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ON THE DISTRIBUTION OF VALUES OF FUNCTIONS IN SOME FUNCTION CLASSES IN THE ABSTRACT HARDY SPACE THEORY

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In this note we shall study some properties of functions in some function classes in the abstract Hardy space theory, developed by König [2], especially the distribution of values of functions in such a class. We shall give first a generalization of a classical Löwner's lemma and its precise form (Theorem 1). From it follows a generalization of a theorem of R. Nevanlinna on inner functions in the unit disc U of the complex plane $\boldsymbol{C}$ to abstract Hardy spaces (Corollary 1). Using the real-analyticity of a function arising in Theorem 1 we shall investigate the distribution of values of bounded functions in abstract Hardy spaces (Theorem 2 and its corollaries). One of them can be stated in the classical case as follows: Let $f(z)$ be a bounded holomorphic function in U such that $|f(z)|<1$ in U and its boundary function value $f(e^{i\theta})$ is real or $|f(e^{i\theta})|=1$ a.e. on T, the boundary of U with the normalized Lebesgue measure L. Then it holds $L\{e^{i\theta}: f(e^{i\theta}) \in E \cup E^*\} > 0$ for every measurable set $E \subset T$ with $L(E)>0$ or $f(z)$ is a constant, where $E^*=\{t \in T: t^* \in E\}$. In Section 6 corresponding results are given for the class H^+: a class of functions with nonnegative real part, which is defined in the next section. We improve also a uniqueness theorem for functions in H^+ (Satz 7 in [9]). Some applications to domains in the n-dimensional complex vector space are given in Section 7. The author would like to acknowledge several helpful conversations with Professor Heinz König.

1. Let (X, Σ, m) be a probability measure space and $L(m)$ be the set of all measurable functions on X. We assume further H is a weak* closed subalgebra of $L^\infty(m)$ with 1 and φ; $\varphi(u)=\int u(x)dm(x)$ $(u \in H)$ is multiplicative on H. Let $L^\#$ be the set of all functions $f \in L(m)$ such that there exists a sequence of functions $u_n \in H$ with $|u_n| \leqq 1$, $u_n \to 1$ and $u_n f \in L^\infty(m)$. Let $H^\#$ be the set of all functions $u \in L(m)$ such that there exists a sequence of functions $u_n \in H$ with $u_n \to u$ and $|u_n| \leqq$ some

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$F \in L^\sharp$. Thus $H^\sharp \subset L^\sharp \subset L(m)$ is a complex subalgebra of $L^\sharp$ with $H \subset H^\sharp$. One proves that for $u \in H^\sharp$ there exists a sequence of functions $u_n \in H$ with $u_n \to u$ and $|u_n| \leqq |u|$. Therefore $H^\sharp \cap L^\infty(m) = H$. Further $u \in L(m)$ and $u_n \in H$ with $u_n \to u$ and $|u_n| \leqq$ some $F \in L^\sharp$ implies that $u \in H^\sharp$. Furthermore there exists a unique extension of φ; $H \to C$ to a multiplicative linear functional φ; $H^\sharp \to C$ which is continuous in the sense that $f_n, f \in H^\sharp, f_n \to f$ and $|f_n| \leqq$ some $F \in L^\sharp$ implies that $\varphi(f_n) \to \varphi(f)$. We define next a subclass of $H^\sharp$. Let

$$H^+ = \{f \in L(m);\ \mathrm{Re}\, f \geqq 0 \text{ a.e. and } e^{-tf} \in H \text{ for all } t > 0\}\,.$$

It is already known that $H^+ \subset H^\sharp$ and

$$\begin{aligned} H^+ &= \{f \in L(m);\ \mathrm{Re} f \geqq 0 \text{ a.e. and } 1/(f+t) \in H \text{ for all } t > 0\} \\ &= \{f = (1+u)/(1-u);\ u \in H \text{ with } |u| \leqq 1, u \neq 1\} \end{aligned}$$

and if $f \in H^+$ and $f \neq 0$, $1/f$ is also in H^+. We state the following lemma, whose proof is due to König.

LEMMA 1.
(i) *Let* $f, g \in H^+$ *and* $\mathrm{Re} fg \geqq 0$ a.e.. *Then* fg *is in* H^+.
(ii) *Let* $f, g \in H^+$ *and* $0 < \alpha < 1$. *Then* $f^\alpha g^{1-\alpha}$ *is also in* H^+.

PROOF. i) We may assume $g \neq 0$. Clearly we have $f + tg^{-1} \in H^+$ and $\neq 0$ for every $t > 0$. Hence we have $g^{-1}/(f + tg^{-1}) \in H^\sharp$. Since $\mathrm{Re} fg \geqq 0$, we see that $g^{-1}/(f + tg^{-1}) = 1/(fg + t)$ is bounded. Hence it is in H, which shows that $fg \in H^+$. ii) It is known that f^α, $g^{1-\alpha}$ are well-defined and in H^+ ([9] Satz 3). Clearly we have $\mathrm{Re}\, f^\alpha g^{1-\alpha} \geqq 0$ a.e.. We apply i).
q.e.d.

