

ON THE JAMES, SAMELSON AND WHITEHEAD PRODUCTS

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Introduction. Let C_W be the mapping cone on the Whitehead product map $W: A*B \rightarrow SA \vee SB$, then we define in §2 maps $h: C_W \rightarrow SA \times SB$ and $k: S(A*B) \rightarrow SA \# SB$ each of which is a homotopy equivalence if A and B are CW complexes. Given that Y has a multiplication with homotopy identity, we construct in §3 a homotopy product $[SA, Y] \times [SB, Y] \rightarrow [S(A*B), Y]$ which is James's product in case A and B are spheres and Y has strict identity. With restrictions on P and Q , two Samelson products $[P, Y] \times [Q, Y] \rightarrow [P \# Q, Y]$ are defined in §4. In case A and B have non degenerate base point and $P = SA$ and $Q = SB$, these Samelson products correspond to the James product (theorem 5.1) under the homomorphism $k^*: [SA \# SB, Y] \rightarrow [S(A*B), Y]$. In §6 linearity properties and the Jacobi identity are given for the James and Samelson product.

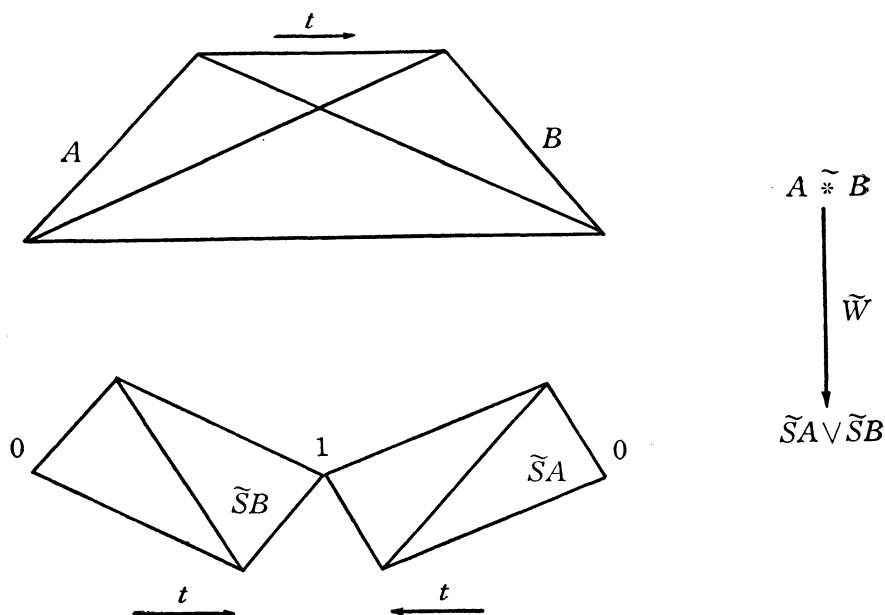
1. Preliminaries. The functors of unreduced join, suspension and cone are denoted $\widetilde{*}$, \widetilde{S} and \widetilde{C} ; the unreduced mapping cone on $f: X \rightarrow Y$ is denoted \widetilde{C}_f . The symbols used for the reduced join, suspension, cone and mapping cone in the base point category are $*$, S , C and C_f ; in this category \vee and $\#$ denote the wedge and collapsed product, and $X \# I = X \times I / * \times I$. The category of based spaces having the base point preserving homotopy type of CW complexes is denoted \mathcal{W} .

2. The functorial equivalences. Let $\widetilde{W}: A \widetilde{*} B \rightarrow \widetilde{S}A \vee \widetilde{S}B$ be the projection defined by

$$\widetilde{W}(a, b, t) = \begin{cases} (*, (b, 2t)) & t \leq 1/2 \\ ((a, 2-2t), *) & t \geq 1/2 \end{cases}$$

where $\widetilde{S}A$ and $\widetilde{S}B$ are joined together at the 1-end of the suspensions: then \widetilde{W} is an unbased Whitehead product (c. f.¹⁾ 2.3 of [1]).

1) Our definition of Whitehead product corresponds to taking the commutator $x+y-x-y$ as in [8], whereas the definition in [1] corresponds to the commutator $-x-y+x+y$ (see 2.2 of [1]).



