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ON RIEMANNIAN MANIFOLDS WITH CERTAIN CUT LOCI II

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1. Introduction. It is interesting to investigate the structure of a complete Riemannian manifold whose first conjugate locus $Q(p)$ or cut locus $C(p)$ with respect to a point $p \in M$ satisfies certain conditions. The structures of M satisfying suitable conditions for the first conjugate locus have been studied by many people. Especially, Warner [8] has proved that if there exists a point p in a compact and simply connected Riemannian manifold M for which each point of the spherical conjugate locus is regular, then that has the same multiplicity as conjugate point and the multiplicity is greater than or equal to 1, and M is homeomorphic to a sphere of the same dimension as M or M has the same integral cohomology ring as one of the compact irreducible symmetric spaces of rank 1. As for the structures of M satisfying suitable conditions for the cut locus $C(N)$ with respect to a submanifold N of M ($\dim N \geqq 0$), Ōmori has shown in [6] that if a connected, compact and real analytic Riemannian manifold M has a connected, compact and real analytic Riemannian submanifold N of $\dim N \geqq 0$ in such a way that the distance between N and every point of the cut locus $C(N)$ of N is constant, then M has a decomposition $M = D_N \cup \phi D_{N'}$, where N' is a real analytic submanifold of M which coincides with $C(N)$ as a set and D_N, $D_{N'}$ are normal disc bundles of N, N' respectively.

Recently, the authors have studied in [5] some structures of M admitting a fixed point p on it where the distance between p and every point of its cut locus $C(p)$ is a constant l. More recently the structures of some Sasakian manifolds with minimal diameter have been investigated by Harada [2] who has proved that the Sasakian manifold is isometric to the sphere under certain conditions. And the second author [7] has investigated the structure of a complete and non-compact Riemannian manifold M of non-negative curvature with compact totally geodesic hypersurface N every point of whose cut locus $C(N)$ has a constant distance to N.

Our Main Theorem obtained in the present paper is stated as follows.

MAIN THEOREM. *Let M be an n-dimensional, connected and compact Riemannian manifold of class C^∞. Assume that there exists a point p at* *which the distance between p and every point of its cut locus $C(p)$ is equal to $\pi/\sqrt{\operatorname{Max} K(P)}$, where $K(P)$ is a sectional curvature with respect to the plane section P. Then every geodesic segment starting from p with length $2\pi/\sqrt{\operatorname{Max} K(P)}$ is a geodesic loop at p, and we have for any point q belonging to the first conjugate locus $Q(p)$ of p, the multiplicity of p and q as conjugate pair is constant λ, where $\lambda=0, 1, 3, 7, n-1$. Moreover we have*

(1) *If M is not simply connected, M has the same (co)homology group as that of a real projective space PR^n and the universal covering manifold of M is homeomorphic to S^n, where $\lambda=0$ holds.*

(2) *If M is simply connected, the integral cohomology ring $H^*(M, \boldsymbol{Z})$ is a truncated polynomial ring generated by an element. In particular, if $\lambda=n-1$, M is isometric to the sphere $S^n(\operatorname{Max} K(P))$ of constant curvature* $\operatorname{Max} K(P)$.

2. Preliminaries. Throughout this paper, let M be an n-dimensional, connected and compact Riemannian manifold of class C^∞ and p be a fixed point of M such that the distance between p and every point of its cut locus $C(p)$ is constant. It will turn out that every geodesic starting from the point p has a conjugate point to p along it. Therefore the maximum value of sectional curvatures must be positive, from which we can consider M satisfying $K(P)\leqq 1$ for every plane section P. We use definitions and notations as those of [5].

