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GENERATORS OF FINITE W^*-ALGEBRAS

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In recent years a great deal has been learned about generators in properly infinite W^*-algebras [1, 2, 3, 7]. The main tool in the study of generating properties have always been matrix techniques. Unfortunately, however, one cannot extend methods using infinite matrices to finite W^*-algebras. In this note we introduce the class of W^*-algebras of type II_1 with property A, to which such matrix techniques can be applied in some sense. For example, a factor $\mathfrak{A}$ of type II_1 has property A, if it can be written as a tensor product $\mathfrak{B} \otimes \mathfrak{C}$ of two factors of type II_1. W^*-algebras with property A are singly generated. Moreover most of the results [1, 2, 3] on special generators in properly infinite W^*-algebras carry over to W^*-algebras with property A.

Throughout all Hilbert spaces are separable and all W^*-algebras are assumed to act on separable Hilbert spaces. For a W^*-algebra $\mathfrak{A}$ we denote by $M_k(\mathfrak{A}) = \mathfrak{A} \otimes M_k$ the algebra of all k by k matrices with entries from $\mathfrak{A}$. In this notation M_∞ stands for the algebra of all bounded linear operators on a separable infinite dimensional Hilbert space. For $A_1, A_2, \ldots \in B(H)$, the algebra of all bounded linear operators on the Hilbert space H, $\mathfrak{R}(A_1, A_2, \ldots)$ denotes the W^*-algebra generated by $A_1, A_2, \ldots$. The operator $T \in B(H)$ will be called n-nilpotent if $T^{n-1} \neq 0$ and $T^n = 0$.

Let $\mathfrak{A}$ be a continuous W^*-algebra and let e be a projection in $\mathfrak{A}$, then e can be decomposed into two orthogonal equivalent projections f and g with sum e. This decomposition we denote by $e = \operatorname{dec} f, g$. Now define $1 = \operatorname{dec} e_1, f_1$ and inductively $e_n = \operatorname{dec} e_{n+1}, f_{n+1}$. Let $v_n^* v_n = e_n$ and $v_n v_n^* = f_n$ then

$$(1) \qquad v_n v_m = 0 \quad \text{for all} \quad m \leqq n \quad \text{and} \quad f_n v_m = \delta_{n,m} v_m$$

DEFINITION. A factor $\mathfrak{A}$ of type II_1 has property A if there exist two factors $\mathfrak{B}$ and $\mathfrak{C}$ of type II_1 such that $\mathfrak{A}$ is isomorphic to $\mathfrak{B} \otimes \mathfrak{C}$. A W^*-algebra $\mathfrak{A}$ of type II_1 with central decomposition $\mathfrak{A} = \int \mathfrak{A}(\zeta) d\zeta$ has

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property A if $d\zeta$ almost all factors $\mathfrak{A}(\zeta)$ have property A.

The hyperfinite factor has property A, but it is not known whether every factor of type II_1 has this property. Let $\mathscr{A}$ denote the class of all W^*-algebra of type II_1 on a separable Hilbert space with property A. Then we can show:

LEMMA 1. a) *$\mathfrak{A} \in \mathscr{A}$ then $M_n(\mathfrak{A}) \in \mathscr{A}$ for all finite $n \geqq 2$.*

b) *$\mathfrak{A} \in \mathscr{A}$ then for any finite $n \geqq 2$ there exists a $\boldsymbol{D} \in \mathscr{A}$ such that $\mathfrak{A} = M_n(\boldsymbol{D})$.*

PROOF. a) Assume $\mathfrak{A}$ is a factor and $\mathfrak{A} \cong \mathfrak{B} \otimes \mathfrak{C}$, where $\mathfrak{B}$ and $\mathfrak{C}$ are factors of type II_1. Then $M_n(\mathfrak{A}) = \mathfrak{A} \otimes M_n \cong \mathfrak{B} \otimes (\mathfrak{C} \otimes M_n)$. If $\mathfrak{A}$ is an arbitrary W^*-algebra with property A and $\mathfrak{A} = \int \mathfrak{A}(\zeta) d\zeta$ is its central decomposition, then $\mathfrak{A} \otimes M_n = \int \mathfrak{A}(\zeta) \otimes M_n d\zeta$ [4; ch. II, § 3.4]. Since $d\zeta$ almost all $\mathfrak{A}(\zeta) \otimes M_n$ have property A, the algebra $\mathfrak{A} \otimes M_n$ has property A also.