2. We shall state a precise form of a generalization of the classical Löwner's lemma*).

THEOREM 1. *Let* $u \in H$ *with* $|u(x)| \leqq 1$ a.e. *and* $u \neq e^{i\alpha}$ (α: *real*). *Let* $\int u dm = b$. *Then for any Lebesgue measurable set* $E \subset T$, *we have*

$$(1)\quad \int_{\{x;\, u(x) \in U\}} dm(x) \int_E \frac{1-|u(x)|^2}{|e^{i\theta} - u(x)|^2} d\theta = \int_E d\theta \int_{\{x;\, u(x) \in U\}} \frac{1-|u(x)|^2}{|e^{i\theta} - u(x)|^2} dm(x)$$
$$= \int_E \frac{1-|b|^2}{|e^{i\theta} - b|^2} d\theta - 2\pi m\{x;\ u(x) \in E\}\,.$$

In particular we have

$$(2)\qquad m\{x;\ u(x) \in E\} \leqq \frac{1+|b|}{1-|b|} L(E)\,.$$

*) Cf. [7] p. 322.

PROOF. We have first

$$\frac{1-r^2|u(x)|^2}{|e^{i\theta}-ru(x)|^2} = \frac{e^{i\theta}}{e^{i\theta}-ru(x)} + \frac{re^{i\theta}\overline{u(x)}}{1-re^{i\theta}\overline{u(x)}}$$

$$= \sum_{n=0}^{\infty}(re^{-i\theta}u(x))^n + \sum_{n=1}^{\infty}(re^{i\theta}\overline{u(x)})^n \qquad \text{for } 0 \leqq r < 1,\ e^{i\theta} \in T\,.$$

Since $u \in H$, by integrating the above equality, we have

$$\int \frac{1-r^2|u(x)|^2}{|e^{i\theta}-ru(x)|^2}dm(x) = \frac{1-r^2|b|^2}{|e^{i\theta}-rb|^2} \qquad \text{for } 0 \leqq r < 1,\ e^{i\theta} \in T\,. \tag{3}$$

Hence, letting $r \to 1$ we see by Fatou's lemma that

$$\int_{\{x;v(x)\in U\}} \frac{1-|u(x)|^2}{|e^{i\theta}-u(x)|^2}dm(x) \leqq \frac{1-|b|^2}{|e^{i\theta}-b|^2} \leqq \frac{1+|b|}{1-|b|} \qquad \text{for all } e^{i\theta} \in T\,.$$

Therefore, since the Lebesgue measure on T is outer regular, it suffices to show 1) or 2) for open sets on T. Now let A be an open set on T. Then we have $A = \bigcup_j A_j$ (A_j: open arc on T, $A_j \cap A_k = \emptyset$ if $j \neq k$). Put

$$g_r(u(x)) = \int_A \frac{1-r^2|u(x)|^2}{|e^{i\theta}-ru(x)|^2}d\theta \qquad (0<r<1)\,.$$

Then we see easily by the properties of the Poisson kernel that

$$\begin{aligned} &|g_r(u(x))| \leqq 2\pi \ (x \in X)\,, \\ &\lim_{r\to 1} g_r(u(x)) = 2\pi \qquad (u(x) \in A)\,, \\ &\qquad = \pi \quad \text{or} \quad 2\pi \qquad \Big(u(x) \in \bigcup_j (\bar{A}_j - A_j)\Big)\,, \\ &\qquad = 0 \qquad \Big(u(x) \in T - \bigcup_j \bar{A}_j\Big)\,, \\ &\qquad = \int_A \frac{1-|u(x)|^2}{|e^{i\theta}-u(x)|^2}d\theta \qquad (u(x) \in U)\,. \end{aligned}$$

Hence, integrating the equality 3) with respect to θ on A and letting $r \to 1$, we have

$$2\pi m\{x;\ u(x) \in A\} \leqq \int_A \frac{1-|b|^2}{|e^{i\theta}-b|^2}d\theta \leqq 2\pi\frac{1+|b|}{1-|b|}L(A)\,.$$

This shows the inequality 2). Hence we have

$$m\Big\{x;\ u(x) \in \bigcup_j (\bar{A}_j - A_j)\Big\} = 0\,.$$

Therefore, by the dominated convergence theorem we have the equality

1) for open sets, which completes the proof.

As immediate consequences of this theorem we have the following corollaries.

COROLLARY 1. *Let $u(x)$, b be the same as in Theorem* 1. *Further suppose* $|u(x)| = 1$ a.e.. *Then we have for any measurable set* $E \subset T$

$$m\{x;\ u(x) \in E\} = \frac{1}{2\pi}\int_E \frac{1-|b|^2}{|e^{i\theta}-b|^2}\,d\theta\ ,$$

and hence

$$\frac{1-|b|}{1+|b|}\,L(E) \leqq m\{x;\ u(x) \in E\} \leqq \frac{1+|b|}{1-|b|}\,L(E)\ .$$

In particular, if $\int u dm = 0$, *we have*

$$m\{x;\ u(x) \in E\} = L(E)\ .$$

COROLLARY 2. *Let $u(x)$, b be the same as in Theorem* 1. *Further suppose $u(x) \in T$ or real* a.e.. *Then we have for any measurable set* $E \subset T$

$$m\{x;\ u(x) \in E\} - m\{x;\ u(x) \in E^*\} = \frac{1-|b|^2}{2\pi}\int_E (|e^{i\theta}-b|^{-2} - |e^{-i\theta}-b|^{-2})d\theta\ ,$$

where $E^* = \{e^{i\theta};\ e^{-i\theta} \in E\}$.