Consider now the induced cofibre sequence

$$A \tilde{*} B \xrightarrow{\tilde{W}} \tilde{S}A \vee \tilde{S}B \xrightarrow{j} \tilde{C}_W \longrightarrow S(A \tilde{*} B) \rightarrow \dots$$

We show that this sequence, less the first term, is continuously bijective with a cofibre sequence induced by the inclusion $i: \tilde{S}A \vee \tilde{S}B \rightarrow \tilde{S}A \times \tilde{S}B$.

PROPOSITION 2.1. *There are functorial continuous bijections¹⁾ $\tilde{h}: \tilde{C}_W \rightarrow \tilde{S}A \times \tilde{S}B$ and $\tilde{k}: \tilde{S}(A \tilde{*} B) \rightarrow \tilde{S}A \# \tilde{S}B$ making the following diagram commutative.*

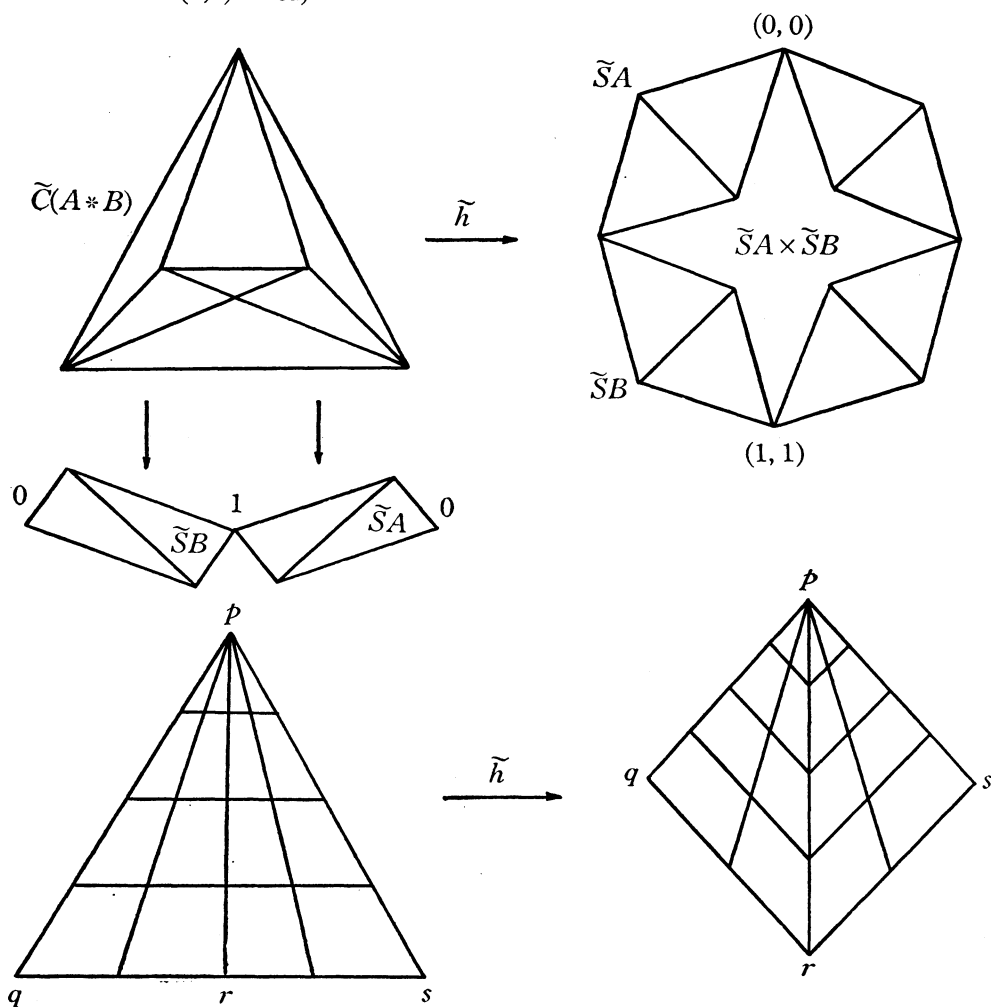
$$\begin{array}{ccccccc}
 A \tilde{*} B & \xrightarrow{\tilde{W}} & \tilde{S}A \vee \tilde{S}B & \xrightarrow{\tilde{j}} & \tilde{C}_W & \xrightarrow{\tilde{P}} & \tilde{S}(A \tilde{*} B) \\
 & & \downarrow 1 & & \downarrow \tilde{h} & & \downarrow \tilde{k} \\
 & & \tilde{S}A \vee \tilde{S}B & \xrightarrow{\tilde{i}} & \tilde{S}A \times \tilde{S}B & \longrightarrow & \tilde{S}A \# \tilde{S}B
 \end{array}$$