b) Again let $\mathfrak{B}$ and $\mathfrak{C}$ be factors of type II_1 and let $\mathfrak{A} = \mathfrak{B} \otimes \mathfrak{C}$. Let p be a nonzero projection in $\mathfrak{A}$ and let q be a nonzero projection in $\mathfrak{C}$ with $\dim p = \dim 1 \otimes q$. Then $p\mathfrak{A}p \cong (1 \otimes q)\,\mathfrak{A}\,(1 \otimes q) = \mathfrak{B} \otimes q\mathfrak{C}q$ and $p\mathfrak{A}p$ has property A also. Now let $\mathfrak{A} \in \mathscr{A}$ be a W^*-algebra with central decomposition $\mathfrak{A} = \int \mathfrak{A}(\zeta) d\zeta$, then there exists a projection p in $\mathfrak{A}$ such that $p^\natural = 1/n\, 1$, where $\natural$ is the canonical centervalued trace [4; ch. III § 4.4]. Clearly $p\mathfrak{A}p = \int p(\zeta)\mathfrak{A}(\zeta)p(\zeta) d\zeta$ has property A, because $d\zeta$ almost all $p(\zeta)\mathfrak{A}(\zeta)p(\zeta)$ have property A also. Since $\mathfrak{A} \cong M_n(p\mathfrak{A}p)$ the algebra $p\mathfrak{A}p = \boldsymbol{D}$ has the required properties.

The following lemma is a slight improvement of some of Wogen's results [7].

LEMMA 2. *Let $\mathfrak{A}$ be a W^*-algebra, which is generated by n selfadjoint operators $a_1, \dots, a_n$. Then $\mathfrak{A} \otimes M_k$ is generated by $m \geqq 2$ selfadjoint operators $A_1, \dots, A_m$ if $(m-1)k^2 + 1 \geqq n$.*

PROOF. a) We may assume the a_i to be positive and invertible contractions. Then define $A_1 = \operatorname{diag}(a_1, a_2 + 2, \cdots, a_k + k)$ and

$$A_2 = \begin{bmatrix} a_{k+1} & a_{2k+1} & & & \\ a_{2k+1} & a_{k+2} & \ddots & & \text{\#} \\ & \ddots & \ddots & \ddots & \\ & & \ddots & a_{2k-1} & a_{3k-1} \\ \text{\#} & & & a_{3k-1} & a_{2k} \end{bmatrix}$$

By $+\!\!+$ we mean that the remaining matrix elements of A_2 are only restricted by the symmetry requirement, $(A_2)_{i,j} = (A_2)^*_{j,i}$ for $i \geqq 2$. If $n < 3k - 1$ we set $a_{n+1} = \dots = a_{3k-1} = 1$. Thus we can place $k + (k-1) + (k-1)(k-2)$ of the a_i in the matrix A_2. The matrices $A_3, \dots, A_m$ are only restricted by the symmetry requirement $(A_l^*)_{i,j} = (A_l)^*_{j,i}$ with $3 \leqq l \leqq m$. Hence each of these matrices can accomodate k^2 of the a_i, and all A_l can accomodate $(m-1)k^2 + 1$ of the a_i.

b) Let $C = C^* = (c_{i,j})^k_{i,j=1} \in \mathfrak{R}(A_1, \dots, A_m)' = \mathfrak{R}'$. Then $CA_1 = A_1C$ and [2, Lemma 1(a)] show that C is diagonal, $C = \operatorname{diag}(c_1, \dots, , c_k)$. The relation $CA_2 = A_2C$ and [2, Lemma 1(b)] imply $c_1 = \dots = c_k$. Hence $C = c_1 \otimes 1$ and because of $A_lC = CA_l$, with $1 \leqq l \leqq m$, $c_1 \in \mathfrak{R}(a_1, \dots, a_n)' = \mathfrak{A}'$. Thus $\mathfrak{R}(A_1, \dots, A_m)' = \mathfrak{A}' \otimes 1$ or $M_k(\mathfrak{A}) = \mathfrak{R}(A_1, \dots, A_m)$.