COROLLARY 3. *Let $u(x)$, b be the same as in Theorem* 1. *Further suppose $u(x) \in T_+$ or real* a.e., *where* $T_+ = \{e^{i\theta};\ 0 \leqq \theta \leqq \pi\}$. *Then we have for any measurable set* $E \subset T_+$

$$m\{x;\ u(x) \in E\} = \frac{1-|b|^2}{2\pi}\int_E (|e^{i\theta}-b|^{-2} - |e^{-i\theta}-b|^{-2})d\theta\ .$$

We state next a lemma which we need in the next section.

LEMMA 2. *Let $u(x)$, b be the same as in Theorem* 1. *Then, if*

$$m\{x;\ u(x) \in E\} = 0$$

for some measurable set $E \subset T$ of positive measure, we have

$$\int_{\{x;u(x)\in U\}} \frac{1-|u(x)|^2}{|e^{i\theta}-u(x)|^2}\,dm(x) = \frac{1-|b|^2}{|e^{i\theta}-b|^2} \qquad \text{a.e. } e^{i\theta} \in E\ .$$

PROOF. We have already seen that

$$\int_{\{x;u(x)\in U\}} \frac{1-|u(x)|^2}{|e^{i\theta}-u(x)|^2}\,dm(x) \leqq \frac{1-|b|^2}{|e^{i\theta}-b|^2} \qquad \text{for all } e^{i\theta} \in T\ .$$

Combining this with 1) of Theorem 1 we have the desired conclusion.

q.e.d.

3. We investigate next some properties of the integrated function

$$\int_{\{u(x)\in U\}} \frac{1-|u(x)|^2}{|e^{i\theta}-u(x)|^2}\,dm(x)\ .$$

LEMMA 3. *Let $f(\theta)=e^{i\theta}/(e^{i\theta}-a)$ and $|a|\neq 1$. Then f is indefinitely differentiable in θ and we have*

$$(*)\qquad |f^{(n)}(\theta)|\leqq\begin{cases}2^n n!\,|e^{i\theta}-a|^{-n} & (|e^{i\theta}-a|<1,\ n=0,1,2,\cdots)\ .\\ 2^n n! & (|e^{i\theta}-a|\geqq 1,\ n=0,1,2,\cdots)\ .\end{cases}$$

In particular, $f(\theta)$ is real-analytic in $(-\infty,\infty)$, i.e., $f(\theta)$ can be expanded in Taylor series at any $\theta_0\in(-\infty,\infty)$ and its convergence radius is larger than $|e^{i\theta_0}-a|/2$.

PROOF. We have first the following formula

$$\frac{d}{d\theta}\frac{e^{in\theta}}{(e^{i\theta}-a)^n}=in\left(\frac{e^{in\theta}}{(e^{i\theta}-a)^n}-\frac{e^{i(n+1)\theta}}{(e^{i\theta}-a)^{n+1}}\right)\qquad n=1,2,\cdots\ .$$

Using this formula, we see easily that $f^{(n)}(\theta)$ is the sum of 2^n terms of the form $ce^{ik\theta}(e^{i\theta}-a)^{-k}$ (c: complex number, k: integer, $0<k\leqq n+1$) such that $|c|\leqq n!$. Hence we have the inequality (*). Since

$$|f^{(n)}(\theta)|\leqq 4^n n!\,|e^{i\theta_0}-a|^{-n}$$

if $|e^{i\theta}-e^{i\theta_0}|\leqq|e^{i\theta_0}-a|/2$, we see that

$$f(\theta)=\sum_{n=0}^{\infty}\frac{f^{(n)}(\theta_0)}{n!}(\theta-\theta_0)^n\qquad\text{in } |\theta-\theta_0|<|e^{i\theta_0}-a|/4\ .$$

Clearly the Taylor series converges in $|\theta-\theta_0|<|e^{i\theta_0}-a|/2$. Hence we have the last assertion. q.e.d.

LEMMA 4. *Let $u(x)\in L(m)$. Let $0<\delta<1$ and*

$$U_\delta(\theta_0)=\{z\in U;\,|z-e^{i\theta_0}|>\delta\}\ .$$

Let

$$f(\theta)=\int_{\{u(x)\in U_\delta(\theta_0)\}}\frac{1-|u(x)|^2}{|e^{i\theta}-u(x)|^2}\,dm(x)\ .$$

Then we have

$$f(\theta)=\sum_{n=0}^{\infty}\frac{f^{(n)}(\theta_0)}{n!}(\theta-\theta_0)^n\qquad \textit{for } |\theta-\theta_0|<\delta/2\ .$$

PROOF. An easy calculation shows that

$$\frac{1-|u(x)|^2}{|e^{i\theta}-u(x)|^2}=\frac{e^{i\theta}}{e^{i\theta}-u(x)}+\frac{e^{i\theta}\overline{u(x)}}{1-e^{i\theta}\overline{u(x)}}\qquad(u(x)\in U)\ .$$