A proof of Main Theorem will be completed if we show that every geodesic segment starting from p with length 2π (the metric tensor of M is changed so as to satisfy $K(P)\leqq 1$ for all P) is a geodesic loop (or a closed geodesic segment without self-intersection). For this purpose, we prepare a Proposition investigated by H. Ōmori:

PROPOSITION (3.4 Prop. of [6]). *Let N be a connected and compact Riemannian manifold of class C^∞ and W be a connected, compact and differentiable Riemannian submanifold of N. Suppose that there is a point $p\in C(W)$ at which $d(p, W)=d(C(W), W)=a$ holds and there are two different shortest geodesics Γ_1, Γ_2 from p to W satisfying $\mathcal{L}(\Gamma_1)=\mathcal{L}(\Gamma_2)=a$ and $\gamma_1'(0)\neq\pm\gamma_2'(0)$. Then we have*

$$\exp_p \frac{a(x\gamma_1'(0)+y\gamma_2'(0))}{\|x\gamma_1'(0)+y\gamma_2'(0)\|}\in W \qquad \textit{for all} \quad x\geqq 0,\ y\geqq 0\,.$$

3. Proof of Main Theorem. In the following let M satisfy the hypothesis of Main Theorem. Our method of proof is essentially due to that of Berger [1] who has proved that if an even dimensional, compact and simply connected Riemannian manifold M satisfying $0<K(P)\leqq 1$ for all plane sections P has its diameter $d(M)=\pi$, then all geodesics in M are closed with length 2π and

the cut locus with respect to every point of M becomes a submanifold of M.

First of all we shall prove the following

PROPOSITION 1. *We obtain either* $C(p)=Q(p)$ *or* $Q(p)=\{p\}$.

PROOF. Suppose that there is a geodesic segment $\Gamma=\{\gamma(t)\}$ $(0\leqq t\leqq\pi)$, $\gamma(0)=p$, $\gamma(\pi)=q\in C(p)$ along which q is not conjugate to p. Then we claim that $Q(p)=\{p\}$. In fact, there exists a small neighborhood $U\subset M_p$ of $\pi\cdot\gamma'(0)$ in which $\exp_p|U$ is a diffeomorphism. For each point r in $\exp_p(U)\cap C(p)$, there is a uniquely determined shortest geodesic $\Gamma_r=\{\gamma_r(t)\}$ $(0\leqq t\leqq\pi)$ in $\exp_p(U)$ such that $\pi\cdot\gamma_r'(0)\in U$, $\gamma_r(0)=p$, $\gamma_r(\pi)=r$ and r is not conjugate to p along Γ_r. By virtue of the Proposition of Ōmori, $\Gamma_r|[0,2\pi]$ is a geodesic loop at P for any point $r\in\exp_p(U)\cap C(p)$, and we see that $\gamma_r(2\pi)=p$ is the first conjugate point to $\gamma_r(0)=p$ along Γ_r. Making use of the discussion stated in Theorem 2.6 of [5], we get $Q(p)=\{p\}$. Q.E.D.

REMARK. We can prove this proposition without the assumption $l=\pi$. We also see that M has the same (co)homology group as that of PR^n and the universal covering manifold of M is homeomorphic to S^n if $Q(p)=\{p\}$ holds.

Now we assume that $Q(p)=C(p)$. Take a point $q\in C(p)$. Let $M_q^\perp$ be defined by $M_q^\perp=\{X\in M_q\mid\|X\|=1,\exp_q\pi X=p\}$. In order to prove the theorem it suffices to show that $M_q^\perp$ becomes the intersection of $S_q^{n-1}(1)$ and a subspace of M_q for every point $q\in C(p)$, where $S_q^{n-1}(1)$ is the unit hypersphere in M_q centered at origin. We note that $M_q^\perp$ has the following properties:

(1) For any $Z\in M_q$, there is $X\in M_q^\perp$ such that $\langle X,Z\rangle\geqq 0$.

(2) For any $X,Y\in M_q^\perp$, such that $X\neq -Y$, we have $\dfrac{\alpha X+\beta Y}{\|\alpha X+\beta Y\|}\in M_q^\perp$ for all $\alpha\geqq 0$, $\beta\geqq 0$.

(3) $M_q^\perp$ is closed in M_q.

