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ON CONFORMALLY FLAT SPACES SATISFYING A CERTAIN CONDITION ON THE RICCI TENSOR

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1. Introduction. The Riemannian curvature tensor R of a locally symmetric Riemannian manifold (M, g) satisfies

$$(*) \qquad R(X, Y)\cdot R = 0 \quad \text{for any tangent vectors } X \text{ and } Y,$$

where the endomorphism $R(X, Y)$ operates on R as a derivation of the tensor algebra at each point of M.

Let R_1 be the Ricci tensor of (M, g). Then (*) implies in particular

$$(**) \qquad R(X, Y)\cdot R_1 = 0 \quad \text{for any tangent vectors } X \text{ and } Y.$$

In the present paper we shall prove

THEOREM A. *Let M^m $(m \geqq 3)$ be an m-dimensional connected complete conformally flat space satisfying the condition (**). Then M^m is one of the following manifolds:*

(I) *A space of constant curvature.*

(II) *A locally product space of a space of constant curvature K $(\neq 0)$ and a space of constant curvature $-K$.*

(III) *A locally product space of a space of constant curvature K $(\neq 0)$ and a 1-dimensional space.*

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2. Conformally flat cases of dimension $m > 3$. Let M^m $(m > 3)$ be a connected conformally flat spaces, then the curvature tensor R of M^m is given by

(2.1) $R(X, Y) = (1/(m-2))(AX\wedge Y + X\wedge AY) - (\mathrm{trace}\, A/(m-1)(m-2))X\wedge Y,$

for any tangent vectors X and Y of M^m, where A denotes a field of symmetric endomorphism which corresponds to the Ricci tensor R_1, that is, $R_1(X, Y) = g(AX, Y)$, and $X\wedge Y$ denotes the endomorphism which maps Z upon $g(Y, Z)X - g(X, Z)Y$.

At a point of M^m, let $\{e_1, e_2, \cdots, e_m\}$ be an orthonormal basis of the tangent space such that $Ae_i = \lambda_i e_i$, $1 \leqq i \leqq m$. Then the equation (2.1) implies

$$R(e_i, e_j) = ((m-1)(m-2))^{-1}\left((m-1)(\lambda_i + \lambda_j) - \sum_{k=1}^{m} \lambda_k\right) e_i \wedge e_j .$$

Now by the equation (**) and

$$[R(e_i, e_j)\cdot R_1](e_k, e_h) = -R_1(R(e_i, e_j)e_k, e_h) - R_1(e_k, R(e_i, e_j)e_h) ,$$

we have

$$(2.2) \qquad (\lambda_i - \lambda_j)\left((m-1)(\lambda_i + \lambda_j) - \sum_{k=1}^{m} \lambda_k\right) = 0, \quad \text{for} \quad i \neq j.$$

In this paper, the indices $i, j, k, h, \cdots$ run from 1 to m.

LEMMA 2.1. *At each point of M^m, the rank of R_1 is m, $m-1$, or 0.*

PROOF. If there exists an integer r $(1<r<m)$ such that $\lambda_1 = \cdots = \lambda_r = 0$, $\lambda_{r+1} \neq 0, \cdots, \lambda_m \neq 0$, and if we put $\Lambda = \sum_{k=1}^{m} \lambda_k$, then (2.2) implies

$$(m-1)\lambda_{r+1} - \Lambda = 0 ,$$
$$\cdots\cdots$$
$$(m-1)\lambda_m - \Lambda = 0 .$$

Hence $\lambda_{r+1} = \cdots = \lambda_m = \lambda \neq 0$. Again (2.2) implies $(m-1)\lambda - (m-r)\lambda = (r-1)\lambda = 0$, that is, $\lambda = 0$ which is a contradiction. Q. E. D.