In the same way as in the proof of Lemma 3 we have similar inequalities for the differential coefficients of the last term to the inequalities (*) for the first term. Hence for any fixed θ: $|e^{i\theta}-e^{i\theta_0}|<\delta$ we have

$$\left|\frac{d^n}{d\theta^n}\frac{1-|u(x)|^2}{|e^{i\theta}-u(x)|^2}\right| \leqq 2^{n+1}n!\,|e^{i\theta}-e^{i\gamma}|^{-n}$$

$$(n=0,1,2,\cdots,\ u(x)\in U_\delta(\theta_0))\ ,$$

where $|e^{i\gamma}-e^{i\theta_0}|=\delta$ and $|e^{i\theta}-e^{i\gamma}|<\delta$. Hence, since m is a probability measure, we have

$$|f^{(n)}(\theta)|\leqq 2^{n+1}n!\,|e^{i\theta}-e^{i\gamma}|^{-n} \qquad (n=0,1,2,\cdots)\ .$$

The rest of the proof follows along the same lines as that of Lemma 3. q.e.d.

By the above lemma we can state the following fundamental lemma.

LEMMA 5. *Let* $u(x)\in L(m)$. *Let* $A=\{e^{i\theta}\in T;\ \alpha<\theta<\beta\}$ *and* W *be an open set in* C *containing* A. *Let* $V=U\cap W^c$ *and* $F=\{x;\ u(x)\in V\}$. *Then the function*

$$\int_F \frac{1-|u(x)|^2}{|e^{i\theta}-u(x)|^2}\,dm(x)$$

is real-analytic in $(\alpha<\theta<\beta)$.

PROOF. This follows immediately from Lemma 4.

4. Combining the results in Sections 2 and 3, we can investigate the distribution of values of bounded functions in H.

THEOREM 2. *Let* A, W, V *be the same as in Lemma* 5. *Let* $u\in H$, $u(x)\in T\cup V$ a.e. *and* $u\neq e^{i\gamma}$(γ: *real*). *Then, if for some measurable set* $E\subset A$ $(L(E)>0)$ *we have* $m\{x;\ u(x)\in E\}=0$, *it follows that*

$$m\{x;\ u(x)\in A\}=0\ .$$

PROOF. Put

$$f(\theta)=\frac{1-|b|^2}{|e^{i\theta}-b|^2}-\int_{\{u(x)\in V\}}\frac{1-|u(x)|^2}{|e^{i\theta}-u(x)|^2}\,dm(x)\ ,$$

where $b=\int u\,dm$. Then by Lemma 2 we have $f(\theta)=0$ a.e. on E. Since E is of positive measure, E contains a non-empty perfect set F. Hence we have

$$f^{(n)}(\theta)=0 \qquad (e^{i\theta}\in F,\ n=0,1,2,\cdots)\ .$$

By Lemma 5 we see that $f(\theta)$ is real-analytic in $\alpha<\theta<\beta$. Hence

$f(\theta) = 0$ in $\alpha < \theta < \beta$. Combining this with the equality 1) in Theorem 1, we have $m\{x; u(x) \in A\} = 0$. q.e.d.

Combining this theorem with the corollaries to Theorem 1 we have the following result.

THEOREM 3. *Let* $u \in H$. *Further suppose* $u(x) \in T \cup (-1, 1)$ a.e.. *Let* $\int u\, dm = b$. *Then we have*

1) *If* $m\{x; u(x) \in E\} = m\{x; u(x) \in F\} = 0$ *for some measurable sets* $E \subset T_+$ and $F \subset T_- = T - T_+$ $(L(E), L(F) > 0)$, *then* u *is constant.*

2) *Let* $\operatorname{Im} b = 0$. *Then if* $m\{x; u(x) \in E\} = 0$ *for some measurable set* $E \subset T$ $(L(E) > 0)$, u *is constant.*

3) *Let* $\operatorname{Im} b > 0$. *Then if* $m\{x; u(x) \in E\} = 0$ *for some measurable set* $E \subset T_+$ $(L(E) > 0)$, u *is constant.*

4) *If* $m\{x; u(x) \in E\} = 0$ *for some measurable set* $E \subset T_+$ $(L(E) > 0)$, *we have* $m\{x; u(x) \in T_+\} = 0$ *and that* $m\{x; u(x) \in F\} > 0$ *for any measurable set* $F \subset T_- \cup [-1, 1]$ *of positive measure or* u *is constant.*

5) *If* $m\{x; u(x) \in E\} = 0$ *for some* $E \subset [-1, 1]$ *of positive measure, then we have* $m\{x; u(x) \in F\} > 0$ *for any measurable set* $F \subset T$ *with* $L(F) > 0$ *or* u *is constant.*

PROOF. 1) By Theorem 2 u is a real-valued function. From the equality

$$\int (u - b)^2\, dm = \int u^2\, dm - 2b \int u\, dm + b^2 = 0$$

it follows that $u = b$.

2) By Corollary 2 we have $m\{x; u(x) \in E^*\} = m\{x; u(x) \in E\} = 0$. Hence u is constant by 1).

3) If u is not constant, we have by Corollary 2

$$0 \leqq m\{x; u(x) \in E^*\} < m\{x; u(x) \in E\},$$

which is a contradiction.

