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MAXIMAL SETS OF AMBIGUOUS POINTS

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Let D be the open unit disk and K be the unit circle in the complex plane. We denote the Riemann sphere by Ω, and the chordal distance between points a and b on Ω by $d(a, b)$. Suppose that $f(z)$ is a function defined on D whose values belong to Ω. As is customary in cluster-set theory (see, e.g., [10]), if $\zeta \in K$, we denote the cluster set and the range of f at ζ relative to D by $C(f, \zeta)$, $R(f, \zeta)$, respectively; and if Λ is an arc at ζ, then $C_\Lambda(f, \zeta)$ represents the cluster set of f at ζ relative to Λ. The principal cluster set of f at ζ is the set

$$\Pi(f, \zeta) = \bigcap_{\Lambda} C_\Lambda(f, \zeta) ,$$

where Λ ranges over all arcs at ζ; the chordal principal cluster set of f at ζ is the set

$$\Pi_\chi (f, \zeta) = \bigcap_{X} C_X(f, \zeta) ,$$

where X ranges over all chords of the unit circle at ζ. Evidently

$$\Pi (f, \zeta) \subseteqq \Pi_\chi (f, \zeta) \subseteqq C(f, \zeta) .$$

A point $\zeta \in K$ is called an ambiguous point of f, provided that there exist two arcs, Λ_1, Λ_2, at ζ such that

$$C_{\Lambda_1}(f, \zeta) \cap C_{\Lambda_2}(f, \zeta) = \varnothing . \tag{1}$$

If one of the two arcs satisfying (1) can be taken to be the radius at ζ, then we shall say that ζ is a radioambiguous point of f. If there exist two chords, Λ_1, Λ_2, at ζ satisfying (1), then we call ζ a chordally ambiguous point of f.

I have shown [1, p. 380, Theorem 2] that an arbitrary function in D has at most enumerably many ambiguous points. In view of this, it is reasonable to say that a function has a *maximal set of ambiguous points*, if its set of ambiguous points is everywhere dense on K. I recently obtained, in an incidental way, the following sufficient condition for a function to have a maximal set of ambiguous points [4, Corollary]:

(I) *If $f(z)$ is a nonconstant holomorphic function in D, and if ω*

is a finite complex number such that $\prod_\chi(f, \zeta) = \{\omega\}$ *for every* $\zeta \in K$, *then* f *has a maximal set of ambiguous points.*

McMillan [8, p. 10, Theorem 4] has shown that if $f(z)$ is holomorphic in D, and if f has no ambiguous points on an open subarc of K, then the set of points $\zeta \in K$ for which $\prod(f, \zeta) \neq \varnothing$ is residual on that arc. This can be reformulated as follows:

(II) *If* $f(z)$ *is holomorphic in* D, *and if the set of points* $\zeta \in K$ *for which* $\prod(f, \zeta) = \varnothing$ *is everywhere of second category, then* f *has a maximal set of ambiguous points.*

We shall now prove a theorem which provides sufficient conditions for a meromorphic function in D to have a maximal set of ambiguous points. This theorem contains (I) as a special case, and it does not impose the stringent requirement on $\prod(f, \zeta)$ of being empty, as (II) does.

When we say that $\prod(f, \zeta)$ is uniformly bounded away from a value $\omega \in \Omega$ for almost every ζ belonging to some subarc K^* of K, we mean that there exists a number $b > 0$ such that, for all $\zeta \in K^*$ except for a set of points of measure zero, either $\prod(f, \zeta) = \varnothing$ or else $\prod(f, \zeta) \neq \varnothing$ and $d(\prod(f, \zeta), \omega) \geqq b$.

THEOREM. *Let* $f(z)$ *be a meromorphic function in* D. *If there exists a value* $\omega \in \Omega$ *such that*

(i) *for an everywhere dense set of points* $\zeta \in K$,

$$\omega \in [\Omega - R(f, \zeta)] \cap C(f, \zeta) ; \tag{2}$$

(ii) *the set of points* $\zeta \in K$ *for which*

$$\omega \in \prod(f, \zeta) \tag{3}$$

is nowhere dense;

(iii) *every arc of* K *contains an open subarc on which* $\prod(f, \zeta)$ *is uniformly bounded away from* ω *for almost every* ζ;

then f *has a maximal set of ambiguous points.*

PROOF. Our proof is based on a theorem of McMillan's [8, p. 4, Theorem 1]. We aim to show that every open subarc K_0 of K contains an ambiguous point of f.