LEMMA 2.2. *If all the λ_i's have the same sign at a point of M^m, then $\lambda_1 = \lambda_2 = \cdots \lambda_m = \lambda$, at the point.*

PROOF. If there exists an integer r $(1 \leqq r < m)$ such that $\lambda_1 = \cdots = \lambda_r = \lambda$, $\lambda_{r+1} \neq \lambda, \cdots, \lambda_m \neq \lambda$, then (2.2) implies

$$(m-1)(\lambda+\lambda_{r+1})-\Lambda=0\,,$$

$$\cdots\cdots$$

$$(m-1)(\lambda+\lambda_m)-\Lambda=0\,.$$

Heace $\lambda_{r+1}=\cdots=\lambda_m=\mu\neq 0$. Again (2. 2) implies $(m-1)(\lambda+\mu)-r\lambda-(m-r)\mu=0$, that is,

$$(m-r-1)\lambda=(1-r)\mu\,. \tag{2. 3}$$

Then, as $m>3$, from (2. 3) we have $r\neq 1$, $m-1$. But from (2. 3) we have also $\lambda\mu<0$. This is a contradiction. Q. E. D.

Now we have

PROPOSITION 2. 3. *Let M^m $(m>3)$ be a connected conformally flat space satisfying the condition* (**). *If the Ricci form R_1 is definite at least at one point of M^m, then M^m is a space of constant curvature.*

PROOF. If the Ricci form R_1 is positive (resp. negative) definite at some point $x_0\in M^m$, then, by the continuity argument for the characteristic polynomial of A, R_1 is positive (resp. negative) definite near x_0 in M^m. Thus, let $W=\{x\in M^m\,;\,R_1$ is positive (resp. negative) definite at $x\}$, which is an open set. Let W_0 be a connected component of x_0 in W. Then by lemma 2. 2, $\lambda_1=\cdots=\lambda_m=\lambda$, on W_0 and $\lambda(x)$ is a differentiable function on W_0, since $\lambda(x)=\operatorname{trace}A/m$. Now, the open submanifold W_0 becomes a conformally flat space by the Riemannian metric which is the restriction of g to W_0. Thus W_0 becomes an Einstein space by the induced metric from M^m. As $m>3$, $\lambda(x)$ is a constant function on W_0. Hence, (2. 1) implies that W_0 is a space of constant curvature $\lambda/(m-1)$. Therefore, by the connectivity of M^m and the continuity argument for the characteristic polynomial of A, it follows that $W_0=M^m$. Q. E. D.

Next, we assume that the Ricci form R_1 is non-degenerate and indefinite at some point $x_0\in M^m$. Then, from the proof of lemma 2. 2, there exists an integer r $(1<r<m-1)$ such that $\lambda_1=\cdots=\lambda_r=\lambda>0$, and $\lambda_{r+1}=\cdots=\lambda_m=\mu<0$, at x_0. By the continuity argument for the characteristic polynomial of A, let $W=\{x\in M^m$; R is non-degenerate and indefinite at $x\}$, which is an open set. Let W_0 be a connected component of x_0 in W.

Then it follows that r is constant on W_0 and non-zero eigenvalues, $\lambda(x)>0$, and $\mu(x)<0$ are differentiable functions on W_0, since, if $m\neq 2r$, then $\lambda(x)=F(x)$, $\mu(x)=G(x)$, or $\lambda(x)=G(x)$, $\mu(x)=F(x)$ $x\in W_0$, where

$$F(x) = ((1-r)/(m-1)(m-2r)) \text{ trace } A,$$

$$G(x) = ((m-r-1)/(m-1)(m-2r)) \text{ trace } A,$$

and if $m=2r$, then $\lambda(x) = \sqrt[2r]{(-1)^r \det A}$, $\mu(x) = -\sqrt[2r]{(-1)^r \det A}$, $x \in W_0$.

We define two distributions on W_0 as follows:

$$T_1(x) = \{X \in M_x^m\,;\ AX = \lambda(x)X\},$$

$$T_2(x) = \{X \in M_x^m\,;\ AX = \mu(x)X\}.$$

LEMMA 2.4. *$T_1(x)$ and $T_2(x)$ are differentiable on W_0.*

Proof is given by the slight modifications of the arguments in [3].

By lemma 2.4, for any $x \in W_0$ we may choose a differentiable field of orthonormal basis $\{X_1, X_2, \cdots, X_m\}$ near x in W_0 in such a way that $\{X_1, \cdots, X_r\}$ and $\{X_{r+1}, \cdots, X_m\}$ are bases near x in W_0 for T_1 and T_2, respectively.

