(n,k)<m<4\delta_e(n,k)$, then*

$$HP^n \not\subseteq_k R^{\binom{4n+k}{k}-1+m}.$$

Since Pontrjagin class of $\tau_k(HP^n)$ can be easily calculated in similar manner of Corollary 2.3 and Theorem 2.6, we omit its calculation.

Now we consider higher order tangent bundles, characteristic classes and non-immersions of lens space.

Let p be an integer larger than one and let $L^n(p)=L(p;1,\cdots,1)$ be $2n+1$-dimensional lens space of mod p. The following fact is well known (see [3])

$$\tau(L^n(p))+1=(n+1)\pi^! rh_C,$$

where $\pi: L^n(p) \longrightarrow CP^n$ is canonical S^1-bundle.

By Lemma 2.1 we have

THEOREM 2.8.

$$\tau_k(L^n(p))+1=\sum_{j=0}^{\left[\frac{k-1}{2}\right]}\binom{n+j}{j}\binom{n+k-j}{k-j}\psi_R^{k-2j}(\sigma)+\binom{2n+1+k}{k},$$

where $\sigma=\pi^!(y)$.

We have the following corollary on k-th order characteristic classes of $L^n(p)$. H. Suzuki informed me of this result.

COROLLARY 2.9. *If p is odd, the Pontrjagin class of $\tau_k(L^n(p))$ is given by*

$$P(\tau_k(L^n(p)))=\prod_{j=0}^{\left[\frac{k-1}{2}\right]}(1+(k-2j)^2u^2)^{\binom{n+j}{j}\binom{n+k-j}{k-j}},$$

and if p is even, the Stiefel-Whitney class of $\tau_k(L^n(p))$ is given by

$$W(\tau_k(L^n(p)))=\begin{cases}(1+\bar{u})^{\frac{1}{2}\binom{2n+1+k}{k}} & (k:\text{ odd})\\ 1 & (k:\text{ even}),\end{cases}$$

where u is generator of $H^2(L^n(p); Z)$ and $\bar{u} = u \bmod 2$.

PROOF. $P(\psi_R^j(\sigma)) = \pi^*(1+j^2x^2) = 1+j^2(\pi^*x)^2 = 1+j^2u^2$. Since p is odd, $H^*(L^n(p); Z)$ has no 2-torsion element (see 5.2. Theorem of [5]).

Thus, the proof for the Pontrjagin class is completed.

$$W(\psi_R^j(\sigma)) = 1 + j\bar{\pi}^*\bar{x} = 1 + j\bar{u} = \begin{cases} 1+\bar{u} (j: \text{odd}) \\ 1 \ (j: \text{even}). \end{cases}$$

This completes the proof for the Stiefel-Whitney class.

Next by Theorem 2.8 and Corollary 2.9, we consider k-th order non-immersions of $L^n(3)$ in real affine spaces.

$\pi^! h_C^3 = 1$ in $K(L^n(3))$ and by Lemma 1.2 and Theorem 1.3, $rz^{-1} = rz$. Hence $\psi_R^2(\sigma) = \psi_R^1(\sigma) = \sigma$.

Let $K_{n,k}^0, K_{n,k}^1$, and $K_{n,k}^2$ be the following integers

$$K_{n,k}^0 = \sum_{j=0}^{[\frac{k-1}{6}]} \binom{n+3j}{3j}\binom{n+k-3j}{k-3j},$$

$$K_{n,k}^1 = \sum_{j=0}^{[\frac{k-3}{6}]} \binom{n+3j+1}{3j+1}\binom{n+k-3j-1}{k-3j-1},$$

$$K_{n,k}^2 = \sum_{j=0}^{[\frac{k-5}{6}]} \binom{n+3j+2}{3j+2}\binom{n+k-3j-2}{k-3j-2}.$$

Then by Theorem 2.7 we have

$$\tau_k^0(L^n(3)) = (K_{n,k}^\alpha + K_{n,k}^\beta)\sigma \quad \text{for} \quad \alpha+\beta \equiv k \ (\text{mod } 3) \ \& \ \alpha \neq \beta,$$

where τ_k^0 is stable class of $\tau_k(L^n(3))$.

We employ k-th order Pontrjagin classes on k-th order non-immersions of $L^n(3)$.

H. Suzuki informed me that this manner is more convenient on this problem than γ-operations.

$$P(\tau_k(L^n(3))) = (1+u^2)^{K_{n,k}^\alpha + K_{n,k}^\beta} \quad \text{for} \quad \alpha+\beta \equiv k (\text{mod} 3) \ \& \ \alpha \neq \beta.$$

Let $d_P(n, k)$, $s_P(n, k)$ be the following integers

$$d_P(n, k) = \max\left\{m\,;\ 1 \leqq m \leqq \frac{n}{2}\,,\ \binom{K^{\alpha}_{n,k} + K^{\beta}_{n,k}}{m} \not\equiv 0 (\bmod 3)\right\},$$

$$s_P(n, k) = \max\left\{m\,;\ 1 \leqq m \leqq \frac{n}{2}\,,\ \binom{K^{\alpha}_{n,k} + K^{\beta}_{n,k} + m - 1}{m} \not\equiv 0 (\bmod 3)\right\}.$$

By similar argument on Pontrjagin classes to Theorem (1.1) of [6], we have the following theorem.

THEOREM 2.10. *For* $-2d_P(n, k) < m < 2s_P(n, k)$

$$L^n(3) \not\subseteq {}_kR^{\binom{2n+1+k}{k}-1+m}.$$

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