4) The first assertion follows immediately from Theorem 2. Suppose next that u is not constant and $m\{x; u(x) \in F\} = 0$ for some measurable set $F \subset T_- \cup [-1, 1]$ of positive measure. Let $f(z)$ be a holomorphic function in $U_- = \{|z| < 1; \operatorname{Im} z < 0\}$ mapping U_- on U conformally. Then we see by the Lemma 6 below that $f(u)$ is well-defined, in H and $|f(u(x))| = 1$ a.e.. By a theorem of F. and M. Riesz F is mapped on a subset of T of positive measure. Hence $f(u)$ is constant by Corollary 1, and so u is also constant. It is a contradiction.

5) If there exists a measurable set $F \subset T$ of positive measure such

that $m\{x;\ u(x) \in F\} = 0$, then by 4) u is constant. q.e.d.

LEMMA 6. *Let D_1, D_2 be simply connected domains in C bounded by Jordan curves Γ_1, Γ_2 respectively. Let $u \in H$ and $u(x) \in \bar{D}_1$* a.e.. *Let $f(z)$ be a conformal mapping function from D_1 on D_2. Then $f(u)$ is well-defined, in H and $f(u(x)) \in \bar{D}_2$* a.e..

PROOF. We see that $f(z)$ is continuous on $\bar{D}_1$ and maps $\bar{D}_1$ onto $\bar{D}_2$ one-to-one and topologically. Hence $f(u)$ is well-defined and $f(u(x)) \in \bar{D}_2$ for $x;\ u(x) \in \bar{D}_1$. By a theorem of Walsh there exists a sequence of polynomials $P_n(z)$ which converges to $f(z)$ uniformly on $\bar{D}_1$. Since $P_n(u)$ is clearly in H and H is weak* closed, $f(u)$ is also in H. q.e.d.

REMARK. By Lemma 6, Theorem 2 holds if we replace T by any closed rectifiable Jordan curve and Lebesgue measure by the measure defined by means of the arc length of that curve. 4) and 5) of Theorem 3 hold if we replace T, $(-1, 1)$ and Lebesgue measures by any closed rectifiable Jordan curve, any rectifiable Jordan arc and the measures defined by means of the arc length of their arcs respectively.

Combining Lemma 6 and Theorem 2 we shall prove

THEOREM 4. *Let A and B be two disjoint compact sets in C such that $(A \cup B)^c$ is connected. Let Γ be a Jordan arc joining a boundary point a of A with a boundary point b of B such that $\Gamma \cap (A \cup B) = \{a, b\}$. Then if $u \in H$ and $u(x) \in A \cup B \cup \Gamma$* a.e., *$u(x) \in A$* a.e. *or $u(x) \in B$* a.e. *or u is a constant.*

PROOF. We suppose first u is not constant. Let Γ_1 be a Jordan arc joining a with b such that Γ_1 does not intersect $A \cup B$ and the jointed curve of Γ and Γ_1 surrounds $A \cup B$. Let $f(z)$ be a conformal mapping function from the simply connected domain bounded by Γ and Γ_1 on the rectangle $\{0 < \operatorname{Im} z < 1,\ -1 < \operatorname{Re} z < 1\}$ such that a point $c \in \Gamma$ $(c \neq a, b)$ is mapped to the origin and $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$. We may assume $f(a) < 0 < f(b)$, say. We map further that rectangle by $g(z) = z^2$. Then there exists a point $d \in \boldsymbol{R}$ such that $d = \alpha^2 = \beta^2$ for some $\alpha, \beta \in f(\Gamma)$ $(\alpha < 0 < \beta)$. Let Γ_2 be a Jordan arc joining the origin with d such that $(0, d)$ and Γ_2 surround $f^2(A) \cup f^2(B) \cup (f^2(\Gamma) - (0, d))$ and Γ_2 does not intersect $f^2(A) \cup f^2(B) \cup f^2(\Gamma)$. Let D be the simply connected domain bounded by $(0, d)$ and Γ_2. We map conformally D on U by an $h(z)$. Then by Lemma 6 we see that $f(u) \in H$ and $f^2(u)$ is clearly in H and again by Lemma 6 we have $h(f^2(u)) \in H$. The image of Γ_2 by $h(z)$ is a non-empty arc I on T. Since $m\{x;\ h(f^2(u(x))) \in I\} = 0$, we have

$$m\{x;\ h(f^2(u(x))) \in h((0, d))\} = 0\ ,$$

by Theorem 2. Since h maps $\bar{D}$ on $\bar{U}$ one-to-one, we have

$$m\{x;\ f^2(u(x)) \in (0, d)\} = 0\ .$$

This shows that $m\{x;\ f(u(x)) \in (\alpha, \beta)\} = 0$. By the remark above we have $m\{x;\ f(u(x)) \in (f(a), f(b))\} = 0$. Again by Lemma 6, we have