THEOREM 1. *Let $\mathfrak{A}$ be a factor of type II_1 with property A on the separable Hilbert space H, then is singly generated.*

PROOF. a) We can write $\mathfrak{A} = \mathfrak{B} \otimes \mathfrak{C}$. As above construct for $\mathfrak{B}$ a system of partial isometries and projections $\{v_n, e_n, f_n\}$ and let $\{c_n\}$ be a countable family of positive invertible contractive generators of $\mathfrak{C}$. Then consider the operators

$$A = \sum_{n=1}^{\infty} (v_n \otimes c_n) 2^{-n} \quad \text{and} \quad B = \sum_{n=1}^{\infty} (v_n \otimes 1) 2^{-n} .$$

Since the e_n and f_m commute, they lie in some maximal abelian *-subalgebra $\boldsymbol{D}$ of $\mathfrak{B}$. Let D be a self-adjoint generator of this subalgebra, and let $C = D \otimes 1$. Then $\boldsymbol{D} \otimes 1 \subset \mathfrak{R}(A, B, C) = \mathfrak{R}$ and

$$(f_m \otimes 1) A = \sum f_m v_n \otimes c_n 2^{-n} = v_m \otimes c_m \cdot 2^{-m} \in \mathfrak{R} .$$

Also $(f_m \otimes 1) B = v_m \otimes 1 \cdot 2^{-m} \in \mathfrak{R}$. Thus $\mathfrak{R}$ contains $v_m \otimes c_m$ and $v_m \otimes 1$ for all m. Hence $(v_m \otimes c_m)(v_m \otimes 1)^* + (v_m \otimes c_m)^*(v_m \otimes 1) = e_{m-1} \otimes c_m \in \mathfrak{R}$. Therefore also $(v_{m-1} \otimes 1)(e_{m-1} \otimes c_m) = v_{m-1} \otimes c_m \in \mathfrak{R}$. Repeating the same procedure as before we see $v_{m-2} \otimes c_m \in \mathfrak{R}$ and finally $1 \otimes c_m \in \mathfrak{R}$. Hence $1 \otimes \mathfrak{C} \subset \mathfrak{R}$.

b) Interchanging the role of $\mathfrak{B}$ and $\mathfrak{C}$ we find by the same method as above a triple A', B' and C' with $\mathfrak{R}(A', B', C') \supset \mathfrak{B} \otimes 1$. Actually we only need A', the analogue of A, because C', $B' \in \mathfrak{R}(A, B, C)$. Thus $\mathfrak{A} = \mathfrak{B} \otimes \mathfrak{C}$ is generated by seven self adjoint operators.

c) Now write $\mathfrak{A} = M_3(\boldsymbol{C})$. Since $\boldsymbol{C}$ has property A also, $\boldsymbol{C}$ is generated by seven selfadjoint operators. Hence by Lemma 2 $\mathfrak{A}$ is singly generated.

It is obvious that the above proof works also for all W^*-algebras $\mathfrak{A}$, which can be written as $\mathfrak{A} = \mathfrak{B} \otimes \mathfrak{C}$, where $\mathfrak{B}$ and $\mathfrak{C}$ are continuous W^*-

algebras or W^*-algebras of type I_∞. In particular this would give a new proof of Wogen's result [7, Theorem 2].

Now we want to extend Theorem 1 to arbitrary W^*-algebras $\mathfrak{A}$ with property A. To do so we need a result of P. Willing (private communication).

