Tôhoku Math. Journ.
25 (1973), 237-262.

ON MAXIMAL FAMILIES OF COMPACT COMPLEX SUBMANIFOLDS OF COMPLEX FIBER SPACES

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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(Received December 15, 1972)

Introduction. The notion of maximal families of compact complex submanifolds of complex manifolds was introduced by Kodaira [3]. In [4], we have proved the existence of maximal families. In this paper, we generalize the notion of maximal families and prove the following theorem. (For the definitions of terminologies, see §1.)

THEOREM. *Let* (X, π, S) *be a family of complex manifolds. Let* o *be a point of* S *and let* V *be a compact complex submanifolds of* $\pi^{-1}(o)$. *Then there exists a maximal family* (Y, μ, T, f) *of compact complex submanifolds of* (X, π, S) *with a point* $t_o \in T$ *such that* $f(t_o) = o$ *and* $\mu^{-1}(t_o) = V$.

The method of the proof is similar to that of [4].

As an application, we give a proof of Kodaira's theorem (Theorem 1, [2]) on the stability of compact complex submanifolds of complex manifolds.

1. Definitions.

By an analytic space, we mean a reduced, Hausdorff, complex analytic space. By *a complex fiber space,* we mean a triple (X, π, S) of analytic spaces X and S, and a surjective holomorphic map $\pi: X \to S$.

DEFINITION 1.1. A complex fiber space (X, π, S) is called *a family of complex manifolds* if and only if there are an open covering $\{X_\alpha\}$ of X, open sets Ω_α of C^n, open sets S_α of S and holomorphic isomorphisms

$$\eta_\alpha: X_\alpha \to \Omega_\alpha \times S_\alpha$$

such that the diagram

$$\begin{tikzcd} X_\alpha \arrow[rr, "\eta_\alpha"] \arrow[dr, "\pi"'] & & \Omega_\alpha \times S_\alpha \arrow[dl, "\mathrm{proj}"] \\ & S_\alpha & \end{tikzcd}$$

is commutative for each α. S is called *the parameter space of the family*

(X, π, S). If π is a proper map, we say that (X, π, S) is *a family of compact complex manifolds*.

Let (X, π, S) be a family of complex manifolds. Let T be an analytic space and let $f: T \to S$ be a holomorphic map. We put

$$f^*X = \{(x, t) \in X \times T \mid \pi(x) = f(t)\} .$$

Let $\mu: f^*X \to T$ be the restriction of the projection map $X \times T \to T$. Then it is easy to see that (f^*X, μ, T) is a family of complex manifolds. This family is called *the induced family of* (X, π, S) *over* f.

DEFINITION 1.2. Let (X, π, S) be a family of complex manifolds. A quadruplet (Y, μ, T, f) is called *a family of compact complex submanifolds of fibers of the family* (X, π, S) if and only if

1) f is a holomorphic map of T into S,
2) Y is a subvariety of f^*X,
3) μ is the restriction of the map

$$\mu: f^*X \to T ,$$

where (f^*X, μ, T) is the induced family of (X, π, S) over f, and

4) (Y, μ, T) is a family of compact complex manifolds.

T is called *the parameter space of the family* (Y, μ, T, f).

REMARK. Each fiber $\mu^{-1}(t), t \in T$, of (Y, μ, T, f) is of the form $V \times t$ where V is a compact complex submanifold of $\pi^{-1}(f(t))$. We identify $V \times t$ with V.

DEFINITION 1.3. A family (Y, μ, T, f) of compact complex submanifolds of fibers of a family (X, π, S) is said to be *maximal at a point* $t \in T$ if and only if, for any family (Z, λ, R, g) of compact complex submanifolds of fibers of (X, π, S) with a point $r \in R$ such that $f(t) = g(r)$ and $\mu^{-1}(t) = \lambda^{-1}(r)$, there are an open neighborhood U of r in R and a holomorphic map

$$h: U \to T$$

such that

1) $h(r) = t$,
2) $fh = g$, and
3) $\lambda^{-1}(q) = \mu^{-1}(h(q))$ for all $q \in U$.

A maximal family is a family which is maximal at every point of its parameter space.

2. Local expressions of families. Let (X, π, S) be a family of complex manifolds. Let o be a point of S. Let V be a compact complex submanifold of $\pi^{-1}(o)$. Since the problem is local, we may replace S by a small neighborhood of o. Thus we may cover V by a finite number of open sets $\{X_i\}_{i \in I}$ of X having the following property: for each $i \in I$, there is a holomorphic isomorphism

$$\eta_i: X_i \to W_i \times S$$

such that the diagram

$$\begin{tikzcd} X_i \arrow[rr, "\eta_i"] \arrow[dr, "\pi"'] & & W_i \times S \arrow[dl, "\text{proj}"] \\ & S & \end{tikzcd}$$

is commutative, where W_i is an open set of $\boldsymbol{C}^n$. We may assume that there is in W_i a coordinate system

$$(w_i, z_i) = (w_i^1, \cdots, w_i^r, z_i^1, \cdots, z_i^d) , \quad r + d = n ,$$

such that

$$\eta_i(V \cap X_i) = \{(w_i, z_i, o) \in W_i \times o \,|\, w_i = 0\} .$$

We put

$$U_i = \{z_i \in \boldsymbol{C}^d \,|\, (0, z_i) \in W_i\} .$$

Then (U_i, η_i) is a local chart of V. We sometimes identify U_i with $V \cap X_i$. We may assume that

$$W_i = D_i \times U_i$$

where D_i is a polydisc in $\boldsymbol{C}^r$ with the center 0.

Now, let (Y, μ, T, f) be a family of compact complex submanifolds of fibers of (X, π, S). We write V_t instead of $\mu^{-1}(t)$. We assume that there is a point t_0 such that $f(t_0) = o$ and $V_{t_0} = V$. We may replace T by a sufficiently small neighborhood of t_0. We may assume that there is an ambient space Γ of T. Then, by the implicit function theorem, we can show the following proposition. Since the proof is straightforward, we omit it.

PROPOSITION 2.1. *For each $i \in I$, there is a holomorphic map ϕ_i of $U_i \times \Gamma$ into D_i such that, for each fixed $t \in T$,*

$$\eta_i(V_t \cap X_i) = \{(w_i, z_i, f(t)) \in W_i \times S \,|\, w_i = \phi_i(z_i, t)\} .$$

3. Some lemmas. Let (X, π, S) be a family of complex minifolds. Let o be a point of S. Let V be a compact complex submanifold of $\pi^{-1}(o)$. We cover V by a finite number of open subsets $\{\tilde{X}_i\}_{i \in I}$ of X such that, for each $i \in I$, there is a holomorphic isomorphism

$$\eta_i\colon \tilde{X}_i \to \tilde{W}_i \times \tilde{S}$$

such that the diagram

$$\begin{tikzcd} \tilde{X}_i \arrow[r, "\eta_i"] \arrow[dr, "\pi"'] & \tilde{W}_i \times \tilde{S} \arrow[d, "\mathrm{proj}"] \\ & \tilde{S} \end{tikzcd}$$

is commutative, where $\tilde{S}$ is an open neighborhood of o in S and $\tilde{W}_i$ is an open set of C^n. We may assume that there is an ambient space $\tilde{\Omega}$ of $\tilde{S}$. We may assume that $\tilde{\Omega}$ is a polydisc in C^l with the center $o = 0$. Let

$$(s) = (s^1, \cdots, s^l)$$

be the standard coordinate system in C^l.

Now, as in §2, we may assume that there is in $\tilde{W}_i$ a coordinate system

$$(w_i, z_i) = (w_i^1, \cdots, w_i^r, z_i^1, \cdots, z_i^d)\ , \quad r + d = n\ ,$$

such that

$$\eta_i(V \cap \tilde{X}_i) = \{(w_i, z_i, o) \in \tilde{W}_i \times o \,|\, w_i = 0\}\ .$$

We put

$$\tilde{U}_i = \{z_i \in C^d \,|\, (0, z_i) \in \tilde{W}_i\}\ .$$

Then $(\tilde{U}_i, \eta_i)$ is a local chart of V. We sometimes identify $\tilde{U}_i$ with $V \cap \tilde{X}_i$. We may assume that

$$\tilde{W}_i = \tilde{D}_i \times \tilde{U}_i$$

where $\tilde{D}_i$ is a polydisc in C^r with the center 0.

