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# 4-DIMENSIONAL CONFORMALLY FLAT KAHLER MANIFOLDS

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**1. Introduction.** Let $(M, g, J)$ be a Kählerian manifold with (almost) complex structure tensor $J$ and Kählerian metric tensor $g$. Let $m$ be the real dimension of $M$. If $m \geqq 6$, every conformally flat Kählerian manifold is locally flat (Yano and Mogi [9], Yano [10]; notice that the assumption $m \geqq 6$ is dropped in their statements of results).

Recently, Takamatsu and Watanabe [7] proved that, if $m \geqq 6$, every conformally flat $K$-space is Ricci parallel and hence locally symmetric (In this case also, $m \geqq 6$ is dropped in the statements).

It is known that a 4-dimensional $K$-space is Kählerian (Gray [3], Takamatsu [6]). So, first we determine 4-dimensional conformally flat Kählerian manifolds.

THEOREM. *Let $(M, g, J)$ be a 4-dimensional Kählerian manifold which is conformally flat. Then $(M, g, J)$ is ether*

(i) *locally flat, or*

(ii) *locally a product space of two 2-dimensional Kählerian manifolds of constant curvature $K$ and $-K$, respectively.*

On the other hand, Takamatsu and Watanabe [8] classified conformally flat $K$-spaces of dimension $m \geqq 6$. Their results and the above Theorem give a complete classification of local structure of conformally flat $K$-spaces. That is, we have

COROLLARY. *Let $(M, g, J)$ be a conformally flat $K$-space, $m \geqq 4$. Then, it is one of the followings:*

(i) *locally flat,*

(ii) *locally a product space $M_1 \times M_2$, where*

$M_1$: *2-dimensional Kählerian manifold of constant curvature $K > 0$,*

$M_2$: *2-dimensional Kählerian manifold of constant curvature $-K$,*

(iii) *$m = 6$ and of constant curvature $K > 0$,*

(iv) *locally a product space $M_3 \times M_2$, where*

$M_3$: *6-dimensional $K$-space of constant curvature $K > 0$.*

Denote by $CP^n[H]$, $CE^n$, and $CD^n[H]$ complex $n$-dimensional Kählerian space forms of constant holomorphic sectional curvature $H > 0$, $H = 0$ and

$H < 0$, respectively.

COROLLARY. *Let $(M, g, J)$ be a complete conformally flat K-space, $m \geqq 4$. Then, one of the followings holds.*

(i) $(M, g, J) = CE^n/\Gamma_1$,

(ii) $(M, g, J) = (CP^1[H] \times CD^1[-H])/\Gamma_2$,

(iii) $(M, g, J) = S^6[K]$: *as a K-space of constant curvature* $K > 0$,

(iv) $(M, g, J) = (S^6[K] \times CD^1[-K])/\Gamma_3$,

*where $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ denote the fixed point free discrete subgroups of the automorphism groups of $CE^n$, $CP^1[H] \times CD^1[-H]$, and $S^6[K] \times CD^1[-K]$, respectively.*

*The converse is also true.*

2. **Proof of Theorem.** Let $(M, g, J)$ be an almost Hermitian manifold with almost complex structure tensor $J$ and almost Hermitian metric tensor $g$. Then we have

$$J_k{}^j J_j{}^i = -\delta_k{}^i \,, \tag{2.1}$$

$$g_{ji} J_r{}^j J_s{}^i = g_{rs} \,. \tag{2.2}$$

Denote by $\nabla$ the Riemannian connection defined by $g$. By $R^i{}_{jkl}$ and $R_{jk}$ we denote the Riemannian curvature tensor and the Ricci curvature tensor, respectively. Then a $K$-space ($=$ almost Tachibana space, nearly Kählerian space) is defined by

$$\nabla_k J_j{}^i + \nabla_j J_k{}^i = 0 \,. \tag{2.3}$$

A Kählerian manifold is characterized by $\nabla_k J_j{}^i = 0$.