THEOREM 2 (P. Willing). *Let $\mathfrak{A}$ be a W^*-algebra on the separable Hilbert space H and let $\mathfrak{A} = \int_Z \mathfrak{A}(\zeta)d\zeta$ be its central decomposition. Then $\mathfrak{A}$ is singly generated if $d\zeta$ almost all $\mathfrak{A}(\zeta)$ are singly generated.*

PROOF. Since any properly infinite W^*-algebra and any finite W^*-algebra of type I is singly generated, we may assume $\mathfrak{A}$ to be of type II_1. Then $H = \int_Z H_\infty d\zeta$, where H_∞ is a fixed infinite dimensional Hilbert space and where Z is a separable metric space. For details we refer to [5].

Since H_∞ is separable, the unit sphere S of $B(H_\infty)$ is weakly compact. Moreover the weak topology is a metric topology on bounded sets defined by the metric ρ [5, 1.4.8]. Let now $\{A_i\}_{i=1}^\infty$, with $A_i = \int A_i(\zeta)d\zeta$, be a countable sequence of hermitean generators of $\mathfrak{A}$ such that $|A_i| \leqq 1$ and such that the conditions a, b and c of [5, I.5.4] are satisfied. We assume further that the $\{A_i(\zeta)\}$ are dense in the set $\{x \in \mathfrak{A}(\zeta) \mid x = x^*, |x| \leqq 1\}$ for $d\zeta$ almost all ζ. Let $\mathfrak{B}_0$ be the free algebra of the two noncommuting variables z an z^* over the rational complex numbers. $\mathfrak{B}_0$ is countable. For any $f \in \mathfrak{B}_0$ the expression $f(A, A^*)$ shall denote operator which one obtains by replacing in f the variables z and z^* by A and A^*. For any $f \in \mathfrak{B}_0$ and any pair n, m of natural numbers define the subset $E(f, n, m)$ of $Z \times S$ by:

$E(f, n, m)$ consists of all pairs (ζ, A) with

i) $A \in \mathfrak{A}(\zeta) \cap S$

ii) $f(A, A^*) \in \mathfrak{A}(\zeta) \cap S$

iii) $\rho(f(A, A^*) - A_n(\zeta)) \leqq 1/m$

$E(f, n, m)$ is an analytic subset of $Z \times S$. Therefore

$$G = \bigcap_{m=1}^\infty \bigcap_{n=1}^\infty \bigcup_{f \in \mathfrak{B}_0} E(f, n, m)$$

is analytic. We have $(\zeta, A) \in G$ exactly when A is a generator of $\mathfrak{A}(\zeta)$. By assumption each $\mathfrak{A}(\zeta)$ is singly generated, thus the projection $\pi(G)$ of $G \subset Z \times S$ onto the first coordinate is all of Z. Then the principle of measurable choice [5, I.4.7] gives us an operator $A \in \mathfrak{A}$ with

$$A = \int_Z A(\zeta)d\zeta$$

where $A(\zeta)$ is a generator of $\mathfrak{A}(\zeta)\ d\zeta$ a.e..

b) Write $A = B + iC$, $B = B^*$ and $C = C^*$. B lies in some maximal abelian *-subalgebra $\boldsymbol{D}$ of $\mathfrak{A}$. $\boldsymbol{D}$ is singly generated by an operator $D = D^*$. Then $E = D + iC$ is a generator of $\mathfrak{A}$.

An immediate consequence of Theorem 1 and Theorem 2 is the following generalization of Theorem 1.

THEOREM 3. *Any W^*-algebra $\mathfrak{A}$ of type II_1 with property A on a separable Hilbert space is singly generated.*

Lemma 1 and Theorem 3 allow us to extend most results of [1, 2, 3] to W^*-algebras with property A. In fact most of these results are valid for any class of W^*-algebras $\mathscr{B}$, which satisfies the following two conditions:

i) $\mathfrak{A} \in \mathscr{B}$, then $\mathfrak{A}$ is singly generated and $\mathfrak{A}$ has a faithful normal representation on a separable Hilbert space.

ii) $\mathfrak{A} \in \mathscr{B}$, then for any finite n there exists a $\mathfrak{B} \in \mathscr{B}$ with $\mathfrak{A} \cong M_n(\mathfrak{B})$.