For each $i \in I$, let U_i be an open set of V such that

1) $\bar{U}_i$ is compact and is contained in $\tilde{U}_i$,

2) $\bigcup_i U_i = V$.

We may assume that $\tilde{U}_i$ and U_i are connected and Stein for all $i \in I$. For each $i \in I$, let D_i be a polydisc in C^r with the center 0 such that $\bar{D}_i$ contained in $\tilde{D}_i$. Let Ω be a polydisc in C^l with the center $o = 0$ such the $\bar{\Omega}$ is contained in $\tilde{\Omega}$. We put

$$W_i = D_i \times U_i\ ,$$
$$S' = \tilde{S} \cap \Omega\ ,$$
$$X_i = \eta_i^{-1}(W_i \times S')\ .$$

We write S instead of S' to simplify the notation. It is clear that

$$U_i = V \cap X_i\ .$$

Now, we consider the map

$$\eta_{ik} = \eta_i \eta_k^{-1}: \eta_k(\tilde{X}_i \cap \tilde{X}_k) \to \eta_i(\tilde{X}_i \cap \tilde{X}_k) .$$

We want to extend the map η_{ik} to an ambient space of $\eta_k(X_i \cap X_k)$. This is done as follows.

Let P be point of $\bar{U}_i \cap \bar{U}_k$. Then it is clear that there is an open neighborhood $W_P \times S_P$ of $\eta_k(P)$ in $\eta_k(\tilde{X}_i \cap \tilde{X}_k)$ such that

1) $S_P = \Omega_P \cap S$ where Ω_P is a polydisc in C^l contained in Ω with the center $o = 0$, and

2) $W_P = D_P \times U_P$ where D_P is a polydisc in C^r with the center 0 contained in D_k and U_P is an open neighborhood of P in V contained in $\tilde{U}_i \cap \tilde{U}_k$.

We cover $\eta_k(\bar{U}_i \cap \bar{U}_k)$ by open sets $\{W_P \times S_P\}_P$ in $\eta_k(\tilde{X}_i \cap \tilde{X}_k)$ satisfying the above conditions 1) and 2). We choose a finite subcovering $\{W_\lambda \times S_\lambda\}_{\lambda=1,\dots,q}$ of $\{W_P \times S_P\}$, where $S_\lambda = \Omega_\lambda \cap S$ and $W_\lambda = D_\lambda \times U_\lambda$. Then $\{U_\lambda\}_{\lambda=1,\dots,q}$ covers $\eta_k(\bar{U}_i \cap \bar{U}_k)$. The following lemma will be proved in §7.

LEMMA 3.1. *There is a Stein open set U in $\tilde{U}_k$ such that*

$$\bar{U}_i \cap \bar{U}_k \subset U \subset \bigcup_\lambda U_\lambda .$$

Let Ω_o be a polydisc in C^l with the center $o = 0$ contained in $\bigcap_\lambda \Omega_\lambda$. We put $S_o = \Omega_o \cap S$. Let D_o be a polydisc in C^r with the center 0 contained in $\bigcap_\lambda D_\lambda$. We put $W_o = D_o \times U$. Then W_o is Stein. It is clear that

$$\eta_k(\bar{U}_i \cap \bar{U}_k) \subset W_o \times o \subset \tilde{W}_k \times o .$$

It is also clear that

$$W_o \times S_o \subset \eta_k(\tilde{X}_i \cap \tilde{X}_k) .$$

The following lemma will be proved in §7.

LEMMA 3.2. *Taking Ω_o and D_o sufficiently small, we have*

$$\eta_k(X_i \cap X_k) \cap (D_o \times U_k \times S_o) \subset W_o \times S_o .$$

We take Ω_o and D_o sufficiently small so that Lemma 3.2 is satisfied. Since $W_o \times S_o \subset \eta_k(\tilde{X}_i \cap \tilde{X}_k)$, the map $\eta_{ik} = \eta_i \eta_k^{-1}$ is defined on $W_o \times S_o$. Since $W_o \times S_o$ is a closed subvariety of the Stein manifold $W_o \times \Omega_o$,

$$\eta_{ik}: W_o \times S_o \to \tilde{W}_i \times S_o$$

is extended to a holomorphic map

$$\eta_{ik}: W_o \times \Omega_o \to \tilde{W}_i \times \Omega_o .$$

The extended map η_{ik} is written as follows:

$$\eta_{ik}(w_k, z_k, s) = (f_{ik}(w_k, z_k, s), g_{ik}(w_k, z_k, s), s) ,$$

where

$$f_{ik}\colon W_o \times \Omega_o \to \tilde{D}_i$$

and

$$g_{ik}\colon W_o \times \Omega_o \to \tilde{U}_i$$

are holomorphic maps.

Henceforth, we assume that, for each $i \in I$,

$$\begin{aligned} U_i &= \{z_i \in \tilde{U}_i \mid |z_i| < 1\} , \\ D_i &= \{w_i \in \tilde{D}_i \mid |w_i| < 1\} , \\ W_i &= \{(w_i, z_i) \in \tilde{W}_i \mid |w_i| < 1, |z_i| < 1\} , \end{aligned}$$

and

$$\Omega = \{s \in \tilde{\Omega} \mid |s| < 1\} ,$$

where $|z_i| = \max_\alpha |z_i^\alpha|$, $z_i = (z_i^1, \cdots, z_i^d)$, and so on. We may assume that there is a positive number ε_o, $0 < \varepsilon_o < 1$, such that

$$\Omega_o = \{s \in \Omega \mid |s| < \varepsilon_o\}$$

and

$$D_o = \{w_k \in D_k \mid |w_k| < \varepsilon_o\} .$$

Let e, $0 < e < 1$, be a small positive number such that the open sets U_i^e, $i \in I$, of V defined by

$$U_i^e = \{z_i \in U_i \mid |z_i| < 1 - e\}$$

again cover V. We put

$$\begin{aligned} W_i^e &= \{(w_i, z_i) \in W_i \mid |w_i| < 1, |z_i| < 1 - e\} \\ &= D_i \times U_i^e , \end{aligned}$$

and

$$X_i^e = \eta_i^{-1}(W_i^e \times S) .$$

For a positive number ε with $0 < \varepsilon < \varepsilon_o$, we put

$$\begin{aligned} \Omega_\varepsilon &= \{s \in \Omega \mid |s| < \varepsilon\} , \\ S_\varepsilon &= S \cap \Omega_\varepsilon , \end{aligned}$$

and

$$D_\varepsilon = \{w_k \in D_k \mid |w_k| < \varepsilon\} .$$

The following Lemmas 3.3, 3.4 and 3.5 will be proved in § 7.

LEMMA 3.3. *There is a small positive number* ε *with* $0 < \varepsilon < \varepsilon_o$ *such that if* $w_k \in D_\varepsilon$ *and* $s \in \Omega_\varepsilon$, *then, for all* $z_k \in U_i^e \cap U_k$, $g_{ik}(w_k, z_k, s)$ *is defined and is a point of* U_i.