$$m\{x;\ u(x) \in \Gamma\} = 0\ .$$

Next suppose $m\{x;\ u(x) \in A\}\ m\{x;\ u(x) \in B\} > 0$. Let $\gamma = \text{ess. inf Re}\ u(x)$. Considering $u + \gamma + 1$, $A + \gamma + 1$, $B + \gamma + 1$ in place of u, A, B respectively, we may assume $\text{Re}\, u(x) \geqq 1$ a.e.. By the assumption for A, B there is a sequence of polynomials $P_n(z)$ converging to 0 uniformly on A and to z uniformly on B in virtue of a theorem of Runge. Since clearly $P_n(u)$ is in H, we see thus that the function u_2: $u_2 = u$ on B', $=0$ on A' is in H, where $B' = \{x;\ u(x) \in B\}$ and $A' = \{x;\ u(x) \in A\}$. Hence we have

$$u_1 = u - u_2 \in H\ .$$

Let $\int u_1\, dm = s$ and $\int u_2 dm = t$. Then since Re $u \geqq 1$ a.e., we have $st \neq 0$. Now since $u_1, u_2 \in H$, we have

$$s^n + t^n = \int u_1^n\, dm + \int u_2^n\, dm = \int (u_1 + u_2)^n\, dm = (s + t)^n \quad (n = 1, 2, \cdots)\ ,$$

which is clearly not true. q.e.d.

As a special case of the above theorem we have

COROLLARY 4. *Let Γ be a Jordan arc with end points a, b in* C *($a \neq b$). Then if $u \in H$ and $u(x) \in \Gamma$* a.e., *u is a constant.*

We conclude this section with the following easy consequence of Theorem 2.

COROLLARY 5. *Let $u \in H$. Further suppose $|u(x)| = 1$ or $|u(x)| \leqq r$ (for some $0 < r < 1$)* a.e.. *Then we have $m\{x;\ u(x) \in E\} > 0$ for any measurable set $E \subset T$ with $L(E) > 0$ or $|u(x)| \leqq r$* a.e..

5. EXAMPLES. We shall give some example functions satisfying the assumptions in the preceding propositions.

(i). Let $X = T$, $m = L$ and H be the set of all bounded functions which are the limit functions $\lim_{r\to 1} f(re^{i\theta})$ of bounded holomorphic functions in U. Let $g(z) = (z - i/2)/(1 + iz/2)$ and put $f(z) = (1+z)/(1-z)\cdot(1+g(z)/(1 - g(z))$. Then we have $f(e^{i\theta}) = -\cot\theta/2 \cot(\theta + \gamma)/2$, where $e^{i\gamma} = (1 - i/2)/(1 + i/2)$. Hence we see easily that $f(e^{i\theta})$ is real-valued and $\{f(e^{i\theta});\ 0 \leqq \theta < 2\pi\} = [-\infty, \infty]$. Let $u(z) = (\sqrt{f(z)} - 1)/(\sqrt{f(z)} + 1)$, where we take the branch of $\sqrt{\ }$ as $\sqrt{1} = 1$. Then we have $|u(e^{i\theta})| = 1$ or real a.e. and $\operatorname{Im} \int u\, dL = \operatorname{Im} u(0) < 0$. This shows that the discussion in 3), 4) of Theorem 3 is meaningful.

(ii). Let $X = T \cup T/2$, where $T/2 = \{z \in \boldsymbol{C};\ |z| = 1/2\}$. Let m be the harmonic measure for $z = 3/4$ and H be the set of all limit functions of bounded holomorphic functions in $\{1/2 < |z| < 1\}$ as in (i). Let $u(t) = t$ for $t \in X$. Then we have $u \in H$ and $|u(t)| = 1$ or $1/2$. This shows that Corollary 5 has a sence.

6. The case H^{+}. We can extend all results in Section 4 to the case H^{+}.

THEOREM 5. *Let I be an open interval on the imaginary axis and $E \subset I$ be a Lebesgue measurable set of positive measure and V be an open set in $\boldsymbol{C}$ containing I. Let $f \in H^{+}$, $f \neq ia$ (a: real) and $f(x) \in i\boldsymbol{R} \cup (S \backslash V)$* ($S = \{\operatorname{Re} z > 0\}$) a.e.. *Further let $m\{x;\ f(x) \in E\} = 0$. Then it follows that $m\{x;\ f(x) \in I\} = 0$.*

PROOF. The function $g = (f-1)/(f+1)$ is in $H^{\#} \cap L^{\infty}(m) = H$. Hence by the conformal mapping $w = (z-1)/(z+1)$ we can apply Theorem 2 and we have the desired result. q.e.d.

THEOREM 6. *Let $f \in H^{+}$, $f^{2}(x)$ be real* a.e. *and $\varphi(f) = \alpha + i\beta$. Then we have*

1) *If $m\{x;\ f(x) \in iE\} = m\{x;\ f(x) \in iF\} = 0$ for some two measurable sets $E \subset \boldsymbol{R}_{+} = [0, \infty)$, $F \subset \boldsymbol{R}_{-} = (-\infty, 0]$ of positive measure, f is a constant.*

2) *Let $\beta = 0$. Then if $m\{x; f(x) \in iE\} = 0$ for some measurable set $E \subset \boldsymbol{R}$ of positive measure, f is a constant.*

3) *Let $\beta > 0$. Then if $m\{x; f(x) \in iE\} = 0$ for some measurable set $E \subset \boldsymbol{R}_{+}$ of positive measure, f is a constant.*