We shall always assume that $\mathscr{B}$ contains all properly infinite W^*-algebras on a separable Hilbert space and every W^*-algebra of type II_1 with property A. If every factor of type II_1 on a separable Hilbert space is singly generated, $\mathscr{B}$ may be chosen to be the class of all W^*-algebras on a separable Hilbert space with no summand of type I-finite.

THEOREM 4. *Let $\mathfrak{A} \in \mathscr{B}$ and let p be a complex polynomial of degree at least three, then there exists a generator T of $\mathfrak{A}$ with $p(T) = 0$.*

PROOF. Use the proof of Corollary 1 of Theorem 1 in [1].

COROLLARY 1. *For any $n \geqq 3$ there exists a generator of $\mathfrak{A}$ with $T^n = 0$ ($T^n = 1$).*

COROLLARY 2. *There exists a generator T of $\mathfrak{A}$, which is similar to a unitary (selfadjoint) operator.*

PROOF. Let T be a generator of $\mathfrak{A}$ with $T^3 = 1$. Then T is similar to a unitary operator U with $U^3 = 1$, $T = QUQ^{-1}$. We can write $U = e^{iA}$, then QAQ^{-1} is a generator of $\mathfrak{A}$, which is similar to the selfadjoint operator A.

COROLLARY 3. *$\mathfrak{A}$ is generated by two commuting idempotents E_1 and E_2 with $E_1 \cdot E_2 = 0$.*

PROOF. Let T, Q and U be as in the proof of Corollary 2 and let $U = P_0 + P_1 e^{2\pi i/3} + P_2 e^{-2\pi i/3}$ be the spectral resolution of U. Then $E_1 = QP_1Q^{-1}$ and $E_2 = QP_2Q^{-1}$ have the required properties.

Theorem 3 of [1] and the corollary are also valid for any $\mathfrak{A} \in \mathscr{B}$. Thus any $\mathfrak{A} \in \mathscr{B}$ arises from a unitary representation of $Z_2 * Z_3$. The results of [2] can be generalized similarly.

THEOREM 5. *Let $\mathfrak{A} \in \mathscr{B}$ then $\mathfrak{A}$ has a dense set of generators and any $T \in \mathfrak{A}$ can be written as the sum of two generators of $\mathfrak{A}$.*

In order to extend the results of [3] we need the following lemmas.

LEMMA 3. *Any continuous or properly infinite W^*-algebra $\mathfrak{A}$ has a transcendental quasinilpotent element.*

PROOF. For $\mathfrak{A}$ construct a system of projections and partial isometries $\{e_n, f_n, v_n\}$ as above. Then let $A_k = \sum_{n \geq k} v_n 2^{-n}$. Because of (1) we have $A_1^k = A_1 \cdot A_2, \cdots A_k$ and thus $| A_1^k | \leqq 2^{-k(k-1)/2}$. Hence A_1 is quasinilpotent. A simple computation shows that A_1 is not nilpotent.

After the author had completed this proof, he learned that a similar construction had been given by Topping [6].

LEMMA 4. *Let $\mathfrak{A}$ be a W^*-algebra of type II_1 on the separable Hilbert space H and let K be a nonempty compact subset of the complex plane, then there exists a normal operator $N \in \mathfrak{A}$ with* $\operatorname{Sp} N = K$.

PROOF. By considering maximal abelian subalgebras of $\mathfrak{A}$ it suffices to show the lemma for the W^*-algebra $\mathscr{L}^\infty([0, 1])$. This however is trivial.