Furthermore \tilde{h} and \tilde{k} are homotopy equivalences provided A and B have the

1) These maps are homeomorphisms if $\tilde{S}A \times \tilde{S}B$ and $\tilde{S}A \# \tilde{S}B$ are given the quotient topology from $A \times I \times B \times I$; this, of course, is the weak topology if A and B are CW complexes.

homotopy type of CW complexes.

PROOF Define \tilde{h} by means of the following diagrams (the second represents a section with (a, b) fixed)



The function \tilde{h} clearly defines $\tilde{k}: \tilde{S}(A \# B) \rightarrow \tilde{S}A \# \tilde{S}B$ by taking the quotient. Since \tilde{C}_w and $\tilde{S}(A \# B)$ have the obvious quotient topologies from $A \times I \times B \times I$, \tilde{h} and \tilde{k} are continuous and the result follows. These maps are related to those of theorems 2.4 and 2.5 of [3]. Now \tilde{h} determines a homeomorphism $\tilde{C}_w \rightarrow \tilde{S}A \times \tilde{S}B$ where $\tilde{S}A \times \tilde{S}B$ is the product $\tilde{S}A \times \tilde{S}B$ given the quotient topology from $A \times I \times B \times I$. Let k be the functor giving the weak topology with respect to compact

subsets and suppose that A and B are CW complexes, then the composite $k(\tilde{S}A \times \tilde{S}B) \rightarrow \tilde{S}A \times \tilde{S}B \rightarrow \tilde{S}A \times \tilde{S}B$ is a homotopy equivalence since $\tilde{S}A \times \tilde{S}B$ has the homotopy type of a CW complex (proposition 3 of [7]), and it follows that $\tilde{S}A \times \tilde{S}B \rightarrow \tilde{S}A \times \tilde{S}B$ is a weak homotopy equivalence. Now $A \ast B$ has the homotopy type of a CW complex by the following lemma and thus so does $\tilde{S}A \times \tilde{S}B$ since it has the homotopy type of a cofibre space induced by a cellular map. It follows that \tilde{h} is a homotopy equivalence and \tilde{k} is a homotopy equivalence by a similar argument.

LEMMA 2.2. *Let A and B be CW complexes, then $A \ast B$ has the homotopy type of a CW complex.*

PROOF. The map $A \ast B \rightarrow \tilde{S}(A \times B)$ gives a domination of $A \ast B$ by $\tilde{S}(A \times B)$ and since $\tilde{S}(A \times B)$ has the homotopy type of a CW complex (proposition 3 of [7]), the lemma follows from results of [12].

Consider now the base point case of the above situation. Clearly \tilde{W} determines a quotient function $W: A \ast B \rightarrow SA \vee SB$. Let $h: C_w \rightarrow SA \times SB$ and $k: S(A \ast B) \rightarrow SA \# SB$ also be the quotients of \tilde{h} and \tilde{k} . Since collapsing a contractible subset of a CW complex is a homotopy equivalence, the following corollary is now proved.

COROLLARY 2.3¹⁾. *There are functorial maps h and k making the following diagram commutative.*

$$\begin{array}{ccccc}
 A \ast B & \xrightarrow{W} & SA \vee SB & \xrightarrow{j} & C_w \xrightarrow{p} S(A \circ B) \\
 & & \downarrow 1 & & \downarrow h & & \downarrow k \\
 & & SA \vee SB & \xrightarrow{i} & SA \times SB & \longrightarrow & SA \# SB
 \end{array}$$

Furthermore h and k are homotopy equivalences for A and B in \mathcal{W} .