4) *Let $\beta < 0$. Then if $m\{x;\ f(x) \in iE\} = 0$ for some measurable set $E \subset \boldsymbol{R}_{+}$ of positive measure, we have $m\{x;\ f(x) \in i\boldsymbol{R}_{+}\} = 0$ and that f is a constant or $m\{x; f(x) \in F\} > 0$ for all measurable set $F \subset \boldsymbol{R}_{+} \cup i\boldsymbol{R}_{-}$ of positieve measure.*

5) *If $m\{x;\ f(x) \in E\} = 0$ for some measurable set $E \subset \boldsymbol{R}_{+}$ of positive measure, then f is constant or $m\{x;\ f(x) \in F\} > 0$ for any measurable set $F \subset i\boldsymbol{R}$ of positive measure.*

PROOF. We apply Theorem 3 to the function $(f-1)/(f+1)$, which is in H. q.e.d.

The following corresponds to Corollary 4.

PROPOSITION 1. *Let Γ be a Jordan arc in* $\{\mathrm{Re}\, z > 0\}$ *one of whose end points lies on* $\{\mathrm{Re}\, z \geqq 0\}$ *and another end point may be the point at infinity. Then, if* $f \in H^+$ *and* $f(x) \in \Gamma$ a.e., f *is a constant.*

As an application of Theorem 5 we have

COROLLARY 6. *Let* $f, g, h \in H^+$. *Further let* $fgh(x)$ *real* a.e. *and* $m\{x; fgh(x) \in E\} = 0$ *for some measurable set* E *on* $(-\infty, 0]$ *of positive measure. Then it follows that* fgh *is a constant.*

PROOF. We see by using Lemma 1 twice that $k = f^{1/3}g^{1/3}h^{1/3}$ is well-defined and in H^+. By the assumption we have

$$k(x) \in i\boldsymbol{R} \cup \boldsymbol{R}_+ \cup e^{i\pi/3}\boldsymbol{R}_+ \cup e^{-i\pi/3}\boldsymbol{R}_+ \qquad \text{a.e.}$$

and there exist four measurable sets E_j $(j = 1, 2, 3, 4)$ such that $m\{x; k(x) \in E_j\} = 0$ and $E_1 \subset i\boldsymbol{R}_+$, $E_2 \subset i\boldsymbol{R}_-$, $E_3 \subset e^{i\pi/3}\boldsymbol{R}_+$, $E_4 \subset e^{-i\pi/3}\boldsymbol{R}_+$ of positive measure respectively. By using Theorem 5 we see that

$$m\,\{x;\, k(x) \in i\boldsymbol{R}\} = 0\,.$$

Since clearly $u(z) = e^{i\pi/6} \in H^+$ and $\mathrm{Re}\, e^{i\pi/6}k(x) \geqq 0$ a.e., we see by Lemma 1 that $e^{i\pi/6}k \in H^+$. Using Theorem 5 again, we have $m\{x; k(x) \in e^{i\pi/3}\boldsymbol{R}_+\} = 0$. In the same way we see that $m\{x; k(x) \in e^{-i\pi/3}\boldsymbol{R}_+\} = 0$. Hence we have $k(x) \in \boldsymbol{R}_+$ a.e.. From Proposition 1 it follows that k is constant, and so fgh is also constant. q.e.d.

REMARK. For more than three functions Corollary 6 does not hold. An example is given in the case of the disc algebra. Let $f_j(z) = f(z) = (1+z)/(1-z)$ $(j=1, 2, 3, 4)$. Then we see easily that $f \in H^+$ and $f^4(z)$ is positive-valued on T.

As a special case of Corollary 6 we have

COROLLARY 7 (Uniqueness theorem for H^+). *Let* $f \in H^+$, $\neq 0$. *Further suppose* $g \in H^+$, $g/f(x)$ *be real* a.e. *and* $m\,\{x; g/f(x) \in E\} = 0$ *for some Lebesgue measurable set* E *on* $(-\infty, 0]$ *of positive measure. Then it follows that* $g = af$ *for some real constant* a.

PROOF. From the assumption it follows that $1/f \in H^+$. Clearly the constant function 1 is in H^+. We apply Corollary 6. q.e.d.

We have thus improved our former result (Satz 7 in [9]) completely in the abstract setting.

7. **Some applications.** Let $H = \{f^*(w) = \lim_{r\to 1} f(rw);\ f \in H^\infty(U^n)\}$ $(n \geqq 1)$. This function class on T^n satisfies the conditions for H in

Section 1. In this case we get

$$H^+ = \{f^*(w);\ f(z) \text{ holomorphic and } \operatorname{Re} f(z) > 0 \text{ for } z \in U^n\} .$$

Hence Theorem 5 in [8] is improved as follows.