The property (2) is guaranteed by Ōmori's Proposition and (1), (3) are evident. Developing the same argument as that of Berger [1], we see that there exists $X_0\in M_q^\perp$ satisfying $-X_0\in M_q^\perp$. Suppose that there exists a point $q\in C(p)$ at which there is a vector $-X\in M_q^\perp$ such that $X\notin M_q^\perp$. We may suppose $\langle -X,X_0\rangle=0$ and $X\notin M_q^\perp$ by virtue of the property (2). Let Γ_0 be the closed geodesic segment at p defined by $\gamma_0'(\pi)=X_0$. The map $\phi:(0,\pi)\times(0,\pi)\to M$ defined by $\phi(s,t):=\exp_q t(X_0\cos s-X\sin s)$ gives a piece of totally geodesic surface with boundary Γ_0 which is isometric to an open hemisphere of constant curvature 1. After developing a local argument in a convex normal ball at p, we see that Γ_0 becomes a closed geodesic, and hence the unit parallel vector field $X(t)$ along Γ_0 defined by $X(\pi)=-X$ has the properties $X(0)=X(2\pi)$ and $K(X(t),\gamma_0'(t))=1$ for all $t\in[0,2\pi]$. Therefore, as is stated in Lemma 1 of Berger [1], there exist small positive numbers ε and η satisfying $d(p, \exp_q tZ) < \pi$ for any $Z \in M_q$, $\|Z\|=1$, $<Z, X> \geqq 1-\eta$ and any $t \in (0, \varepsilon)$.

LEMMA 2. *There exists* $Y \in M_q{}^{\perp}$ *satisfying* $<Y, X> > 0$, *where* $-X \in M_q{}^{\perp}$, $X \notin M_q{}^{\perp}$ *and* $<X_0, X> = 0$.

PROOF. Let $-k^2$ be the minimum value of sectional curvatures. If $k=0$, the proof is concluded in that of Lemma 2 of [1]. We only consider $k > 0$. Since $d(p, \exp_q tX) < \pi$ holds for all $t \in (0, \varepsilon)$, let us denote by Σ, Λ_t and Ψ_t the geodesics such that $\sigma'(0)=X$, $\lambda_t(\pi-s) = \sigma(t)$, $\lambda_t(\pi) = p$, $d(p, \sigma(t)) = s$, and $\psi_t(0)=q$, $\psi_t(u)=\lambda_t(0)$, $d(q, \lambda_t(0))=u$ respectively. The statement mentioned in the last paragraph of Lemma 2 shows that $<X, \psi_t'(0)> < 1-\eta$ for any $t \in (0, \varepsilon)$. Putting $\cos \alpha_t = <\lambda_t'(\pi-s), \sigma'(t)>$, this fact and the assumption of X imply that $\overline{\lim_{t\to 0}} \alpha_t < \pi$ and $\underline{\lim_{t\to 0}} \alpha_t > 0$. We get a family of small geodesic triangles with vertices $(q, \sigma(t), \psi_t(u))$, $t \in (0, \varepsilon)$ shrinking to a point q as $t \to 0$ in such a way that the angles at vertices q and $\sigma(t)$ take limits in $(0, \pi)$ as $t \to 0$. We can choose a subsequence $\{\overline{\Lambda}_t\}$ of the family $\{\Lambda_t\}$ $(0 < t < \varepsilon)$ converging to Λ_0 which connects q to p with length π. Then we observe that $\sin \measuredangle (X, \psi_t'(0)) \leqq (\pi-s)/t$ for each small $t \in (0, \varepsilon)$, and hence $\lim_{t\to 0} (\pi-s)/t \geqq \lim_{t\to 0} \sin \measuredangle (X, \psi_t(0)) \geqq [1-(1-\eta)^2]^{1/2} > 0$. Making use of the basic theorem on triangles, we obtain the following

$$\cos \alpha_t \geqq \frac{\cosh \pi/k - \cosh s/k \cdot \cosh t/k}{\sinh s/k \cdot \sinh t/k}.$$