THEOREM 6. *Let $\mathfrak{A} \in \mathscr{B}$ and let K be an arbitrary nonempty compact set in the unit disc and let $\varepsilon > 0$ be arbitrary. Then there exists a generator T of $\mathfrak{A}$ with $| T | \leqq 1 + \varepsilon$ and* $\operatorname{Sp} T = K$. *If $K = \{0\}$ the generator T may be chosen to be a transcendental quasinilpotent (n-nilpotent) partial isometry, with $n \geqq 4$ arbitrary.*

PROOF. a) For $K \neq \{0\}$ use Lemma 3 and the proof of Theorem 4 in [3]. Thus we may assume $K = \{0\}$. Write $\mathfrak{A} = \mathfrak{B} \otimes M_2$ and let $T = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$, where a is a transcendental quasinilpotent $((n-1)$ – nilpotent) generator of $\mathfrak{B}$ with $| a | < 1$ and where $b = (1 - a^* a)^{1/2}$. Then T is a transcendental quasinilpotent (n-nilpotent) generator of $\mathfrak{A}$.

b) Thus it remains to show that any $\mathfrak{B} \in \mathscr{B}$ has a transcendental quasinilpotent generator a. To do this write $\mathfrak{B} = \mathfrak{C} \otimes M_3$ and let

$$a = \begin{bmatrix} h & e & 0 \\ & 0 & f \\ & & 0 \end{bmatrix}$$

where e and f are positive invertible with $\mathfrak{R}(e, f) = \mathfrak{C}$. h is a transcendental quasinilpotent element in $\mathfrak{C}$. That a has all the required properties is shown by simple matrix computation.

The extension of Theorem 2 in [3] we present in a slightly strengthened version.

THEOREM 7. *Let $\mathfrak{A} \in \mathscr{B}$ and let K be a compact set containing* 0 *inside the disc of radius $1 - \varepsilon$, with $1 > \varepsilon > 0$ arbitrary; then there exists a partial isometry T such that TT^* and T^*T commute with* $\operatorname{Sp} T = K$ *and $\mathfrak{R}(T) = \mathfrak{A}$. If $K = \{0\}$ T may be chosen nilpotent for any finite $n \geqq 5$.*

PROOF. a) Write $\mathfrak{A} = \mathfrak{B} \otimes M_3$ and set

$$T = \begin{bmatrix} 0 & a & c \\ & b & d \\ & & 0 \end{bmatrix}$$

where $\mathfrak{R}(b) = \mathfrak{B}$, $\operatorname{Sp} b \cup \{0\} = K$, $|\, b \,| \leqq 1 - \varepsilon/2$ and $b = (bb^*)^{1/2}u$, with u unitary. The operators a, c and d are defined as $a = u^*(1 - bb^*)^{1/2}u$, $c = u^*(bb^*)^{1/2}u$ and $d = (1 - bb^*)^{1/2}u$. Then $TT^* = \operatorname{diag}(1, 1, 0)$, $T^*T = \operatorname{diag}(0, 1, 1)$ and $\operatorname{Sp} T = \operatorname{Sp} b \cup \{0\} = K$. $\mathfrak{R}(T) = \mathfrak{A}$ is shown as in [3].

b) Thus it remains to show that we can find such a $b \in \mathfrak{B}$. To see this write $\mathfrak{B} = \mathfrak{C} \otimes M_3$ and set

$$b = \begin{bmatrix} h & e & 0 \\ & 0 & f \\ & & 0 \end{bmatrix}$$

with h normal and $\operatorname{Sp} h = K$ if $K \neq \{0\}$. e and f are positive invertible operators with $\mathfrak{R}(e, f) = \mathfrak{C}$ and $|\, e \,|, |\, f \,| < \varepsilon/4$. We may further assume that e and h commute. Then b has all the required properties.

c) If $K = \{0\}$ we need a b with $|\, b \,| < 1$, $\mathfrak{R}(b) = \mathfrak{B}$, $b^{n-2} = 0$ and $b = (bb^*)^{1/2}u$, where u is a unitary operator. Write $\mathfrak{B} = \mathfrak{C} \otimes M_{n-2}$ and set

$$b = \frac{1}{2}\begin{bmatrix} 0 & e & & & \\ & 0 & f & & \\ & & 0 & 1 & \\ & & & & 1 \\ & & & & 0 \end{bmatrix}$$

with e and f positive invertible such that $\mathfrak{R}(e, f) = \mathfrak{C}$ and $|e|, |f| < 1$. Then b has all required properties.

By a slightly more complicated construction one can even show that the b in (c) may be chosen transcendental quasinilpotent.

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