Tôhoku Math. Journ.
24 (1972), 441-447.

# NOTES ON HARMONIC TRANSFORMATIONS

SHINSUKE YOROZU

(Received Dec. 14, 1971)

1. In [2]*) J. Eells, Jr. and J. H. Sampson defined harmonic mappings of Riemannian manifolds. They called a mapping $f: M \rightarrow M'$ ($M$ is a compact orientable Riemannian manifold and $M'$ is a complete Riemannian manifold) harmonic if $f$ is a solution of Euler-Lagrange equation, $\tau(f) = 0$, for energy functional

$$E(f) = \frac{1}{2}\int_M \langle g, f^*g' \rangle *1 .$$

If $M = S^1$ (1-dimensional sphere), then $f$ is harmonic if and only if the image of $f$ is a closed geodesic in $M'$. The harmonic mapping also is a generalization of harmonic function in complex analysis.

If $f$ is an isometric immersion, then we know that $f$ is harmonic if and only if it is a minimal immersion (cf. [2] p. 119). If $f$ is a Riemannian fibration, then we know that $f$ is harmonic if and only if all fibers are minimal submanifolds (cf. [2] p. 127).

In this paper, we define harmonic transformations and show examples of harmonic transformations.

The author expresses his hearty thanks to Professors S. Sasaki and S. Tanno for their kind guidances.

2. Let $M$ and $M'$ denote Riemannian manifolds of dimension $n$ and $m$ respectively. Let $g$ and $g'$ denote Riemannian metrics of $M$ and $M'$ respectively. We assume that manifolds, vector fields, tensor fields, mappings, functions, etc. are smooth (i.e. of class $C^\infty$). Let $(x^1, \cdots, x^n)$ denote local coordinates on $M$ in a neighborhood of a point $P_0$, and let $(y^1, \cdots, y^m)$ denote local coordinates on $M'$ in a neighborhood of a point $f(P_0)$. Then $g$ and $g'$ are written in these local coordinates as

$$ds^2 = g_{ij}dx^i dx^j \qquad 1 \leqq i, j \leqq n ,$$

$$ds'^2 = g'_{\alpha\beta}dy^\alpha dy^\beta \qquad 1 \leqq \alpha, \beta \leqq m .$$

Let $\Gamma^h_{ij}$ and $\Gamma'^\gamma_{\alpha\beta}$ denote the Christoffel symbols with respect to $g_{ij}$ and $g'_{\alpha\beta}$ respectively. For a mapping $f: M \rightarrow M'$, $f(P) = (f^1(P), \cdots, f^m(P))$,

*) The numbers in brackets refer to the references at the end of this paper.

we set $f_i^\alpha = \partial f^\alpha/\partial x^i$ and covariant derivatives $f^\alpha_{;ij}$ are

$$f^\alpha_{;ij} = \partial^2 f^\alpha/\partial x^i \partial x^j - \Gamma^k_{ij} f^\alpha_k .$$

Let $f: M \to M'$ be a mapping and $\tau(f) = (\tau(f)^1, \cdots, \tau(f)^m)$ be defined by

$$\tau(f)^\gamma(P) = g^{ij}(P) f^\gamma_{;ij}(P) + g^{ij}(P) \Gamma'^\gamma_{\alpha\beta}(f(P)) f^\alpha_i(P) f^\beta_j(P) , \tag{1}$$

for $\gamma = 1, 2, \cdots, m$ and for any point $P \in M$. We remark that $\tau(f)^\gamma$ is unaffected by any transformation of the local coordinates on $M$ at $P$, and that $\tau(f)^\gamma(P)$ transforms as a contravariant vector tangent space at $f(P)$ for any transformation of the local coordinates on $M'$ at $f(P)$.

DEFINITION. $f$ is a harmonic mapping, if $\tau(f)^\gamma = 0$ for every $1 \leqq \gamma \leqq m$.

REMARK. If $M$ is compact and oriented, then a harmonic mapping $f: M \to M'$ is an extremal of the energy functional

$$\begin{aligned} E(f) &= \frac{1}{2} \int_M \langle g, f^* g' \rangle *1 \\ &= \frac{1}{2} \int_M g^{ij} f^\alpha_i f^\beta_j g'_{\alpha\beta} \sqrt{det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n . \end{aligned} \tag{2}$$