By making use of (2.1) and (2.3), we have

LEMMA 2.5. *With respect to the basis $\{X_1, X_2, \cdots, X_m\}$, we have*

$$\text{(2.4)} \qquad \begin{aligned} R(X_a, X_b) &= KX_a \wedge X_b, \\ R(X_u, X_v) &= -KX_u \wedge X_v, \end{aligned}$$

and otherwise zero, where $K = (\lambda - \mu)/(m-2)$ and $1 \leqq a, b, c, \cdots \leqq r$, $r+1 \leqq u, v, w, \cdots \leqq m$.

Now, in general, for a differentiable local field of orthonormal basis $\{X_1, X_2, \cdots, X_m\}$ in a Riemannian manifold (M, g), we may put

$$\text{(2.5)} \qquad \nabla_{X_i} X_j = \sum_{k=1}^{m} \gamma_{ijk} X_k,$$

where ∇_X denotes covariant differentiation for the Riemannian connection constructed by g, and $\gamma_{ijk} = -\gamma_{ikj}$.

PROPOSITION 2.6. *Let M^m $(m>3)$ be a connected conformally flat space satisfying the condition (**). If the Ricci form R_1 is non-degenerate and indefinite of signature $2r-m$ at least at one point of M^m, then M^m is a locally product space of an r-dimensional space of constant curvature K and an $(m-r)$-dimensional space of constant curvature $-K$, where $1<r<m-1$.*

PROOF. Taking account of (2. 4) and (2. 5), we have

$$
\begin{aligned}
(\nabla_{X_u} R)(X_a, X_b) &= X_u K X_a \wedge X_b + K \sum_{j=1}^{m} \gamma_{u\,ai} X_i \wedge X_b \\
&\quad + K \sum_{i=1}^{m} \gamma_{u\,bi} X_a \wedge Y_i - K \sum_{c=1}^{r} \gamma_{u\,ac} X_c \wedge X_b \\
&\quad - K \sum_{c=1}^{r} \gamma_{u\,bc} X_a \wedge X_c \\
&= X_u K X_a \wedge X_b + K \sum_{v=r+1}^{m} \gamma_{u\,av} X_v \wedge X_b + K \sum_{v=r+1}^{m} \gamma_{u\,bv} X_a \wedge X_v, \\
(\nabla_{X_a} R)(X_b, X_u) &= K \sum_{v=r+1}^{m} \gamma_{a\,bv} X_v \wedge X_u - K \sum_{c=1}^{r} \gamma_{a\,uc} X_b \wedge X_c, \\
(\nabla_{X_b} R)(X_u, X_a) &= -K \sum_{c=1}^{r} \gamma_{b\,uc} X_c \wedge X_a + K \sum_{v=r+1}^{m} \gamma_{b\,av} X_u \wedge X_v.
\end{aligned}
$$

By the second Bianchi identity, we have $X_u K = 0$, and $\gamma_{u\,va} = 0$. Similarly we have $X_a K = 0$, and $\gamma_{a\,bu} = 0$. Where $a, b = 1, \cdots, r$; $u, v = r+1, \cdots, m$. Thus W_0 is a locally product space of an r-dimensional space of constant curvature K and an $(m-r)$-dimensional space of constant curvature $-K$. Therefore, by the connectivity of M^m and the continuity argument for the characteristic polynomial of A, it follows that $W_0 = M^m$. Q. E. D.

Lastly, we assume that the rank of the Ricci form R_1 is $m-1$ at some point $x_0 \in M^m$, and furthermore M^m is complete. Then, from the proofs of lemma 2. 1 and lemma 2. 2, and the continuity argument for the characteristic polynomial of A, the rank of R_1 is $m-1$ near x_0 in M^m. Thus let $W = \{x \in M^m;$ the rank of R_1 is $m-1$ at $x\}$, which is an open set. Let W_0 be a connected component of x_0 in W. From the proof of lemma 2. 1, we see that all the non-zero eigenvalues of A at each point of W_0 are equal to each other, say, λ, and the non-zero eigenvalue $\lambda(x)$ is a differentiable function on W_0, since $\lambda(x) = \text{trace } A/(m-1)$.