According to (ii), there exists an open subarc K_1 of K_0 such that (3) holds for no point $\zeta \in K_1$.

There is no Koebe arc of f whatever for the value ω, because every interior point ζ of such an arc would satisfy (3), whereas by (ii) the set of points satisfying (3) is nowhere dense on that arc.

Then if f has the asymptotic value ω in the arc K_1, f must have the asymptotic value ω at some point $\zeta_0 \in K_1$. If $\prod(f, \zeta_0) \neq \varnothing$, then $\omega \in \prod(f, \zeta_0)$, which is incompatible with our choice of K_1. Hence $\prod(f, \zeta_0) = \varnothing$, which implies the existence of an arc Λ_0 at ζ_0 such that $\omega \notin C_{\Lambda_0}(f, \zeta_0)$, and ζ_0 is thus an ambiguous point of f on the arc K_0.

If, however, f does not have the asymptotic value ω in the arc K_1, then according to (iii), K_1 contains an open subarc K_2 such that, for almost every $\zeta \in K_2$,

$$\begin{cases} \text{either } \prod(f, \zeta) = \varnothing \\ \text{or else } \prod(f, \zeta) \neq \varnothing \text{ and } d(\prod(f, \zeta), \omega) \geqq b > 0 \,. \end{cases} \tag{4}$$

In view of (i), K_2 contains a point ζ satisfying (2). Then by McMillan's theorem, the set of points $\zeta \in K_2$ at which f has an asymptotic value λ_ζ satisfying

$$0 < d(\omega, \lambda_\zeta) < b \tag{5}$$

is of positive measure, and consequently there exists such a point $\zeta_1 \in K_2$ which satisfies both (4) and (5) (for $\zeta = \zeta_1$). If $\prod(f, \zeta_1) \neq \varnothing$, then $\lambda_{\zeta_1} \in \prod(f, \zeta_1)$, which is impossible because of (4) and (5). Hence $\prod(f, \zeta_1) = \varnothing$, and this again implies that ζ_1 is an ambiguous point of f on the arc K_0, and the proof of the theorem is complete.

COROLLARY. *If $f(z)$ is a normal meromorphic function in D satisfying conditions* (i), (ii), *and* (iii), *then f has a maximal set of radioambiguous points.*

For, the proof of the Theorem has produced on every open subarc of K an ambiguous point of f such that, on one of the arcs of ambiguity in question, f actually has an asymptotic value which is not a cluster value of f on the other arc of ambiguity. But, according to Lehto and Virtanen [7, p. 53, Theorem 2], if a normal meromorphic function has an asymptotic value α at a point $\zeta \in K$, then the function has α as a radial asymptotic value at ζ, and the Corollary follows.

REMARK 1. It would be interesting to find weaker conditions than those given in the Theorem for the existence of a maximal set of ambiguous points. That there must be weaker conditions is evident from the fact that, as already mentioned, the Theorem actually provides for the existence of ambiguous points at which one of the pair of arcs of ambiguity is an asymptotic path for the function, whereas there exists [3, p. 14, Theorem 3] a bounded holomorphic function in D having a maximal set of chordally ambiguous points. Conditions guaranteeing the existence of a maximal set of chordally ambiguous points would also be of interest.

REMARK 2. Let $f(z)$ be the elliptic modular function in D, and take $\omega = \infty$. Then (i) is satisfied, as is well known; and (ii) and (iii) are satisfied, because [5, p. 30, Theorem 3] $\prod(f, \zeta) = \varnothing$ for every $\zeta \in K$ ((iii) is satisfied also because (cf. [2, p. 404, Theorem 7]) to every $\zeta \in K$ there corresponds an arc at ζ on which $|f(z)| \leqq 2$). Thus $f(z)$ provides an example of a function to which the Corollary applies.

REMARK 3. In connection with the Corollary, we note that Lappan [6, p. 185, Theorem 5] asserts that if $f(z)$ is a normal holomorphic function in D, then f automatically satisfies our condition (ii) with $\omega = \infty$. This, however, is false, because, as McMillan has noted [9, p. 196, Example 2 and p. 188, Corollary 3], there exists a univalent holomorphic function $f(z)$ in D for which the set of points $\zeta \in K$ satisfying (3) with $\omega = \infty$ is a residual subset of K. (The error occurs in the first sentence of Lappan's proof.)

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