PROPOSITION 2. *If the ranges of f and g, holomorphic in U^n, are contained in some open wedge of angular measure $\alpha\pi$ $(0 < \alpha < 2)$ with vertex at the origin, then the proposition f^*/g^* is real* a.e. *on T^n and $m_n\{w \in T^n;\ f^*/g^*(w) \in E\} = 0$ for some measurable set E on $(-\infty, 0]$ of positive measure implies that $f = ag$ for some real constant a.*

PROOF. We see easily that we may assume $|\arg f/g(z)| \leqq \pi$ in U^n. We may also assume $|\arg f/g(z)| < \pi$ in U^n. Otherwise f/g is trivially constant. We consider the function $h = f^{1/2}g^{-1/2}$, where we take the branch of $z^{1/2}$ as $1^{1/2} = 1$. Then we see that $h \in H^+$ and $h^{*2}(w)$ is real-valued and there exists a measurable set $F \subset i\boldsymbol{R}_+$ of positive measure such that $m_n\{w \in T^n;\ h^*(w) \in F \cup (-F)\} = 0$. By Theorem 5, 1) we see that h is constant. Hence f/g is constant and is clearly real. q.e.d.

For U^n a consequence of Theorem 3 is as follows.

PROPOSITION 3. *Let $f(z)$ be a bounded holomorphic function on U^n, bounded by 1 and its boundary value $f^*(w)$ be real or $|f^*(w)| = 1$* a.e. *on T^n. Then it holds $m_n\{w;\ f^*(w) \in E \cup E^*\} > 0$ for every measurable set $E \subset T$ with $L(E) > 0$ or f is a constant.*

This proposition is in a sense sharp. Indeed, let $f(z)$ be a conformal mapping function from U onto the upper half disc $\{z \in U;\ \operatorname{Im} z > 0\}$. Then $\operatorname{Arg} f(e^{i\theta})$ does not take any value on $(-\pi, 0)$. Hence we can not replace $E \cup E^*$ for instance by E. However as a consequence of 3) in Theorem 3 we have

PROPOSITION 4. *Let $f(z)$ be a bounded holomorphic function on U^n such that $|f(z)| < 1$ in U^n and $f(0)$ is real. Then if $f^*(w)$ is real or $|f^*(w)| = 1$* a.e. *on T^n, we have $m_n\{w;\ f^*(w) \in E\} > 0$ for every measurable set $E \subset T$ with $L(E) > 0$ or f is a constant.*

These results above can be formulated also for other domains in the complex plane or in the n-dimensional complex vector space $\boldsymbol{C}^n$, for example the unit ball in $\boldsymbol{C}^n$ etc.

8. Localization in the classical case. Unfortunately we have no strong result as for inner functions in Seidel [6]. We shall state, however, the following weak proposition.

PROPOSITION 5. *Let $f(z)$ be a bounded holomorphic function, bounded by 1 in U. Further suppose $f(e^{i\theta})$ is real or* $|f(e^{i\theta})|=1$ a.e. *on an open arc $A\subset T$ and $L\{e^{i\theta}\in A;\ u<|\mathrm{Arg}\, f(e^{i\theta})|<v\}=0$ for some $u, v>0$. Then $(1+f(z))^2/(1-f(z))^2$ can be continued meromorphically across the arc A and has poles of order at most 2 only on A.*

This is clearly equivalent to the following.

PROPOSITION 5′. *Let $f(z)$ be holomorphic in U so that $|\arg f(z)|<\pi$ in U. Further suppose $f(e^{i\theta})$ is real* a.e. *on an open arc*

$$A=(e^{ia}, e^{ib})\subset T\ (a<b)$$

and $L\{e^{i\theta}\in A;\ u<f(e^{i\theta})<v\}=0$ for some $u<v<0$. Then $f(z)$ can be continued meromorphically across A and has poles of order at most 2 only on A.

PROOF. Consider the function $g(z)=(f(z)-(u+v)/2)^{-1}$. Then we see that $|\arg g(z)|<\pi$ in U and $g(e^{i\theta})$ is real a.e. on A and $|g(e^{i\theta})|<2/(v-u)<\infty$ on A. Let $h(z)=g^{1/3}(z)$, where we take the branch of $z^{1/3}$ as $1^{1/3}=1$. Then we see easily that $h\in H^1(U)$ and $h(e^{i\theta})$ is bounded on A. This shows that $h(z)$ is bounded in any set $D=\{z\in U;\ c<\arg z<d\}$ $(a<c<d<b)$. Hence $g(z)$ is also bounded in D and real a.e. on (e^{ic}, e^{id}). Therefore $g(z)$ can be continued analytically across (e^{ic}, e^{id}) and so across A. Hence $f(z)$ can be continued meromorphically across A so that $f(z)=\overline{f(1/\bar{z})}$. So $f(z)$ has poles only on A. If $f(z)$ has a pole of order more than 2, we see easily that $|\arg f(z)|>\pi$ for some z sufficiently near that pole. q.e.d.

REMARK. In Proposition 5 $f(z)$ may have branch points on A.

Example: Let $g(z)=(1+z^2)/(1-z)^2$, $h(z)=g^{1/2}(z)$ and

$$f(z)=(h(z)-1)/(h(z)+1)\ .$$

Then we have $h(e^{i\theta})>0$ on $(e^{i\pi/2}, e^{i\pi})$ and $h(e^{i\theta})$ is pure-imaginary with $0<-ih(e^{i\theta})<1$ on $(e^{i\pi/3}, e^{i\pi/2})$. Hence $f(z)$ satisfies the assumption in Proposition 5. But $f(z)$ has a branch point at $z=i$, i.e. $f(z)$ can not be continued meromorphically across the arc $(e^{i\pi/3}, e^{i\pi})$.

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