3. The generalized James product. In this section we define the generalized James product and prove that it is equivalent to the generalized Samelson product. We assume from here onwards that all maps preserve base points.

DEFINITION. An M -space is a space Y with a multiplication $m: Y \times Y \rightarrow Y$

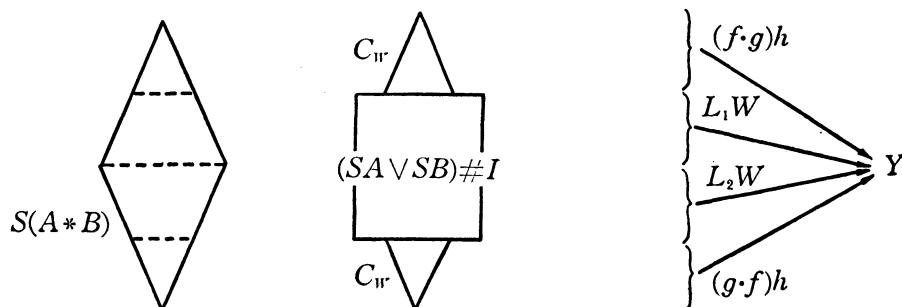
1) c. f. Theorem 4.2 of [1].

having homotopy identity, that is, $m|Y \vee Y$ is homotopic to the folding map.

Now let Y be an M -space and $f: SA \rightarrow Y$ and $g: SB \rightarrow Y$ be maps; then $f \cdot g, g \cdot f: SA \times SB \rightarrow Y$ will denote the products of the two maps $SA \times SB \xrightarrow{p_1} SA \xrightarrow{f} Y$ and $SA \times SB \xrightarrow{p_2} SB \xrightarrow{g} Y$ using the multiplication in Y . Let $H: mi \sim c: Y \vee Y \rightarrow Y$ be a given homotopy and let $L_1 = c(H(j_1 f \# 1) \vee H(j_2 g \# 1))$ and $L_2 = c(H(j_2 f \# 1) \vee H(j_1 g \# 1))$ be the maps $(SA \vee SB) \# I \rightarrow Y$ where $j_1, j_2: Y \rightarrow Y \vee Y$ are the inclusions. Then, denoting track addition in $(A * B) \# I$ by $+$, it is clear that the maps $(f \cdot g)h + L_1(W \# 1)$ and $(g \cdot f)h + L_2(W \# 1): C(A \# B) \rightarrow Y$ agree on the base of the cone and thus their difference is defined

$$d(f \cdot g, g \cdot f) = (f \cdot g)h + L_1(W \# 1) - L_2(W \# 1) - (g \cdot f)h: S(A * B) \rightarrow Y$$

as in the following diagram.



LEMMA 3.1. *The class of $d(f \cdot g, g \cdot f)$ depends only on the classes of f and g .*

PROOF. If $f_0 \sim f_1$ and $g_0 \sim g_1$, then $d(f_t \cdot g_t, g_t \cdot f_t)$ gives a homotopy $d(f_0 \cdot g_0, g_0 \cdot f_0) \sim d(f_1 \cdot g_1, g_1 \cdot f_1)$.

Definition. The James product¹⁾ is the function $\langle, \rangle_J: [SA, Y] \times [SB, Y] \rightarrow [S(A * B), Y]$ given by $\langle f, g \rangle_J = \{d(f \cdot g, g \cdot f)\}$.

The homotopy $H: mi \sim c: Y \vee Y \rightarrow Y$ is used in the above definition, however the construction \langle, \rangle_J is independent of the choice of this homotopy.