LEMMA 3.4. *Given any* δ, $0 < \delta \leqq 1$, *there is a small positive number* ε *with* $0 < \varepsilon < \varepsilon_o$ *such that if* $w_k \in D_\varepsilon$ *and* $s \in \Omega_\varepsilon$, *then, for all* $z_k \in U_i^e \cap U_k$, $f_{ik}(w_k, z_k, s)$ *is defined and*

$$|f_{ik}(w_k, z_k, s)| < \delta .$$

LEMMA 3.5. *There is a small positive number* ε *with* $0 < \varepsilon < \varepsilon_o$ *such that if* $w_k \in D_\varepsilon$ *and* $s \in S_\varepsilon$, *then*

$$\eta_k^{-1}(w_k, z_k, s) \in X_i \cap X_k \quad \textit{for all} \quad z_k \in U_i^e \cap U_k .$$

Let e', $0 < e < e' < 1$, be a small positive number such that the open sets $U_i^{e'}$, $i \in I$, of V defined by

$$U_i^{e'} = \{z_i \in U_i \mid |z_i| < 1 - e'\}$$

again cover V. We put

$$\begin{aligned} W_i^{e'} &= \{(w_i, z_i) \in W_i \mid |w_i| < 1, |z_i| < 1 - e'\} \\ &= D_i \times U_i^{e'} \end{aligned}$$

and

$$X_i^{e'} = \eta_i^{-1}(W_i^{e'} \times S) .$$

The following lemma will be proved in §7.

LEMMA 3.6. *There is a small positive number* ε *with* $0 < \varepsilon < \varepsilon_0$ *such that if* $w_k \in D_\varepsilon$, $s \in S_\varepsilon$ *and* $\eta_k^{-1}(w_k, z_k, s) \in X_i^{e'} \cap X_k$, *then*

$$z_k \in U_i^e \cap U_k .$$

The set U in Lemma 3.1 depends on the indices i and k. On the other hand, we may assume that ε_o is independent of indices, for the set of indices is a finite set. Thus Ω_o, S_o and D_o are independent of indices. We write

$$U = U_{(ik)}$$

and

$$W_o = W_{o(ik)} .$$

Then $\eta_{jk}^{-1}(W_{o(ij)} \times \Omega_o)$ is an open set of $W_{o(jk)} \times \Omega_o$ and contains $\bar{U}_i \cap \bar{U}_j \cap \bar{U}_k$. The following lemma will be proved in §7.

LEMMA 3.7. *There is a small positive number* ε *with* $0 < \varepsilon < \varepsilon_o$ *such that if* $w_k \in D_\varepsilon$ *and* $s \in \Omega_\varepsilon$, *then*

1) $(w_k, z_k, s) \in \eta_{jk}^{-1}(W_{o(ij)} \times \Omega_o)$ *for all* $z_k \in U_i \cap U_j \cap U_k$,

2) $g_{ik}(w_k, z_k, s) \in U_i^{e/2} \cap U_j^{e/2}$ *for all* $z_k \in U_i^e \cap U_j^e \cap U_k$ *where* $U_i^{e/2} = \{z_i \in U_i \mid |z_i| < 1 - e/2\}$ *and* $U_j^{e/2} = \{z_j \in U_j \mid |z_j| < 1 - e/2\}$.

4. **Banach spaces $C^p(|\ |)$.** We use the same notations as in §3. Henceforth we assume that $\tilde{S} \subset \tilde{\Omega}$ is a neat imbedding of $\tilde{S}$ at o, [1]. Thus l is equal to the dimension of the Zariski tangent space T_oS at o. We assume that $\tilde{S}$ is defined in $\tilde{\Omega}$ as the common zeros of holomorphic functions

$$e_1(s), \cdots, e_m(s) .$$

It is easy to see that

$$e_\alpha(o) = 0 , \quad \alpha = 1, \cdots, m , \tag{1}$$

$$(\partial e_\alpha/\partial s^\beta)(o) = 0 , \quad \alpha = 1, \cdots, m, \beta = 1, \cdots, l . \tag{2}$$

In §3, we extended the map

$$\eta_{ik} = \eta_i \eta_k^{-1} \colon W_o \times S_o \to \tilde{W}_i \times S_o$$

to the map

$$\eta_{ik} \colon W_o \times \Omega_o \to \tilde{W}_i \times \Omega_o .$$

We wrote the extended map η_{ik} as follows:

$$\eta_{ik}(w_k, z_k, s) = (f_{ik}(w_k, z_k, s), g_{ik}(w_k, z_k, s), s) .$$

LEMMA 4.1. *Let z_k be a point of $U_i \cap U_k$. Then the matrices*

$$(\partial f_{ik}/\partial w_k)_{(0, z_k, o)} \quad and \quad (\partial f_{ik}/\partial s)_{(0, z_k, o)}$$

are independent how to extend the map η_{ik}.

PROOF. The first assertion is ovbious. We prove the second assertion. In a neighborhood of $(0, z_k, o)$ in $W_o \times \Omega_o$, another extension of η_{ik} is written as

$$w_i = f'_{ik}(w_k, z_k, s) = f_{ik}(w_k, z_k, s) + \sum_{\alpha=1}^m a_{ik}^\alpha(w_k, z_k, s) e_\alpha(s) ,$$

$$z_i = g'_{ik}(w_k, z_k, s) = g_{ik}(w_k, z_k, s) + \sum_{\alpha=1}^m b_{ik}^\alpha(w_k, z_k, s) e_\alpha(s) ,$$

where a_{ik}^α and b_{ik}^α are vector valued holomorphic functions in the neighborhood. Hence

$$\begin{aligned}(\partial f'_{ik}/\partial s)_{(0, z_k, o)} &= (\partial f_{ik}/\partial s)_{(0, z_k, o)} + \sum_{\alpha=1}^m (\partial a_{ik}^\alpha/\partial s)_{(0, z_k, o)} e_\alpha(o) \\ &\quad + \sum_{\alpha=1}^m a_{ik}^\alpha(0, z_k, o)(\partial e_\alpha/\partial s)_o \\ &= (\partial f_{ik}/\partial s)_{(0, z_k, o)}\end{aligned}$$

by 1) and 2) above. q.e.d.

LEMMA 4.2. *Let* z_k *be a point of* $U_i \cap U_j \cap U_k$. *Then*

$$(\partial f_{ik}/\partial s)_{(0,z_k,o)} = (\partial f_{ij}/\partial w_j)_{(0,z_j,o)}(\partial f_{jk}/\partial s)_{(0,z_k,o)} + (\partial f_{ij}/\partial s)_{(0,z_j,o)} ,$$

where $z_j = g_{jk}(0, z_k, o)$.

PROOF. Let z_k be a point of $U_i \cap U_j \cap U_k$. Then there are a neighborhood Y of $(0, z_k, o)$ in $W_{o(jk)} \times \Omega_o$ and vector valued holomorphic functions

$$d^\alpha(w_k, z_k, s) , \qquad \alpha = 1, \cdots, m$$

on Y such that $\eta_{ij} \circ \eta_{jk}$ is defined on Y and

$$f_{ik}(w_k, z_k, s) = f_{ij}(f_{jk}(w_k, z_k, s), g_{jk}(w_k, z_k, s), s) + \sum_{\alpha=1}^{m} d^\alpha(w_k, z_k, s)e_\alpha(s) . \tag{3}$$

Hence, noting that $f_{ij}(0, z_j, o) = 0$, we have

$$\begin{aligned}(\partial f_{jk}/\partial s)_{(0,z_k,o)} = {} & (\partial f_{ij}/\partial w_j)_{(0,z_j,o)}(\partial f_{jk}/\partial s)_{(0,z_k,o)} + (\partial f_{ij}/\partial s)_{(0,z_j,o)} \\ & + \sum_{\alpha=1}^{m} (\partial d^\alpha/\partial s)_{(0,z_k,o)}e_\alpha(o) + \sum_{\alpha=1}^{m} d^\alpha(0, z_k, o)(\partial e_\alpha/\partial s)_{(o)} .\end{aligned}$$

The third and the fourth terms vanish by (1) and (2) above. q.e.d.

Differentiating (3) above with respect to w_k, we get

LEMMA 4.3. *Let* z_k *be a point of* $U_i \cap U_j \cap U_k$. *Then*

$$(\partial f_{ik}/\partial w_k)_{(0,z_k,o)} = (\partial f_{ij}/\partial w_j)_{(0,z_j,o)}(\partial f_{jk}/\partial w_k)_{(0,z_k,o)} ,$$

where $z_j = g_{jk}(0, z_k, o)$.