Because of $\lim_{x\to 0} \dfrac{\sinh bx}{\sinh x} = b$, we obtain

$$\begin{aligned}\lim_{t\to 0} \cos \alpha_t &\geqq 2 \coth \frac{s}{k} \cdot \lim_{t\to 0} \frac{\sinh \dfrac{(1+c)t}{2k}}{\sinh \dfrac{t}{k}} \cdot \sinh \frac{(c-1)t}{2k} + \lim_{t\to 0} \frac{\sinh ct/k}{\sinh t/k} \\ &= c > 0,\end{aligned}$$

where we put $c=[1-(1-\eta)^2]^{1/2}$. Hence we get $\lambda_0(0) \in M_q{}^{\perp}$ and $<X, \lambda_0'(0)> = \lim_{t\to 0} \cos \alpha_t \geqq c > 0$. Q.E.D.

PROOF OF MAIN THEOREM. We have proved the following property for $M_q{}^{\perp}$.

(4) For any $-X \in M_q{}^{\perp}$ such that $X \notin M_q{}^{\perp}$, there exists $Z \in M_q{}^{\perp}$ satisfying $<X, Z> \geqq c > 0$.

Now let M_q^1 be the subset of $M_q^\perp$ defined by $M_q^1 = \{X \in M_q^\perp | <X_0, X> = 0\}$. The set M_q^1 is contained in an $(n-2)$-dimensional unit sphere $S_q^{n-2}(1)$ defined by $S_q^{n-2}(1) = \{v \in M_q | \|v\| = 1, <v, X_0> = 0\}$. We see that there exists $X_1 \in M_q^1$ satisfying $-X_1 \in M_q^1$. In fact, suppose that we have $-X \notin M_q^1$ for any $X \in M_q^1$. Then it follows from the assumption and the properties (2) and (3) for $M_q^\perp$ that M_q^1 is contained in an open hemisphere of $S_q^{n-2}(1)$. We see that there exists a small $\varepsilon > 0$ such that the ε-neighborhood V of M_q^1 in $S_q^{n-2}(1)$ is contained entirely in the open hemisphere. Then we obtain

$$\inf_{X \in M_q^1} \sup_{v \in M_q^1} d(v, X) + \varepsilon \leqq \inf_{X \in M_q^1} \sup_{v \in V} d(v, X) \leqq \pi/2 .$$

There is $Y_1 \in M_q^1$ satisfying $d(v, Y_1) = \inf_{X \in M_q^1} \sup_{v \in M_q^1} d(v, X)$ by the property (3). Making use of (4) for Y_1, there is $Z \in M_q^\perp$ satisfying $<Z, -Y_1> \geqq c > 0$. Putting $Z_1 = (Z - <Z, Y_1> Y_1)/\|Z - <Z, Y_1> Y_1\|$, it follows from $\pm X_0 \in M_q^\perp$ together with $\pm X_0 \neq Z_1$ that the vector $X_1 = (Z_1 - <X_0, Z_1> X_0)/\|Z_1 - <X_0, Z_1> X_0\| \in M_q^\perp$ is orthogonal to both X_0 and Y_1. Hence we have $X_1 \in M_q^1$. Therefore we must have $d(X_1, Y_1) = \pi/2 \leqq \pi/2 - \varepsilon$, which is a contradiction. Let M_q^2 be the subset of M_q^1 given by $M_q^2 = \{X \in M_q^1 | <X, X_1> = 0\}$. Then there is $X_2 \in M_q^1$ such that $-X_2 \in M_q^2$ in the same way as X_1 in M_q^1. We find that $M_q^\perp$ is the intersection of $S_q^{n-1}(1)$ and a subspace in M_q after developing the inductive argument for $M_q, M_q^1, M_q^2, \cdots, M_q^k$, which is analogous as the proof of Theorem C in [7]. The last statement of (2) in Main Theorem is already shown in [4].

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