We define two distributions on W_0 as follows:

$$
\begin{aligned}
T_1(x) &= \{X \in M_x^m;\ AX = \lambda(x)X\}, \\
T_0(x) &= \{X \in M_x^m;\ AX = 0\}, \qquad x \in W_0.
\end{aligned}
$$

Corresponding to lemma 2. 4, we have

LEMMA 2.7. *$T_1(x)$ and $T_0(x)$ are differentiable on W_0.*

Thus, for any $x \in W_0$, we may choose a differentiable field of orthonormal basis $\{X_1, X_2, \cdots, X_m\}$ near x in W_0 in such a way that $\{X_1, \cdots, X_{m-1}\}$ and $\{X_m\}$ are bases near x in W_0 for T_1 and T_0, respectively. Corresponding to lemma 2.5, we have

LEMMA 2.8. *With respect to the basis $\{X_1, X_2, \cdots, X_m\}$, we have*

$$R(X_a, X_b) = KX_a \wedge X_b, \tag{2.6}$$

and otherwise zero, where $K = \lambda/(m-2)$, and $1 \leqq a, b, c, \cdots \leqq m-1$.

LEMMA 2.9. *T_1 is involutive.*

PROOF. Taking account of (2.5) and (2.6), we have

$$(\nabla_{X_c} R)(X_a, X_b) = X_c K X_a \wedge X_b + K\gamma_{c\,am} X_m \wedge X_b + K\gamma_{c\,bm} X_a \wedge X_m,$$

$$(\nabla_{X_a} R)(X_b, X_c) = X_a K X_b \wedge X_c + K\gamma_{a\,bm} X_m \wedge X_c + K\gamma_{a\,cm} X_b \wedge X_m,$$

$$(\nabla_{X_b} R)(X_c, X_a) = X_b K X_c \wedge X_a + K\gamma_{b\,cm} X_m \wedge X_a + K\gamma_{b\,am} X_c \wedge X_m.$$

By the second Bianchi identity, we have

$$X_c K = 0, \qquad c = 1, \cdots, m-1. \tag{2.7}$$

$$\gamma_{a\,bm} - \gamma_{b\,am} = 0, \qquad \text{for} \quad a \neq b, \quad a, b = 1, \cdots, m-1. \tag{2.8}$$

By (2.8), T_1 is involutive. Q. E. D.

For each $x \in W_0$, we denote by $M_1(x)$ the maximal integral manifold through x of T_1. Then, by (2.7), K is constant on each $M_1(x)$.

LEMMA 2.10. *Each trajectory of X_m is a geodesic.*

PROOF. From (2.5) and lemma 2.8, we have

$$(\nabla_{X_m} R)(X_a, X_b) = X_m K X_a \wedge X_b + K\gamma_{m\,am} X_m \wedge X_b + K\gamma_{m\,bm} X_a \wedge X_m,$$

$$(\nabla_{X_a} R)(X_b, X_m) = -K \sum_{c=1}^{m-1} \gamma_{a\,mc} X_b \wedge X_c,$$

$$(\nabla_{X_b} R)(X_m, X_a) = -K \sum_{c=1}^{m-1} \gamma_{b\,mc} X_c \wedge X_a.$$

By the second Bianchi identity, we have

$$\gamma_{m\,ma} = 0, \tag{2.9}$$

$$\gamma_{a\,mb} = 0, \tag{2.10}$$

$$X_m K + K(\gamma_{a\,ma} + \gamma_{b\,mb}) = 0, \quad \text{for } a \neq b, \quad a, b = 1, \cdots, m-1. \tag{2.11}$$

Thus, from (2.9), it follows that $\nabla_{X_m} X_m = 0$. Q. E. D.