We define a matrix valued holomorphic function $F_{ik}(z_k)$ on $U_i \cap U_k$ by

$$F_{ik}(z_k) = (\partial f_{ik}/\partial w_k)_{(0,z_k,o)} .$$

Then, by Lemma 4.3, we have

$$F_{ik}(z_k) = F_{ij}(z_j)F_{jk}(z_k) ,$$

where $z_k \in U_i \cap U_j \cap U_k$ and $z_j = g_{jk}(0, z_k, o)$. The holomorphic vector bundle F on V defined by the transition matrices $\{F_{ik}\}$ is called *the normal bundle of* V *in* $\pi^{-1}(o)$.

We define a matrix valued holomorphic function $N_{ik}(z_k)$ on $U_i \cap U_k$ by

$$N_{ik}(z_k) = \begin{pmatrix} F_{ik}(z_k) & (\partial f_{ik}/\partial s)_{(0,z_k,o)} \\ 0 & 1 \end{pmatrix} ,$$

where 1 is the $(l \times l)$-identity matrix. Then by Lemmas 4.2 and 4.3, we have

$$N_{ik}(z_k) = N_{ij}(z_j)N_{jk}(z_k) ,$$

where $z_k \in U_i \cap U_j \cap U_k$ and $z_j = g_{jk}(0, z_k, o)$.

DEFINITION 4.1. By *the normal bundle of V in X*, we mean the holomorphic vector bundle N on V defined by the transition matrices $\{N_{ik}\}$.

From the definitions of F and N, we have

LEMMA 4.4. *There is the following exact sequence:*

$$0 \longrightarrow F \longrightarrow N \longrightarrow V \times T_oS \longrightarrow 0 ,$$

where $V \times T_oS$ is the trivial bundle on V with the fiber T_oS.

We do not use the bundle N in the sequel.

Now, we refer some results in §2 of [4]. We define additive groups C^p, $p = 0, 1, 2, \cdots$, as follows.

An element $\xi = \{\xi_{i_0\ldots i_p}\} \in C^p$ is a function which associates to each $(p+1)$-ple $(i_0, \cdots, i_p)$ of indices of I a holomorphic section $\xi_{i_0\ldots i_p}$ of the normal bundle F on $U^e_{i_0} \cap \cdots \cap U^e_{i_{p-1}} \cap U_{i_p}$. In particular, an element $\xi = \{\xi_i\} \in C^0$ is a function which associates to each index $i \in I$ a holomorphic section ξ_i of F on U_i.

We define the coboundary map

$$\delta \colon C^p \longrightarrow C^{p+1}$$

by

$$(\delta\xi)_{i_0\ldots i_{p+1}}(z) = \sum_\nu (-1)^\nu \xi_{i_0\ldots i_{\nu-1} i_{\nu+1}\ldots i_{p+1}}(z) \quad \text{for } z \in U^e_{i_0} \cap \cdots \cap U^e_{i_p} \cap U_{i_{p+1}} .$$

Then it is easy to see that

$$\delta^2 = 0 .$$

We introduce a norm $|\ |$ in C^p. For each $\xi = \{\xi_{i_0\ldots i_p}\} \in C^p$, we define $|\xi|$ by

$$\begin{aligned} |\xi| = \sup\{|\xi^\lambda_{i_0\ldots i_p}(z)| \colon & \lambda = 1, \cdots, r , \\ & z \in U^e_{i_0} \cap \cdots \cup U^e_{i_{p-1}} \cap U_{i_p}, (i_0, \cdots, i_p) \in I^{p+1}\} , \end{aligned}$$

where $\xi^\lambda_{i_0\ldots i_p}$ is the representation of the component $\xi_{i_0\ldots i_p}$ of ξ with respect to the coordinate (w_{i_0}, z_{i_0}). In particular, we define $|\xi|$ for $\xi \in C^0$ by

$$|\xi| = \sup\{|\xi^\lambda_i| \colon \lambda = 1, \cdots, r, i \in I, z \in U_i\}$$

where ξ^λ_i is the representation of ξ_i with respect to the coordinate (w_i, z_i). Note that we denoted $|\ |_e$ in [4] instead of $|\ |$.

We put

$$C^p(|\ |) = \{\xi \in C^p \,|\, |\xi| < +\infty\} .$$

It is easy to see that $C^p(|\ |)$ is a Banach space and the coboundary map δ maps $C^p(|\ |)$ continuously into $C^{p+1}(|\ |)$.

We put, for $p = 0, 1, 2, \cdots$,

$$Z^p(|\ |) = \{\xi \in C^p(|\ |) \mid \delta\xi = 0\} ,$$
$$B^p(|\ |) = \delta C^{p-1} \cap C^p(|\ |)$$

and

$$H^p(|\ |) = Z^p(|\ |)/B^p(|\ |) .$$

It is clear that $H^0(|\ |)$ is canonically isomorphic to the 0-th cohomology group $H^0(V, F)$ of F.

By Lemmas 2.3 of [4] and 2.4 of [4], there are continuous linear maps

$$E\colon B^2(|\ |) \to C^1(|\ |)$$

and

$$E_0\colon B^1(|\ |) \to C^0(|\ |)$$

such that

$$\delta E = \text{the identity map on } B^2(|\ |) ,$$
$$\delta E_0 = \text{the identity map on } B^1(|\ |) .$$

We put

$$\Lambda = 1 - E\delta .$$

Then Λ is a projection map of $C^1(|\ |)$ onto $Z^1(|\ |)$.

By Lemma 2.5 of [4], $B^1(|\ |) = \delta C^0(|\ |)$ and is closed in $Z^1(|\ |)$. Again, by Lemma 2.5 of [4], $H^1(|\ |)$ is canonically isomorphic to $H^1(V, F)$, the first cohomology group of F. Thus there is a subspace $H^1(|\ |)^*$ of $Z^1(|\ |)$ isomorphic to $H^1(V, F)$ such that $Z^1(|\ |)$ splits into the direct sum of $B^1(|\ |)$ and $H^1(|\ |)$:

$$Z^1(|\ |) = B^1(|\ |) \oplus H^1(|\ |) .$$

Let

$$B\colon Z^1(|\ |) \to B^1(|\ |)$$

and

$$H\colon Z^1(|\ |) \to H^1(|\ |)$$

be the projection maps corresponding to the splitting.

By Lemma 4.2, $\{(\partial f_{ik}/\partial s)_{(0, z_k, o)}\}$ is an element of $Z^1(|\ |)$. Thus we have

* We use the same notation for the convenience.

a continuous linear map

$$\sigma\colon T_oS \to Z^1(|\ |)$$

defined by

$$\sigma(a)_{ik}(z_i) = \sum_{\alpha=1}^{l} a^\alpha(\partial f_{ik}/\partial s^\alpha)_{(0,z_k,o)}\ ,$$

for $z_i \in U_i^e \cap U_k$, where

$$a = \sum_{\alpha=1}^{l} a^\alpha(\partial/\partial s^\alpha)_o$$

and

$$z_k = g_{ki}(0, z_i, o)\ .$$

5. **Proof of the theorem.** We use the same notations as in §3 and §4. We consider the product

$$C^0(|\ |) \times T_oS$$

of the Banach space $C^0(|\ |)$ introduced in §4 and the Zariski tangent space T_oS. We introduce a norm $|\ |$ in $C^0(|\ |) \times T_oS$ as follows:

$$|(\phi, s)| = \max\{|\phi|, |s|\} \quad \text{for} \quad (\phi, s) \in C^0(|\ |) \times T_oS\ ,$$

where

$$|s| = \max_\alpha |a_\alpha| \qquad \text{if } s = \sum_{\alpha=1}^{l} a^\alpha\left(\frac{\partial}{\partial s^\alpha}\right)_o .$$

Then $C^0(|\ |) \times T_oS$ is a Banach space.