3. Let us set $M' = M$ and $g' = g$. We consider a transformation $f: M \to M$. Hereafter, we assume that the indices $h, i, j, k, \cdots, \alpha, \beta, \gamma, \cdots$ run over 1 to $n$. Let ${}^*\Gamma^h_{ij}$ denote a Christoffel symbol on $M$ with respect to the induced metric $(f^*g)_{ij}$. We put

$$W^h_{ij} = {}^*\Gamma^h_{ij} - \Gamma^h_{ij} . \tag{3}$$

We compute (1) in this case. Then we have

$$\begin{aligned} \tau(f)^\gamma &= g^{ij} f^\gamma_{;ij} + g^{ij} \Gamma^\gamma_{\alpha\beta} f^\alpha_i f^\beta_j \\ &= g^{ij}(\partial^2 f^\gamma/\partial x^i \partial x^j - \Gamma^h_{ij} f^\gamma_h + \Gamma^\gamma_{\alpha\beta} f^\alpha_i f^\beta_j) \\ &= g^{ij}({}^*\Gamma^h_{ij} - \Gamma^h_{ij}) f^\gamma_h . \end{aligned}$$

Therefore we have

$$\tau(f)^\gamma = g^{ij} W^h_{ij} f^\gamma_h . \tag{4}$$

PROPOSITION 1. *$f$ is a harmonic transformation, if and only if* $g^{ij} W^h_{ij} = 0$.

"An almost isometric transformation with respect to $g$" defined in [1] or [11] is a harmonic transformation in our sense.

4. We consider a conformal transformation $f$ of $M$, then $(f^*g)_{ij} = e^{2\rho} g_{ij}$, where $\rho$ is a real valued function on $M$. Then we have the rela-

tions $W^h_{ij} = \delta^h_j \rho_i + \delta^h_i \rho_j - g^{hk} g_{ij} \rho_k$, where $\rho_k = \partial\rho/\partial x^k$ and $\delta^h_j$ denotes Kronecker's delta. Therefore we have

$$g^{ij} W^h_{ij} = (2 - n)\, g^{hk} \rho_k \,. \tag{5}$$

Thus we have

THEOREM 2. (cf. [2] p. 126) *If $f$ is a conformal transformation of $M$ and* $\dim M = n = 2$, *then $f$ is a harmonic transformation.*

THEOREM 3. *Let $f$ be a conformal transformation of $M$ and* $\dim M = n > 2$. *Then $f$ is a harmonic transformation if and only if it is a homothetic transformation.*

COROLLARY 4. *Every homothetic (isometric) transformation is a harmonic transformation.*

If $f$ is an affine transformation, then it is clear that $W^h_{ij} = 0$. Thus we have

THEOREM 5. *Every affine transformation is a harmornic transformation.*

Next we consider a projective transformation $f$ of $M$. Then there exists a covariant vector field $\psi$ on $M$ such that $W^h_{ij} = \psi_i \delta^h_j + \psi_j \delta^h_i$. Therefore we have

$$g^{ij} W^h_{ij} = 2 g^{hk} \psi_k \,. \tag{6}$$

Thus we have

THEOREM 6. *Let $f$ be a projective transformation of $M$. Then $f$ is a harmonic transformation if and only if it is an affine transformation.*

5. In this section, we show some examples of harmonic transformations of odd dimensional Riemannian manifolds.

Let $M$ be an almost contact Riemannian manifold with structure tensors $\phi$, $\xi$, $\eta$ and $g$ which satisfy

$$\phi^i_j \xi^j = 0 \,, \qquad \eta_j \phi^j_i = 0 \,, \qquad \eta_i \xi^i = 1 \,, \tag{7}$$

$$\phi^j_k \phi^k_i = -\delta^j_i + \eta_i \xi^j \,, \tag{8}$$

$$g_{ij} \xi^j = \eta_i \tag{9}$$

and

$$g_{hk} \phi^h_i \phi^k_j = g_{ij} - \eta_i \eta_j \,. \tag{10}$$

An almost contact Rimannian manifold $M$ is called a contact Riemannian manifold, if

$$2 g_{ik} \phi^k_j = 2 \phi_{ij} = \eta_{j;i} - \eta_{i;j} \,. \tag{11}$$

A contact Riemannian manifold $M$ is called a $K$-contact Riemannian manifold, if

(12) $$\eta_{j;i} + \eta_{i;j} = 0 \quad \text{(i.e. } \xi \text{ is a Killing vector field)} .$$

A contact Riemannian manifold $M$ is called a Sasakian manifold, if

(13) $$\phi_{ij;k} = \eta_i g_{jk} - \eta_j g_{ik} .$$

A Sasakian manifold is necessarily a $K$-contact Riemannian manifold (cf. [6]).