PROPOSITION 3.2. \langle, \rangle_J is independent of the homotopy $H: mi \sim c:$

1) A similar product is defined by K. Tsuchida in §4 of [11] in case Y has strict identity and all spaces are path connected countable CW complexes. However it differs from ours by involutions because of our different conventions for commutators.

$$Y \vee Y \rightarrow Y.$$

PROOF. By considering the difference in $[S(A*B), Y]$ of the $d(f.g, g.f)$ for different H^* 's, it is clearly sufficient to show that the homomorphism $W^*: \pi_1^{SA \vee SB}(Y; (f.g)i) \rightarrow \pi_1^{A*B}(Y; (f.g)i)$ is zero. Since $A*B$ is an H^* -space^{1,2)}, this is equivalent to showing that $\Gamma((f.g)i, W): \pi_1^{SA \vee SB}(Y; *) \rightarrow \pi_1^{A*B}(Y; *)$ is zero where Γ is defined in 3.2 of [9]. However, by the co-primitivity theorem 3.3.3 of [9], $\Gamma((f.g)i, W) = W^*$. The result follows since $\pi_1^{SA \vee SB}(Y; *) \cong \pi_1^{SA}(Y, *) \oplus \pi_1^{SB}(Y, *)$, W^* is linear, and the composite functions $A*B \rightarrow SA \vee SB \rightarrow SA$ and $A*B \rightarrow SA \vee SB \rightarrow SB$ are clearly nulhomotopic.

4. The generalized Samelson products. We now define two Samelson products. We are interested in the case that Y is an M -space such that for suitable K the set of homotopy classes $[K, Y]$ is a loop³⁾.

LEMMA 4.1. (c. f. theorem 1.1 of [5]). *Let K belong to \mathcal{W} and Y be a path connected⁴⁾ M -space, then $[K, Y]$ is a loop under the multiplication on Y .*

Consider now the following sequence which is exact for P and Q having non degenerate base points:

$$\cdots \rightarrow [S(P \times Q), Y] \rightarrow [S(P \vee Q), Y] \rightarrow [P \# Q, Y] \xrightarrow{\pi^*} [P \times Q, Y] \rightarrow [P \vee Q, Y].$$

LEMMA 4.2. *Let P and Q have non degenerate base point and $\pi^*: [P \# Q, Y] \rightarrow [P \times Q, Y]$ be a homomorphism of loops, then π^* is injective.*

PROOF. The proof given here is elementary (c. f. proposition 4 of [2]). The homomorphism $[S(P \times Q), Y] \rightarrow [S(P \vee Q), Y]$ is surjective, hence by exactness the kernel of π^* is zero. Since a homomorphism of loops is injective if and only if its kernel is zero, the result is immediate.

Now let P and Q have non degenerate base point, and let Y be an M -space for which the functions $[P \# Q, Y] \rightarrow [P \times Q, Y] \rightarrow [P \vee Q, Y]$ are homomorphisms of loops: this is true if either P and Q are in \mathcal{W} or if Y has homotopy inverses. Let $f: P \rightarrow Y$ and $g: Q \rightarrow Y$ then $f.g = (fp_1).(gp_2)$ and $g.f = (gp_2).(fp_1): P \times Q \rightarrow Y$ are well defined and $\{f.g\} \cdot r\{g.f\}$ in the loop $[P \times Q, Y]$ depends only

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- 1) An H^* -space has a homotopy associative comultiplication with homotopy identity and homotopy inverses.
 - 2) e. g. it is dominated by $S(A \times B)$: there is in fact a simply defined comultiplication.
 - 3) A set having a binary structure with two sided identity and unique solutions x and y for $xa=b$ and $ay=b$.
 - 4) Or more generally let the M -structure induce a loop structure on the path components of Y .

on the homotopy classes $\{f\}$ and $\{g\}$. Since $i^*: [P \times Q, Y] \rightarrow [P \vee Q, Y]$ is linear, $i^*({f.g}.r\{g.f\}) = \{*\}$, and thus there is a unique $\langle f, g \rangle_{s_1}$ in $[P \# Q, Y]$ with $\pi^* \langle f, g \rangle_{s_1} = \{f.g\}.r\{g.f\}$. Similarly there is a unique $\langle f, g \rangle_{s_2}$ in $[P \# Q, Y]$ with $\pi^* \langle f, g \rangle_{s_2} = l\{g.f\}. \{f.g\}$.

DEFINITION. The right and left Samelson¹⁾ products are the functions²⁾ \langle , \rangle_{s_1} and $\langle , \rangle_{s_2}: [P, Y] \times [Q, Y] \rightarrow [P \# Q, Y]$.