We identify $\tilde{\Omega}$ with an open set of T_oS by

$$(a^1, \cdots, a^l) \in \tilde{\Omega} \to \sum_{\alpha=1}^{l} a^\alpha\left(\frac{\partial}{\partial s^\alpha}\right)_o \in T_oS\ .$$

Let V' be a complex submanifold of $\pi^{-1}(s)$, $s \in S$, such that

1) $V' \subset \bigcup X_i$ and

2) for each $i \in I$, there is a holomorphic map ϕ_i on U_i into D_i such that

$$\eta_i(X_i \cap V') = \{(w_i, z_i, s) \in W_i \times s \mid w_i = \phi_i(z_i)\}\ .$$

For such V', we associate an element

$$(\phi, s) \in C^0(|\ |) \times T_oS\ ,$$

where $\phi = \{\phi_i\} \in C^0(|\ |)$ and $s \in S \subset \Omega \subset T_oS$.

The proof of the following lemma will be given in §7.

LEMMA 5.1. *There is a small positive number* ε, $0 < \varepsilon < 1$, *such that* *if V' corresponds to*

$$(\phi, s) \in C^0(|\ |) \times T_o S$$

with $|(\phi, s)| < \varepsilon$, then

$$V' \subset \bigcup_i X_i^\varepsilon .$$

Using Lemma 5.1, we prove

LEMMA 5.2. *There is a small positive number ε, $0 < \varepsilon < 1$, such that if a complex submanifold V' of $\pi^{-1}(s)$, $s \in S$, corresponds to*

$$(\phi, s) \in C^0(|\ |) \times T_o S$$

with $|(\phi, s)| < \varepsilon$, then V' is compact.

PROOF. Let ε satisfy Lemma 5.1. Let $\{P^\nu\}_{\nu=1,2,\dots}$ be a sequence of points in V'. We want to choose a subsequence converging to a point of V'. By Lemma 5.1, we may assume that

$$\{P^\nu\}_{\nu=1,2,\dots} \subset X_i^\varepsilon$$

for a fixed $i \in I$. We put

$$\eta_i(P^\nu) = (w_i^\nu, z_i^\nu, s) , \qquad \nu = 1, 2, \cdots .$$

Then

$$w_i^\nu = \phi_i(z_i^\nu) , \qquad \nu = 1, 2, \cdots .$$

For each P^ν, we associate a point Q^ν in V defined by

$$\eta_i(Q^\nu) = (0, z_i^\nu, o) .$$

Then $Q^\nu \in U_i^\varepsilon$, $\nu = 1, 2, \cdots$. Thus we may assume that $\{Q^\nu\}_{\nu=1,2,\dots}$ converges to a point $Q \in U_i$. We put

$$\eta_i(Q) = (0, z_i, o) .$$

Now we put

$$P = \eta_i^{-1}(\phi_i(z_i), z_i, s) \in X_i .$$

Then $P \in V'$ and

$$\phi_i(z_i) = \phi_i\Big(\lim_\nu z_i^\nu\Big) = \lim_\nu \phi_i(z_i^\nu) = \lim_\nu w_i^\nu .$$

Hence $\{P^\nu\}_{\nu=1,2,\dots}$ converges to P. q.e.d.

Now, let V' be a compact complex submanifold of $\pi^{-1}(s)$, $s \in S$, such that 1) and 2) above hold. Then the corresponding

$$(\phi, s) \in C^0(|\ |) \times T_o S$$

must satisfy the following compatibility conditions:

3) $s \in S$ and

4) $f_{ik}(\phi_k(z_k), z_k, s) = \phi_i(g_{ik}(\phi_k(z_k), z_k, s))$ for $(\phi_k(z_k), z_k, s) \in \eta_k(X_i \cap X_k) \cap \pi^{-1}(s)$.

Conversely if an element $(\phi, s) \in C^0(|\ |) \times T_oS$ with $|(\phi, s)| < \varepsilon$, ($\varepsilon$ satisfying Lemma 5.2), satisfies 3) and 4), then it is clear that a compact complex submanifold V' of $\pi^{-1}(s)$ is defined by the equations:

$$w_i = \phi_i(z_i) \qquad \text{for } z_i \in U_i,\ i \in I\,,$$

and satisfies 1) and 2).

Henceforth, let ε, $0 < \varepsilon < 1$, be a small positive number which satisfies Lemmas 3.3, 3.4 (for $\delta = 1$), 3.5, 3.6, 3.7 and 5.2. Let B_ε be the open ε-ball of $C^0(|\ |)$ with the center 0. Let Ω_ε be the open ε-ball of T_oS with the center o. We put

$$S_\varepsilon = S \cap \Omega_\varepsilon\,.$$

We assume that S is defined in Ω as the common zeros of holomorphic functions

$$e_1(s), \cdots, e_m(s)\,.$$

We define a holomorphic map

$$e\colon \Omega \longrightarrow \boldsymbol{C}^m$$

by

$$e(s) = (e_1(s), \cdots, e_m(s))\,.$$

Then

$$S_\varepsilon = \{s \in \Omega_\varepsilon \,|\, e(s) = 0\}\,.$$

Now, we define a map

$$K\colon B_\varepsilon \times \Omega_\varepsilon \longrightarrow C^1(|\ |)$$

by

$$K(\phi, s)_{ik}(z_i) = f_{ik}(\phi_k(z_k), z_k, s) - \phi_i(g_{ik}(\phi_k(z_k), z_k, s)) \quad \text{for} \quad z_i \in U_i^e \cap U_k,$$

where $z_k = g_{ki}(0, z_i, o)$. By Lemmas 3.3 and 3.4, $f_{ik}(\phi_k(z_k), z_k, s)$ and $g_{ik}(\phi_k(z_k), z_k, s)$ are defined, and

$$|f_{ik}(\phi_k(z_k), z_k, s)| < 1$$

and

$$|g_{ik}(\phi_k(z_k), z_k, s)| < 1\,.$$

Hence $\phi_i(g_{ik}(\phi_k(z_k), z_k, s))$ is defined and

$$|\phi_i(g_{ik}(\phi_k(z_k), z_k, s))| < \varepsilon .$$

Hence

$$|K(\phi, s)| < 1 + \varepsilon .$$

Thus

$$K: B_\varepsilon \times \Omega_\varepsilon \to C^1(|\ |)$$

is well defined.

Let

$$\beta: C^0(|\ |) \times T_oS \to T_oS$$

be the projection map.

We put

$$M_1 = \{(\phi, s) \in B_\varepsilon \times \Omega_\varepsilon \,|\, K(\phi, s) = 0\}$$

and

$$\begin{aligned} M &= \{(\phi, s) \in B_\varepsilon \times \Omega_\varepsilon \,|\, K(\phi, s) = 0,\ e\beta(\phi, s) = e(s) = 0\} \\ &= \{(\phi, s) \in B_\varepsilon \times S_\varepsilon \,|\, K(\phi, s) = 0\} . \end{aligned}$$

Now we take an element $(\phi, s) \in B_\varepsilon \times \Omega_\varepsilon$ which satisfies 3) and 4) above. Let z_i be any fixed point of $U_i^e \cap U_k$. By Lemma 3.5,

$$(\phi_k(z_k), z_k, s) \in \eta_k(X_i \cap X_k) ,$$

where $z_k = g_{ki}(0, z_i, o)$. Hence, by 4),

$$K(\phi, s)_{ik}(z_i) = 0 .$$

Since $z_i \in U_i^e \cap U_k$ is arbitrary,

$$K(\phi, s) = 0 .$$

Hence $(\phi, s) \in M$.