A transformation $f$ of an almost contact Riemannian manifold $M$ is called a $D$-homothetic transformation, if

(14) $$(f^*g)_{ij} = \alpha g_{ij} + \beta\eta_i\eta_j$$

for some constants $\alpha$ and $\beta$ satisfying $\alpha > 0$ and $\alpha + \beta > 0$. For a $D$-homothetic transformation $f$ of an almost contact Riemannian manifold $M$, we have

(15) $$W^h_{ij} = \frac{\beta}{2\alpha}\eta_j(\eta_{k;i} - \eta_{i;k})g^{hk} + \frac{\beta}{2\alpha}\eta_i(\eta_{k;j} - \eta_{j;k})g^{hk} + \frac{\beta}{2(\alpha+\beta)}\xi^h(\eta_{i;j} + \eta_{j;i}) + \frac{\beta^2}{2(\alpha+\beta)}\xi^h\xi^k(\eta_i\eta_j)_{;k}$$

(cf. [9]). Thus we have

LEMMA 7. *In an almost contact Riemannian manifold $M$, a $D$-homothetic transformation $f$ satisfies*

(16) $$g^{ij}W^h_{ij} = \frac{\beta}{\alpha}\eta_i(\eta_{k;j} - \eta_{j;k})g^{ij}g^{hk} + \frac{\beta}{\alpha+\beta}\xi^j_{;j}\xi^h .$$

If $M$ is a contact Riemannian manifold, then it holds good that (11) and $\xi^j_{;j} = 0$ (cf. [6]). Thus we have

THEOREM 8. *Every $D$-homothetic transformation of a contact Riemannian manifold is a harmonic transformation.*

COROLLARY 9. *Every $D$-homothetic transformation of $K$-contact Riemannian manifold (Sasakian manifold) is a harmonic transformation.*

REMARK. If $\beta \neq 0$, then such harmonic transformations are obviously neither conformal nor projective.

A transformation $f$ of an almost contact Riemannian manifold $M$ is called a $\phi$-transformation, if $f$ leaves the structure tensor $\phi$ invariant (i.e. $f_*\cdot\phi = \phi\cdot f_*$). If $f$ is a $\phi$-transformation of a contact Riemannian manifold $M$, then there exists a positive constant $\alpha$ such that

$$(f^*\eta)_i = \alpha\cdot\eta_i\,,\qquad (f_*\xi)^i = \alpha^{-1}\xi^i\,,\qquad (f^*\omega)_{ij} = \alpha\cdot\omega_{ij}$$

and

$$(f^*g)_{ij} = \alpha g_{ij} + \alpha(\alpha-1)\eta_i\eta_j\,,$$

where $2\omega = d\eta$(cf. [7]). By Theorem 8 and Corollary 9, we have

THEOREM 10. *Every $\phi$-transformation of a contact Riemannian manifold is a harmonic transformation.*

COROLLARY 11. *Every $\phi$-transformation of a K-contact Riemannian manifold (Sasakian manifold) is a harmonic transformation.*

**6.** In their paper [2], Eells and Sampson noticed the following: If $M$ and $M'$ are Kählerian manifolds and $f\colon M\to M'$ is a holomorphic mapping, then $f$ is harmonic relative to the associated real Riemannian structure on $M$ and $M'$. So in this section, we show some examples of harmonic transformations of even dimensional Riemannian manifolds.

Let $M$ be an almost Hermitian manifold with structure tensors $F$ and $g$ which satisfy

$$F^j_k F^k_i = -\delta^j_i\,,\qquad g_{hk}F^h_i F^k_j = g_{ij}\,. \tag{17}$$

An almost Hermitian manifold $M$ is called an almost $A$-space, if

$$F^j_{i;j} = 0\,. \tag{18}$$

An almost Hermitian manifold $M$ is called an almost *0-space, if

$$2{}^*0^{ij}_{kl}F^h_{j;i} = (\delta^i_k\delta^j_l + F^i_k F^j_l)\,F^h_{j;i} = 0\,. \tag{19}$$

An almost Hermitian manifold $M$ is called an almost Kählerian manifold, if

$$F_{hi;j} + F_{ij,h} + F_{jh,i} = 0\,, \tag{20}$$

where $F_{hi} = F^k_h g_{ki}$.

An almost Hermitian manifold $M$ is called an almost Tachibana space, if

$$F^h_{i;j} + F^h_{j,i} = 0\,. \tag{21}$$

An almost Hermitian manifold $M$ is called a Kählerian manifold, if

$$F^h_{i,j} = 0\,. \tag{22}$$

It is well known that relations among these are

$$\begin{array}{ccc} & \nearrow (20) \searrow & \\ (22) & & (19) \Rightarrow (18) \Rightarrow (17)\,. \\ & \searrow (21) \nearrow & \end{array} \tag{23}$$

A transformation $f$ of an almost Hermitian manifold $M$ is called an almost analytic transformation, if $f$ leaves the structure tensor $F$ invariant (i. e. $f_* \cdot F = F \cdot f_*$).