5. The relation between the products. We now prove that, with $P = SA$ and $Q = SB$, the James product $\langle , \rangle_J: [SA, Y] \times [SB, Y] \rightarrow [S(A*B), Y]$ and the Samelson product $\langle , \rangle_{s_1}: [SA, Y] \times [SB, Y] \rightarrow [SA \# SB, Y]$ correspond under the functorial homomorphism $k^*: [SA \# SB, Y] \rightarrow [S(A*B), Y]$ when both products are defined: furthermore that if k^* is also an isomorphism then the Samelson products \langle , \rangle_{s_1} and \langle , \rangle_{s_2} are equal.

THEOREM 5.1. *Let Y be an M -space, and let the James product $\langle , \rangle_J: [SA, Y] \times [SB, Y] \rightarrow [S(A*B), Y]$, and Samelson product $\langle , \rangle_{s_1}: [SA, Y] \times [SB, Y] \rightarrow [SA \# SB, Y]$ both be defined, then $k^* \langle f, g \rangle_{s_1} = \langle f, g \rangle_J$ under the homomorphism³⁾ $k^*: [SA \# SB, Y] \rightarrow [S(A*B), Y]$ ⁴⁾. Moreover, when k^* is an isomorphism, the two Samelson products \langle , \rangle_{s_1} and \langle , \rangle_{s_2} are equal.*

PROOF. Let $\mu: C_W \rightarrow S(A*B) \vee C_W$ denote the coaction. Given any map $q: SA \times SB \rightarrow Y$, we denote by \tilde{q} the composite $C_W \xrightarrow{h} SA \times SB \xrightarrow{q} Y$. Also let $M_i = L_i \cdot (r(\tilde{g} \cdot \tilde{f})j): (SA \vee SB) \# I \rightarrow Y$ ($i = 1, 2$). Then, as maps $C_W \rightarrow Y$, we have

$$(\tilde{f} \cdot \tilde{g}).r(\tilde{g} \cdot \tilde{f}) \sim (\tilde{f} \cdot \tilde{g}).r(\tilde{g} \cdot \tilde{f}) + M_1(W \# 1) \sim d((\tilde{f} \cdot \tilde{g}).r(\tilde{g} \cdot \tilde{f}) + M_1(W \# 1), (\tilde{g} \cdot \tilde{f}).r(\tilde{g} \cdot \tilde{f}) + M_2(W \# 1)), (\tilde{g} \cdot \tilde{f}).r(\tilde{g} \cdot \tilde{f}) + M_2(W \# 1))\mu$$

and thus, since $(\tilde{g} \cdot \tilde{f}).r(\tilde{g} \cdot \tilde{f}) + M_2(W \# 1) \sim *$, we have

$$\begin{aligned} (\tilde{f} \cdot \tilde{g}).r(\tilde{g} \cdot \tilde{f}) &\sim d((\tilde{f} \cdot \tilde{g}).r(\tilde{g} \cdot \tilde{f}) + M_1(W \# 1), (\tilde{g} \cdot \tilde{f}).r(\tilde{g} \cdot \tilde{f}) + M_2(W \# 1)) \circ p \\ &\sim (d(\tilde{f} \cdot \tilde{g} + L_1(W \# 1), \tilde{g} \cdot \tilde{f} + L_2(W \# 1)).d(r(\tilde{g} \cdot \tilde{f}), r(\tilde{g} \cdot \tilde{f}))) \circ p \\ &\sim (d(\tilde{f} \cdot \tilde{g} + L_1(W \# 1), \tilde{g} \cdot \tilde{f} + L_2(W \# 1))) \circ p. \end{aligned}$$

1) After the product defined in [10].

2) In case $[P \times Q, Y]$ is a group, these products differ from one another only by the obvious involutions.

3) If A and B are in \mathcal{M} , then both products are defined and k^* is an isomorphism.

4) In case A and B are spheres and Y has strict identity, the first part of this theorem is similar to proposition 1.3 of [6].