Conversely, let $(\phi, s) \in M$. Then $s \in S_\varepsilon$. Thus 3) is satisfied. Let z_k be a point of U_k. We assume that

$$(\phi_k(z_k), z_k, s) \in \eta_k(X_i^{e'} \cap X_k^{e'}) .$$

Then, by Lemma 3.6,

$$z_k \in U_i^e \cap U_k .$$

Since $K(\phi, s) = 0$, we have

$$f_{ik}(\phi_k(z_k), z_k, s) = \phi_i(g_{ik}(\phi_k(z_k), z_k, s)) \qquad \text{for } (\phi_k(z_k), z_k, s) \in \eta_k(X_i^{e'} \cap X_k^{e'}) .$$

Hence the equations

$$w_i = \phi_i(z_i) , \quad z_i \in U_i^{e'}, i \in I ,$$

define a complex submanifold V' of $\pi^{-1}(s)$. Thus, by the principle of analytic continuation, the equations

$$w_i = \phi_i(z_i) \, , \quad z_i \in U_i, \, i \in I \, ,$$

define V'. Hence (ϕ, s) satisfies 4). V' is compact by Lemma 5.2. Thus the problem is reduced to analyze the set M.

PROPOSITION 5.1. *Let ε be sufficiently small. Then*

$$K: B_\varepsilon \times \Omega_\varepsilon \to C^1(|\ |)$$

is an analytic map, and

$$K'(0, 0) = \delta + \sigma: C^0(|\ |) \times T_o S \to C^1(|\ |) \, ,$$

where δ and σ are continuous linear maps defined in §4, and $\delta + \sigma$ is defined by

$$(\delta + \sigma)(\phi, s) = \delta\phi + \sigma s \, .$$

PROOF. The proof of the first half is similar to that of Lemma 3.4, [4]. Only what we have to note is that we use Cauchy's estimate for holomorphic functions of variables (w, s). The rest goes pararell to the proof of Lemma 3.4, [4]. We prove the second half. Let $(\phi, s) \in B_\varepsilon \times \Omega_\varepsilon$. Since $K(0, o) = 0$,

$$\begin{aligned} K(\phi, s) - K(0, o) &= K(\phi, s) \, . \\ K(\phi, s)_{ik}(z_i) &= f_{ik}(\phi_k(z_k), z_k, s) - \phi_i(g_{ik}(\phi_k(z_k), z_k, s)) \\ &= [f_{ik}(\phi_k(z_k), z_k, s) - f_{ik}(0, z_k, o)] - \phi_i(z_i) \\ &\quad - [\phi_i(g_{ik}(\phi_k(z_k), z_k, s)) - \phi_i(g_{ik}(0, z_k, o))] \, , \end{aligned}$$

where $z_k = g_{ki}(0, z_i, o)$. Hence

$$\begin{aligned} K(\phi, s)_{ik}(z_i) = {} & (\partial f_{ik}/\partial w_k)_{(0, z_k, o)}\phi_k(z_k) + (\partial f_{ik}/\partial s)_{(0, z_k, o)} s \\ & + o(\phi, s) - \phi_i(z_i) - (\partial \phi_i/\partial z_i)_{z_i}(\partial g_{ik}/\partial w_k)_{(0, z_k, o)}\phi_k(z_k) \\ & - (\partial \phi_i/\partial z_i)_{z_i}(\partial g_{ik}/\partial s)_{(0, z_k, o)} s + o(\phi, s) \, , \end{aligned}$$

where $o(\phi, s)$ is some function of (ϕ, s) (and of z_k) such that

$$|o(\phi, s)|/|(\phi, s)| \to 0 \quad \text{as} \quad |(\phi, s)| \to 0 \, .$$

There are constants C_1 and C_2 such that

$$|(\partial g_{ik}/\partial w_k)_{(0, z_k, o)}| \leqq C_1$$

and

$$|(\partial g_{ik}/\partial s)_{(0, z_k, o)}| \leqq C_2 \qquad \text{for } z_k \in U_i^\varepsilon \cap U_k \, .$$

On the other hand, there is a constant C_3 such that

$$|(\partial\phi_i/\partial z_i)_{z_i}| \leqq C_3|\phi| \quad \text{for} \quad z_i \in U_i^e .$$

Hence

$$-(\partial\phi_i/\partial z_i)_{z_i}(\partial g_{ik}/\partial w_k)_{(0,z_k,o)}\phi_k(z_k) = o(\phi, s) ,$$
$$-(\partial\phi_i/\partial z_i)_{z_i}(\partial g_{ik}/\partial s)_{(0,z_k,o)}s = o(\phi, s) .$$

Hence

$$\begin{aligned} K(\phi, s)_{ik}(z_i) &= F_{ik}(z_k)\phi_k(z_k) - \phi_i(z_i) + (\partial f_{ik}/\partial s)_{(0,z_k,o)}s + o(\phi, s) \\ &= (\delta\phi)_{ik}(z_i) + \sigma(s)_{ik}(z_i) + o(\phi, s) . \end{aligned}$$

Thus

$$K(\phi, s) = \delta\phi + \sigma s + o(\phi, s) . \qquad \text{q.e.d.}$$

Now, we define a map

$$L: B_\varepsilon \times \Omega_\varepsilon \to C^0(|\ |) \times T_oS$$

by

$$L(\phi, s) = (\phi + E_0 B \Lambda K(\phi, s) - E_0\delta\phi, s) ,$$

where continuous linear maps E_0, B, Λ and δ are defined in § 4. Then L is analytic by Proposition 5.1. We have $L(0, o) = 0$ and

$$\begin{aligned} L'(0, 0) &= \begin{pmatrix} 1 + E_0B\Lambda\delta - E_0\delta & E_0B\Lambda\sigma \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & E_0B\sigma \\ 0 & 1 \end{pmatrix} . \end{aligned}$$

(We note that $B\Lambda\delta = \delta$ and $\Lambda\sigma = \sigma$.) Thus $L'(0, 0)$ is a continuous linear isomorphism. Hence, by the inverse mapping theorem, there are a small positive number ε', an open neighborhood U of $(0, o)$ in $B_\varepsilon \times \Omega_\varepsilon$ and an analytic isomorphism Φ of $B_{\varepsilon'} \times \Omega_{\varepsilon'}$ onto U such that

$$L \mid U = \Phi^{-1} .$$

We put

$$L(M_1 \cap U) = T_1$$

and

$$L(M \cap U) = T .$$

Then $M_1 \cap U = \Phi(T_1)$ and $M \cap U = \Phi(T)$.

Lemma 5.3. $T_1 \subset (H^0(|\ |) \cap B_{\varepsilon'}) \times \Omega_{\varepsilon'}$.

Proof. Let $(\phi, s) \in M_1 \cap U$. Then

$$L(\phi, s) = (\phi + E_0 B\Lambda K(\phi, s) - E_0\delta\phi, s) = (\phi - E_0\delta\phi, s) .$$

But $\delta(\phi - E_0\delta\phi) = \delta\phi - \delta\phi = 0$. q.e.d.

COROLLARY 1. $T_1 = \{(\xi, s) \in (H^0(|\ |) \cap B_{\varepsilon'}) \times \Omega_{\varepsilon'} \mid K\Phi(\xi, s) = 0\}$.

COROLLARY 2. $T = \{(\xi, s) \in (H^0(|\ |) \cap B_{\varepsilon'}) \times S_{\varepsilon'} \mid K\Phi(\xi, s) = 0\}$.

Corollary 1 follows from the definition of M_1 and Lemma 5.3. Corollary 2 follows from Corollary 1.

Now, let $(\xi, s) \in (H^0(|\ |) \cap B_{\varepsilon'}) \times \Omega_{\varepsilon'}$. We put $(\phi, s) = \Phi(\xi, s)$. Then

$$0 = \delta\xi = \delta(\phi + E_0 B\Lambda K(\phi, s) - E_0\delta\phi) = B\Lambda K(\phi, s) = B\Lambda K\Phi(\xi, s) .$$

Hence

$$\begin{aligned} K\Phi(\xi, s) &= H\Lambda K\Phi(\xi, s) + B\Lambda K\Phi(\xi, s) + E\delta K\Phi(\xi, s) \\ &= H\Lambda K\Phi(\xi, s) + E\delta K\Phi(\xi, s) , \end{aligned}$$

where H and E are continuous linear maps defined in §4.