From (2.11) we have

$$\gamma_{1\,m1} = \gamma_{2\,m2} = \cdots = \gamma_{m-1\,mm-1}. \tag{2.12}$$

Thus, from lemma 2.8, taking account of (2.9), (2.10) and (2.12), we have

$$\begin{aligned} R(X_a, X_m)X_m &= \nabla_{X_a} \nabla_{X_m} X_m - \nabla_{X_m} \nabla_{X_a} X_m - \nabla_{[X_a, X_m]} X_m \\ &= -X_m \gamma_{a\,ma} X_a - \sum_{c=1}^{m-1} \gamma_{a\,ma} \gamma_{m\,ac} X_c \\ &\quad - \sum_{c=1}^{m-1} \gamma_{a\,ma} \gamma_{a\,mc} X_c + \sum_{c=1}^{m-1} \gamma_{m\,ac} \gamma_{c\,mc} X_c \\ &= -X_m \gamma_{a\,ma} X_a - (\gamma_{a\,ma})^2 X_a = 0, \end{aligned}$$

that is

$$X_m \gamma_{a\,ma} + (\gamma_{a\,ma})^2 = 0, \qquad \text{for } a = 1, \cdots, m-1. \tag{2.13}$$

LEMMA 2.11. *Any geodesic whose tangent belongs to T_0 at each $x \in W_0$ is infinitely extendible in W_0.*

PROOF. For any $x \in W_0$, let $L(s)$ be a geodesic with arc length s, whose initial point is x and initial direction at x belongs to T_0. Then, by lemma 2.10, for sufficiently small s each tangent vector at s of $L(s)$ belongs to T_0.

Thus, from (2.11) and (2.12), we have

$$\frac{d^2K}{ds^2} + 2\frac{dK}{ds}\gamma_{a\,ma} + 2K\frac{d}{ds}\gamma_{a\,ma} = 0,$$

that is,

(2.14) $$2K\frac{d^2K}{ds^2}-3\left(\frac{dK}{ds}\right)^2=0.$$

If $K>0$, then (2.14) implies

(2.15) $$\frac{d^2}{ds^2}(1/\sqrt{K})=0.$$

If $K<0$, then (2.14) implies

(2.16) $$\frac{d^2}{ds^2}(1/\sqrt{-K})=0.$$

Therefore, from (2.15) and (2.16), we have

(2.17) $$K=1/(as+b)^2, \text{ and } -1/(as+b)^2, \text{ respectively,}$$

where a and b are certain constants. As a geodesic in M^m, $L(s)$ is infinitely extendible. If this geodesic does not lie in W_0, let s_0 be a point such that $L(s)\in W_0$ for $s<s_0$ but $L(s)\notin W_0$. The characteristic polynomial of A at $L(s)$, $s<s_0$, is $(t-\lambda(s))^{m-1}t$. That of A at $L(s_0)$ is therefore the limit as $s\to s_0$, namely, $(t-\lambda(s_0))^{m-1}t$. But $\lambda(s_0)=\lim_{s\to s_0}\lambda(s)=\lim_{s\to s_0}\pm(m-2)/(as+b)^2=\pm(m-2)/(as_0+b)^2$ can not be 0. This is a contradiction. It follows that $L(s_0)\in W_0$. Q. E. D.

PROPOSITION 2.12. *Let M^m $(m>3)$ be a connected complete conformally flat space satisfying the condition (**). If the rank of the Ricci form R is $m-1$ at least at one point of M^m, then M^m is a locally product space of an $(m-1)$-dimensional space of constant curvature K and a 1-dimensional space.*

PROOF. From lemma 2.11, $K(s)$ must to be defined for any s along $L(s)$. But, if $a\neq 0$ in (2.17), then $1/\lambda$ will be 0 for $s=-b/a$ which is a contradiction. We have thus shown that K is equal to a constant on each $L(s)$. Therefore, K is constant on W_0. Then, from (2.11) and (2.12), we have $\gamma_{a\,ma}=0$, for $a=1,\cdots,m-1$. Thus, from (2.9) and (2.10), T_1 and T_0 are parallel. Therefore, M^m is a locally product space of an $(m-1)$-dimensional space of constant curvature K and a 1-dimensional space. Q. E. D.

3. 3-dimensional cases. Let M be a 3-dimensional connected Riemannian manifold with the metric tensor g. Then the curvature tensor R of M is given by

(3.1) $$R(X,Y)=AX\wedge Y+X\wedge AY-(\text{trace } A/2)X\wedge Y,$$

for any tangent vectors X and Y of M, where A is a symmetric endomorphism satisfying $R_1(X, Y) = g(AX, Y)$. Then (3.1) is obtained by putting $m = 3$ in (2.1). This suggests that we may apply the similar ones as the arguments in § 2 in this section.