Consider the exact sequence

$$[S(SA \vee SB), Y] \xrightarrow{(SW)^*} [S(A*B), Y] \xrightarrow{p^*} [C_w, Y].$$

In the proof of proposition 3.2 we showed $(SW)^* = 0$, and thus p^* is injective since it is a homomorphism of loops having zero kernel. The first part of the theorem now follows. It is clear that by a similar argument to the above, multiplying on the left by $l(\tilde{g}, \tilde{f})$ instead of on the right by $r(\tilde{g}, \tilde{f})$, we obtain $k^* \langle f, g \rangle_{s_1} = \langle f, g \rangle_J$ and the second part of the theorem is immediate.

6. Properties of the products. We now consider the linearity of the James and Samelson products and obtain Jacobi identities for them.

Let Y be an M -space, giving group structures to $[P \times Q, Y]$ and $[P \# Q, Y]$ where P and Q have non-degenerate base points. Let \cdot^c denote the automorphism of $[P \# Q, Y]$ corresponding to the inner automorphism $x \rightarrow c + x - c$ of $[P \times Q, Y]$. The following relations are elementary from corresponding relations between commutators:

$$\langle aa', b \rangle_{s_1} = \langle a', b \rangle_{s_1}^a + \langle a, b \rangle_{s_1},$$

$$\langle a, bb' \rangle_{s_1} = \langle a, b \rangle_{s_1} + \langle a, b' \rangle_{s_1}^b.$$

We now give conditions ensuring the linearity of the James and Samelson products in each of their variables.

THEOREM 6.1. *Let P and Q have non-degenerate base points and let Y have an M structure inducing group structures¹⁾ on $[P \times Q, Y]$ and $[P \# Q, Y]$, then the product $\langle \cdot, \cdot \rangle_{s_1}: [P, Y] \times [Q, Y] \rightarrow [P \# Q, Y]$ satisfies $\langle aa', b \rangle_{s_1} = \langle a, b \rangle_{s_1} + \langle a', b \rangle_{s_1}$ if P has a comultiplication with homotopy identity and satisfies $\langle a, bb' \rangle_{s_1} = \langle a, b \rangle_{s_1} + \langle a, b' \rangle_{s_1}$ if Q has a comultiplication with homotopy identity.*

PROOF. The comultiplication on P induces comultiplications on $P \times Q / * \times Q$ and $P \# Q$ with homotopy identity. Thus $[P \times Q / * \times Q, Y]$ and $[P \# Q, Y]$ are abelian. From the exact sequence $[Q, Y] \leftarrow [P \times Q, Y] \leftarrow [P \times Q / * \times Q, Y]$ it follows that $[a', b]$ and a both lie in the image of $[P \times Q / * \times Q, Y]$ and hence commute. The proof is now clear.

1) e.g. if Y is homotopy associative and P and Q belong to \mathcal{W} , or if Y is homotopy associative with homotopy inverses.

COROLLARY 6.2. *Let A and B be in \mathcal{W} and let Y be a homotopy associative M -space, then the James product $\langle \cdot, \cdot \rangle_J: [SA, Y] \times [SB, Y] \rightarrow [S(A*B), Y]$ is bilinear.*

Given that Y is a homotopy associative M -space and P, Q and R are in \mathcal{W} , then the Samelson product $\langle \cdot, \cdot \rangle_{s_1}$ satisfies a general Jacobi identity

$$\tau_1 \langle \langle -q, p \rangle, r \rangle^q + \tau_2 \langle \langle -r, q \rangle, p \rangle^r + \tau_3 \langle \langle -p, r \rangle, q \rangle^p = 0$$

where τ_1, τ_2 and τ_3 are the obvious twisting functions (c.f. §2 of [8]). The following special case is obtained as in theorem 1 of [8].

THEOREM 6.3. (The Jacobi identity) *Let P, Q and R be in \mathcal{W} and have comultiplications with homotopy identity and let Y be a homotopy associative M -space, then in the group $[P\#Q\#R, Y]$, the Samelson product satisfies*

$$\langle \langle p, q \rangle, r \rangle + \tau_1 \langle \langle r, p \rangle, q \rangle + \tau_2 \langle \langle q, r \rangle, p \rangle = 0$$

where τ_1 and τ_2 are the obvious twisting functions.

COROLLARY 6.4. *If A, B and C are in \mathcal{W} and Y is a homotopy associative M -space, then the James product satisfies the Jacobi identity.*

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