PROPOSITION 5.2. *Let ε' be sufficiently small. Then*

$$T = \{(\xi, s) \in (H^0(|\ |) \cap B_{\varepsilon'}) \times S_{\varepsilon'} \mid H\Lambda K\Phi(\xi, s) = 0\} .$$

COROLLARY. *If $H^1(V, F) = 0$, then*

$$T = (H^0(|\ |) \cap B_{\varepsilon'}) \times S_{\varepsilon'} .$$

PROOF OF PROPOSITION 5.2. The proof is almost similar to that of Lemma 3.6, [4]. Only what we have to note are the following two points. A). By 2) of Lemma 3.7, if $(\phi, s) \in B_\varepsilon \times \Omega_\varepsilon$, then

$$\zeta_j = g_{jk}(\phi_k(z_k), z_k, s) \in U_i^{e/2} \cap U_j^{e/2} \quad \text{for} \quad z_k = g_{ki}(0, z_i, o) \in U_i^e \cap U_j^e \cap U_k .$$

B). We put, for $(\phi, s) \in B_\varepsilon \times \Omega_\varepsilon$,

$$\begin{aligned} R^1(K(\phi, s), \phi, s) &= \{R^1(K(\phi, s), \phi, s)_{ijk}\} \in C^2(|\ |) , \\ R^1(K(\phi, s), \phi, s)_{ijk}(z_i) &= f_{ij}(\phi_j(\zeta_j), \zeta_j, s) - f_{ij}(f_{jk}(\phi_k(z_k), z_k, s), \zeta_j, s) \\ &\quad + F_{ij}(z_j)K(\phi, s)_{jk}(z_j) . \end{aligned}$$

Then

$$R^1(K(\phi, s), \phi, s)_{ijk}(z_i) = f_{ij}(\phi_j(\zeta_j), \zeta_j, s) - f_{ik}(\phi_k(z_k), z_k, s) + F_{ij}(z_j)K(\phi, s)_{jk}(z_j)$$

for $s \in S_\varepsilon$. The rest goes pararell to the proof of Lemma 3.6, [4]. q.e.d.

Now, for each $t = (\xi, s) \in T$, we put

$$\Phi(t) = (\phi(t), f(t)) .$$

Then

$$\phi : T \to C^0(|\ |)$$

and

$$f\colon T \to S$$

are analytic maps. f is actually the projection map

$$t = (\xi, s) \to s\ .$$

If we write

$$\phi(t) = \{\phi_i(z_i, t)\}\ ,$$

then it is easy to see that

$$\phi_i\colon U_i \times T \to \boldsymbol{C}^r$$

is a holomorphic map. For each $t \in T$, we denote V_t the compact complex submanifold of $\pi^{-1}(f(t))$ defined in X_i by the equation

$$w_i = \phi_i(z_i, t) \qquad \text{for } z_i \in U_i\ .$$

Note that

$$V_{(0,o)} = V\ .$$

We put

$$Y = \{(x, t) \in X \times T \mid x \in V_t\}$$

and

$$\mu = \text{the restriction to } Y \text{ of the projection map}$$

$$X \times T \to T\ .$$

LEMMA 5.4. *(Y, μ, T, f) is a family of compact complex submanifolds of fibers of (X, π, S).*

PROOF. Since $V_t \subset \pi^{-1}(f(t))$, we have $Y \subset f^*X$. Next, we note that

$$Y \cap (X_i \times T) = \{(\eta_i^{-1}(w_i, z_i, s), t) \mid w_i = \phi_i(z_i, t),\ s = f(t)\}\ .$$

Hence Y is a subvariety of f^*X. Since the projection

$$(\eta_i^{-1}(\phi_i(z_i, t), z_i, f(t)), t) \to (z_i, t)$$

gives a local isomorphism, (Y, μ, T) is a family of complex manifolds.

q.e.d.

We identify each fiber $\mu^{-1}(t) = V_t \times t$, $t \in T$, with V_t.

LEMMA 5.5. *(Y, μ, T, f) is a maximal family.*

PROOF. Let t_o be a point of T. Let (Z, λ, R, g) be a family of compact complex submanifolds of fibers of (X, π, S) with a point $r_o \in R$ such that $\lambda^{-1}(r_o) = V_{t_0}$. V_{t_0} is covered by $\{X_i\}_{i \in I}$. We introduce a new coordinate system (w_i', z_i, s) in $W_i \times \Omega$ where

$$w_i' = w_i - \phi_i(z_i, t_o) .$$

Then V_{t_o} is given in X_i by the equations

$$w_i' = 0$$

and

$$s = f(t_o) = g(r_o) .$$

By Proposition 2.1, there are an open neighborhood R' of r_o, an ambient space G' of R' and a vector valued holomorphic function ψ_i on $U_i \times G'$ such that, for each fixed $r \in R'$,

$$\eta_i(\lambda^{-1}(r) \cap X_i) = \{(w_i', z_i, s) \in W_i \times S \mid w_i' = \psi_i(z_i, r), s = g(r)\} .$$

We put

$$\phi_i'(z_i, r) = \psi_i(z_i, r) + \phi_i(z_i, t_o)$$

for $r \in R'$ and

$$\phi'(r) = \{\phi_i'(z_i, r)\}_{i \in I} .$$

Then

$$(\phi'(r), g(r)) \in C^0(|\ |) \times S .$$

Note that

$$(\phi'(r_o), g(r_o)) = \Phi(t_o) .$$

It is easy to see that ϕ' is an analytic map of R' into the Banach space $C^0(|\ |)$, provided that R' is sufficiently small. We may assume that

$$(\phi'(r), g(r)) \in U = \Phi(B_{\varepsilon'} \times \Omega_{\varepsilon'}) \qquad \text{for } r \in R' .$$

Since the equations

$$w_i = \phi_i'(z_i, r)$$

and

$$s = g(r)$$

define a compact complex submanifold of $\pi^{-1}(g(r))$,

$$(\phi'(r), g(r)) \in U \cap M \quad \text{for } r \in R' .$$

Hence

$$L(\phi'(r), g(r)) \in T \quad \text{for } r \in R' .$$

We put

$$h(r) = L(\phi'(r), g(r)) .$$

Then h is a holomorphic map of R' into T. We have

$$\Phi(h(r)) = (\phi'(r), g(r))$$

so that

$$V_{h(r)} = \lambda^{-1}(r) \qquad \text{for } r \in R' . \qquad \text{q.e.d.}$$

Lemma 5.5 completes the proof of the theorem.

REMARK. Among maximal families, our maximal family (Y, μ, T, f) is a special one. It is so called effectively parametrized. In other words, the map h with the property:

$$\mu^{-1}(h(r)) = \lambda^{-1}(r) \qquad \text{for } r \in R'$$

is uniquely determined.

6. A stability of compact complex submanifolds of complex manifolds.

DEFINITION 6.1 ([2]). Let V be a compact complex submanifold of a complex manifold W. V is called *a stable submanifold of* W if and only if, for any family (X, π, S) of complex manifolds with a point $o \in S$ such that $\pi^{-1}(o) = W$, there are a neighborhood U of o in S and a closed subvariety N of $\pi^{-1}(U)$ such that

1) (N, π', U) is a family of compact complex manifolds where $\pi' = \pi | N$ and

2) $\pi'^{-1}(o) = V$.

The following theorem is due to Kodaira (Theorem 1, [2]). Here we give another proof.