At a point of M, let $\{e_1, e_2, e_3\}$ be an orthonormal basis of the tangent space such that $Ae_i = \lambda_i e_i$, $i = 1, 2, 3$. Then the condition (**) is equivalent to

$$(3.2) \qquad (\lambda_i - \lambda_j)(2(\lambda_i + \lambda_j) - \sum_{k=1}^{3} \lambda_k) = 0, \qquad \text{for} \quad i \neq j.$$

From (3.2), we can easily show that the following cases are possible:

(i) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda \neq 0$,

(ii) $\lambda_1 = \lambda_2 = \lambda \neq 0, \qquad \lambda_3 = 0$,

(iii) $\lambda = \lambda_2 = \lambda_3 = 0$.

Thus we have

PROPOSITION 3.1. *Let M be a 3-dimensional connected Riemannian manifold satisfying the condition (**). If the rank of R is 3 at least at one point of M, then M is a space of constant curvature.*

REMARK. In general, if a Riemannian manifold (M, g) satisfies the condition (**), then we can see that multiplicity of any non-zero eigenvalue of A is greater than 1.

Next, we assume that the rank of the Ricci form R_1 is 2 at some point $x_0 \in M$. Then we can define two differentiable distributions, T_1 and T_0 corresponding to the eigenvalues, λ and 0 of A respectively on W_0, and furthermore we may choose a differentiable field of orthonormal basis $\{X_1, X_2, X_3\}$ near x in W_0, for any $x \in W_0$, in such a way that $\{X_1, X_2\}$ and $\{X_3\}$ are bases for T_1 and T_0 respectively, where W_0 is the connected component of x_0 in $W = \{x \in M;$ the rank of R_1 is 2 at $x\}$.

With respect to the basis $\{X_1, X_2, X_3\}$, we have

$$(3.3) \qquad R(X_1, X_2) = \lambda X_1 \wedge X_2, \qquad \text{and otherwise zero.}$$

$$(3.4) \qquad R_1(X_1, X_1) = R_1(X_2, X_2) = \lambda, \qquad \text{and otherwise zero.}$$

Now, we assume that M is conformally flat and complete. Then the following equation holds good:

$$(\nabla_Z R_1)(X, Y) - (\nabla_Y R_1)(X, Z) \tag{3.5}$$
$$= (1/4)(Z(\text{trace } A)g(X, Y) - Y(\text{trace } A)g(X, Z)),$$

for any tangent vectors X, Y and Z of M.

We shall prove

PROPOSITION 3.2. *Let M be a 3-dimensional connected complete conformally flat space satisfying the condition (**). If the rank of the Ricci form R_1 is 2, then M is a locally product space of a 2-dimensional space of constant curvature and 1-dimensional space.*

PROOF. From (2.5) and (3.3), by the second Bianchi identity, we have

$$\gamma_{3\,31} = \gamma_{3\,32} = 0. \tag{3.6}$$

By putting $X = X_1$, $Y = X_2$, $Z = X_3$ in (3.5) and using (2.5) and (3.4), we have

$$\gamma_{2\,13} = 0, \qquad \text{similarly,} \quad \gamma_{1\,23} = 0. \tag{3.7}$$

By putting $X = X_3$, $Y = X_1$, $Z = X_3$

$$X_1\lambda = 0, \qquad \text{similarly,} \quad X_2\lambda = 0. \tag{3.8}$$

By putting $X = X_1$, $Y = X_1$, $Z = X_3$

$$X_3\lambda + 2\lambda\gamma_{1\,31} = 0, \qquad \text{similarly,} \quad X_3\lambda + 2\lambda\gamma_{2\,32} = 0. \tag{3.9}$$

By (3.9), we have

$$\gamma_{\ \,31} = \gamma_{2\,32}. \tag{3.10}$$

From the equation $R(X_1, X_3)X_3 = 0$, by making use of (3.6), (3.7) and (3.10), we have

$$X_3\gamma_{1\,31} + (\gamma_{1\,31})^2 = 0.$$

Therefore, from the above discussions, the rest of proof is given by the slight modifications of the arguments in the last case in 2. Q. E. D.

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