PROPOSITION 12. (cf. [10]) *In an almost *O-space $M$, an almost analytic transformation $f$ satisfies*

$$g^{ij} W_{ij}^{h} = 0 . \tag{24}$$

From this we have

THEOREM 13. *Every almost analytic transformation of an almost *O-space is a harmonic transformation.*

COROLLARY 14. *Every almost analytic transformation of an almost Tachibana space (almost Kählerian manifold) is a harmonic transformation. Every analytic transformation of a Kählerian manifold is a harmonic transformation.*

If an almost analytic transformation $f$ of an almost $A$-space is conformal, then $f$ is a harmonic transformation. By Theorem 3.3 of [3] and Theorem 3, we have

COROLLARY 15. *Let $M$ be an almost $A$-space (almost *O-space, almost Tachibana space, almost Kählerian manifold, Kählerian manifold) and $\dim M = n = 2p > 2$. Then every almost analytic (analytic, in Kählerian manifold case) conformal transformation of $M$ is a harmonic transformation.*

7. We state some remarks.

(A) The set of all harmonic transformations does not constitute a group under the natural rule of composition. Because it holds good that $\tau(f' \cdot f)^a = \tau(f)^\gamma f'^a_\gamma + g^{ij} f^\alpha_i f^\beta_j W'^\gamma_{\alpha\beta} f'^a_\gamma$ $(1 \leqq a \leqq n)$ for any transformation $f$ and $f'$ of $M$, where $W'^\gamma_{\alpha\beta} = {}'^*\Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\alpha\beta}$ and ${}'^*\Gamma^\gamma_{\alpha\beta}$ denote the Christoffel symbol with respect to $(f'^*g)_{\alpha\beta}$.

(B) Let $f$ and $f'$ be harmonic transformations. If $f$ is a conformal transformation, then $f' \cdot f$ is a harmonic transformation. But $f \cdot f'$ is not necessarily a harmonic transformation. If $f'$ is an affine transformation, then $f' \cdot f$ is a harmonic transformation. But $f \cdot f'$ is not necessarily a harmonic transformation.

(C) Let $h$ be a harmonic function on $M$ (i.e. $g^{ij} h_{;ij} = 0$) and $f$ be a conformal transformation of $M$. It holds good that $g^{ij}(h \cdot f)_{;ij} = \tau(f)^\gamma(\partial h/\partial x^\gamma)$ Thus we have

(a) Let $h$ be a harmonic function on $M$ and $f$ be a conformal transformation and harmonic transformation of $M$. Then a function $h \cdot f$ on

$M$ is a harmonic function.

(b) Let $f$ be a conformal tansformation of $M$ and $h$ be a harmonic function on $M$ and $\dim M = 2$. Then a function $h \cdot f$ on $M$ is a harmonic function.

(D) If $M$ is compact and oriented, then an isometric transformation $f$ of $M$ is a harmonic transformation and $E(f) = (n/2)\,\mathrm{Vol}(M)$, where $\dim M = n$ and $\mathrm{Vol}(M)$ denotes the volume of $M$. If $M$ is a compact contact Riemannian manifold, then a $D$-homothetic transformation $f$ (i.e. $(f^*g)_{ij} = \alpha g_{ij} + \beta \eta_i \eta_j$) is a harmonic transformation and

$$E(f) = \frac{\alpha n + \beta}{2} \mathrm{Vol}(M) ,$$

where $\dim M = n$.

## REFERENCES


[1] S. S. CHERN and C. C. HSIUNG, On the isometry of compact submanifolds in Euclidean space, Math. Ann. 149 (1963), 278-285.

[2] J. EELLS, Jr. AND J. H. SAMPSON, Harmonic mappings of Riemannian manifolds, Amer. J. Math. 86 (1964), 109-160.

[3] A. GRAY, Some examples of almost Hermitian manifolds, Illinois J. Math. 10 (1966), 353-366.

[4] S. I. GOLDBERG, Curvature and Homology, Academic Press, New York, 1962.

[5] S. KOBAYASHI AND K. NOMIZU, Foundations of Differential Geometry I, Interscience, New York, 1963.

[6] S. SASAKI, Almost contact manifolds I, II, Lecture notes, Math. Inst. Tôhoku Univ., 1965, 1967.

[7] S. TANNO, Some transformations on manifolds with almost contact and contact metric structures I, II, Tôhoku Math. J. (2) 15 (1963), 140-147, 322-331.

[8] ———, Partially conformal transformations with respect to $(m-1)$-dimensional distributions of $m$-dimensional Riemannian manifolds, Tôhoku Math. J. (2) 17 (1965), 358-409.

[9] ———, The topology of contact Riemannian manifolds, Illinois J. Math. 12 (1968), 700-717.

[10] ———, Analytic transformations preserving the Chern's 2-form, Tensor (N.S.) 21 (1970), 373-376.

[11] K. YANO AND M. KONISHI, Notes on almost isometries, Kōdai Math. Sem. Rep. 23 (1971), 238-247.



DEPARTMENT OF MATHEMATICS,
SCHOOL OF LIBERAL ARTS AND SCIENCES,
IWATE MEDICAL COLLEGE