The Weyl's conformal curvature tensor is given by

$$\begin{aligned} C^i{}_{jkl} = R^i{}_{jkl} &- a[R_{jk}\delta_l{}^i - R_{jl}\delta_k{}^i + g_{jk}R^i{}_l - g_{jl}R^i{}_k] \\ &+ bS[g_{jk}\delta_l{}^i - g_{jl}\delta_k{}^i] \,, \end{aligned} \tag{2.4}$$

where $a = 1/(m-2)$, $b = 1/(m-1)(m-2)$, and $S$ is the scalar curvature. Then, it is known that

$$[2/(m-3)]\nabla_i C^i{}_{jkl} = 2a[\nabla_l R_{jk} - \nabla_k R_{jl}] - b[g_{jk}\nabla_l S - g_{jl}\nabla_k S] \,. \tag{2.5}$$

A tensor $T_{a\cdots ij\cdots u}$ is called hybrid in $(i, j)$, if $O^{ij}_{rs} T_{a\cdots ij\cdots u} = 0$, i.e.,

$$T_{a\cdots ij\cdots u} = T_{a\cdots rs\cdots u} J_i{}^r J_j{}^s \,. \tag{2.6}$$

PROPOSITION. *Let $(M, g, J)$ be an almost Hermitian manifold. If a tensor $T_{a\cdots ijk\cdots u}$ is hybrid in $(i, j)$, $(j, k)$ and $(i, k)$, then $T_{a\cdots ijk\cdots u} = 0$.*

PROOF. We apply (2.6) three times.

$$\begin{aligned} T_{a\cdots ijk\cdots u} &= T_{a\cdots rsk\cdots u}J_i{}^rJ_j{}^s \\ &= T_{a\cdots rpt\cdots u}J_s{}^pJ_k{}^tJ_i{}^rJ_j{}^s \\ &= T_{a\cdots qpv\cdots u}J_r{}^qJ_t{}^vJ_s{}^pJ_k{}^tJ_i{}^rJ_j{}^s \\ &= -T_{a\cdots ijk\cdots u} \,. \end{aligned}$$

This implies that $T_{a\cdots ijk\cdots u} = 0$.

PROOF OF THEOREM. Assume that $(M, g, J)$ is Kählerian, conformally flat, and $m = 4$. Then, it is known that the scalar curvature $S = 0$ (cf. Tachibana [5]). By $C^i{}_{jkl} = 0$, $S = 0$ and (2.5), we have

$$\nabla_l R_{jk} = \nabla_k R_{jl} \,. \tag{2.7}$$

This means that $\nabla_i R_{jk}$ is symmetric in all indices.

On the other hand, $R_{jk}$ is hybrid in $(j, k)$, i.e.,

$$R_{jk} = R_{rs}J_j{}^rJ_k{}^s \,. \tag{2.8}$$

Since $J$ is parallel, we have

$$\nabla_i R_{jk} = \nabla_i R_{rs}J_j{}^rJ_k{}^s \,. \tag{2.9}$$

By (2.7), the tensor $\nabla_i R_{jk}$ is hybrid in $(i, j)$, $(j, k)$ and $(i, k)$. Therefore, by Proposition, we have $\nabla_i R_{jk} = 0$.

If $(M, g, J)$ is irreducible, it is an Einstein space. Since $(M, g, J)$ is conformally flat, $(M, g, J)$ is of constant curvature $K$. By a result of Bochner [1], we have $K = 0$. Hence, $(M, g, J)$ is locally flat.

If $(M, g, J)$ is reducible, by a result of Kurita [4] (cf. also Goldberg [2]) we have one of the followings:

(1) $M$: locally flat,

(2) $M$: locally a space $(M_1, g_1) \times (M_2, g_2)$, where $(M_1, g_1)$ is of constant curvature $K > 0$ and $(M_2, g_2)$ is of constant curvature $-K$,

(3) $M$: locally a product space $(M_3, g_3) \times (M_4, g_4)$, where $(M_3, g_3)$ is a line and $(M_4, g_4)$ is of constant curvature $\neq 0$.

If we have (3), there is a parallel vector field $X$ which is locally defined. $JX = (J_j{}^iX^j)$ is also parallel and tangent to $M_4$. This is impossible, since $(M_4, g_4)$ is a space of constant curvature $\neq 0$.

In (2), the fact that $(M, g, J)$ is locally $(M_1, g_1, J_1) \times (M_2, g_2, J_2)$ is proved in a similar way as in [8].

REMARK. Proof in [8] is simplified, if we use the above discussion applying related known results.

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