THEOREM (Kodaira). *Let V be a compact complex submanifold of a complex manifold W. Let F be the normal bundle of V in W. If $H^1(V, F) = 0$, then V is a stable submanifold of W.*

PROOF. Let (X, π, S) be a family of complex manifolds with a point $o \in S$ such that $\pi^{-1}(o) = W$. Let (Y, μ, T, f) be the maximal family of compact complex submanifolds of fibers of (X, π, S) constructed in §5 with respect to V. If $H^1(V, F) = 0$, then, by the corollary of Proposition 5.2,

$$T = (H^0(|\ |) \cap B_{\varepsilon'}) \times S_{\varepsilon'} .$$

We define a map

$$j : S_{\varepsilon'} \to T$$

by

$$j(s) = (0, s) .$$

Then j is a holomorphic injection. Let N be the closed subvariety of $\pi^{-1}(S_{\varepsilon'})$ defined in X_i by the equation:

$$w_i = \phi_i(z_i, j(s)) \quad \text{for} \quad z_i \in U_i .$$

Then it is easy to see that $(N, \pi', S_{\varepsilon'})$, $\pi' = \pi | N$, satisfies 1) and 2) of Definition 6.1. q.e.d.

REMARK. In the above proof, we can take j any holomorphic map

$$j\colon S_{\varepsilon'} \to T$$

such that

$$j(o) = (0, o)$$

and

$$fj = \text{the identity map on } S_{\varepsilon'} .$$

7. Proofs of lemmas in §3 and §5. In order to prove lemmas in §3 and §5, we need the following lemma.

LEMMA 7.1. *Let A be a compact subset of a Hausdorff space X. Let $A(\nu)$, $\nu = 1, 2, \cdots$, be compact subsets of X such that*

1) $A(1) \supset A(2) \supset \cdots \supset A$ *and*

2) $\bigcap_\nu A(\nu) = A$.

Let U be an open neighborhood of A. Then there exists an integer ν such that $U \supset A(\nu)$.

PROOF OF LEMMA 3.1. Let $\{U_i^\nu\}_{\nu=1,2,\ldots}$ be a sequence of Stein open sets in V such that

1) $\tilde{U}_i \Supset U_i^1 \Supset U_i^2 \Supset \cdots \Supset U_i$, where $\tilde{U}_i \Supset U_i^1$ means that $\overline{U_i^1}$ is compact and is contained in $\tilde{U}_i$, and so on, and

2) $\bigcap_\nu \overline{U_i^\nu} = \overline{U_i}$.

In a similar way, let $\{U_k^\nu\}_{\nu=1,2,\ldots}$ be a sequence of Stein open sets in V such that

3) $\tilde{U}_k \Supset U_k^1 \Supset U_k^2 \Supset \cdots \Supset U_k$ and

4) $\bigcap_\nu \overline{U_k^\nu} = \overline{U_k}$.

Then we have

5) $\tilde{U}_i \cap \tilde{U}_k \Supset U_i^1 \cap U_k^1 \Supset \cdots \Supset U_i \cap U_k$ and

6) $\bigcap_\nu (\overline{U_i^\nu} \cap \overline{U_k^\nu}) = \overline{U_i} \cap \overline{U_k}$.

By Lemma 7.1, there is ν such that

$$\bigcup_\lambda U_\lambda \supset \overline{U_i^\nu} \cap \overline{U_k^\nu} \supset U_i^\nu \cap U_k^\nu \supset \overline{U_i} \cap \overline{U_k} .$$

Thus $U = U_i^\nu \cap U_k^\nu$ satisfies the requirement. q.e.d.

PROOF OF LEMMA 3.2. Let $\{\Omega_\nu\}_{\nu=1,2,\ldots}$ be a sequence of polydiscs in Ω_0

with the center o such that

1) $\Omega_0 \supset \Omega_1 \supset \Omega_2 \supset \cdots$ and

2) $\bigcap_\nu \overline{\Omega_\nu} = o$.

We put $S_\nu = \Omega_\nu \cap S$, $\nu = 1, 2, \cdots$. In a similar way, let $\{D_\nu\}_{\nu=1,2,\dots}$ be a sequence of polydiscs in D_0 with the center 0 such that

3) $D_0 \supset D_1 \supset D_2 \supset \cdots$ and

4) $\bigcap_\nu \overline{D_\nu} = 0$.

Then we get

5) $\eta_k(\overline{X_i} \cap \overline{X_k}) \cap (\overline{D_1} \times \overline{U_k} \times \overline{S_1}) \supset \eta_k(\overline{X_i} \cap \overline{X_k}) \cap (\overline{D_2} \times \overline{U_k} \times \overline{S_2}) \supset \cdots \supset \eta_k(\overline{U_i} \cap \overline{U_k})$ and

6) $\bigcap_\nu (\eta_k(\overline{X_i} \cap \overline{X_k}) \cap (\overline{D_\nu} \times \overline{U_k} \times \overline{S_\nu})) = \eta_k(\overline{U_i} \cap \overline{U_k})$.

Thus, by Lemma 7.1, there is ν such that

$$\eta_k(\overline{X_i} \cap \overline{X_k}) \cap (\overline{D_\nu} \times \overline{U_k} \times \overline{S_\nu}) \subset W_o \times S_o .$$

Hence

$$\eta_k(X_i \cap X_k) \cap (D_\nu \times U_k \times S_\nu) \subset W_o \times S_o .$$

Thus we have

$$\begin{aligned} \eta_k(X_i \cap X_k) \cap (D_\nu \times U_k \times S_\nu) &\subset (W_o \times S_o) \cap (D_\nu \times U_k \times S_\nu) \\ &= D_\nu \times U \times S_\nu . \end{aligned}$$

q.e.d.

PROOF OF LEMMA 3.3. It is clear that if $|w_k| < \varepsilon_o$, $|s| < \varepsilon_o$ and $z_k \in U_i^e \cap U_k$, then

$$(w_k, z_k, s) \in W_o \times \Omega_o .$$

Hence $g_{ik}(w_k, z_k, s)$ is defined.

For any fixed $z_k \in \eta_k(\overline{U_i^e} \cap \overline{U_k})$, we have

$$|g_{ik}(0, z_k, o)| \leqq 1 - e .$$

Hence there is an open neighborhood

$$D(z_k) \times U(z_k) \times \Omega(z_k)$$

of z_k in $W_o \times \Omega_o$ such that

1) $D(z_k)$ and $\Omega(z_k)$ are polydiscs with the centers 0 and o respectively and

2) $|g_{ik}(w_k, z'_k, s)| < 1$ for $(w_k, z'_k, s) \in D(z_k) \times U(z_k) \times \Omega(z_k)$.

We cover $\eta_k(\overline{U_i^e} \cap \overline{U_k})$ by such open sets in $W_o \times \Omega_o$. Since $\overline{U_i^e} \cap \overline{U_k}$ is compact, there is a finite subcovering

$$\{D_\lambda \times U_\lambda \times \Omega_\lambda\}_{\lambda=1,\dots,m}$$

of $\eta_k(\overline{U_i^e} \cap \overline{U_k})$. We take ε, $0 < \varepsilon < \varepsilon_o$, so that

$$\Omega_\varepsilon \subset \bigcap_\lambda \Omega_\lambda$$

and

$$D_\varepsilon \subset \bigcap_\lambda D_\lambda .$$

Now if $|w_k| < \varepsilon$ and $|s| < \varepsilon$, then

$$|g_{ik}(w_k, z_k, s)| < 1 \quad \text{for all} \quad z_k \in U_i^e \cap U_k . \qquad \text{q.e.d.}$$

The proof of Lemma 3.4 is almost similar to that of Lemma 3.3 above, so we omit it.

PROOF OF LEMMA 3.5. Since

$$W_o \times S_o \subset \eta_k(\tilde{X}_i \cap \tilde{X}_k) ,$$

we have

$$\eta_k^{-1}(w_k, z_k, s) \in \tilde{X}_i \cap \tilde{X}_k \cap X_k = \tilde{X}_i \cap X_k$$

for $z_k \in U_i^e \cap U_k$, $|w_k| < \varepsilon_o$ and $s \in S_{\varepsilon_o}$. We take ε satisfying Lemmas 3.3 and 3.4 for $\delta = 1$. Then

$$|g_{ik}(w_k, z_k, s)| < 1$$

and

$$|f_{ik}(w_k, z_k, s)| < 1$$

for $z_k \in U_i^e \cap U_k$, $|w_k| < \varepsilon$ and $s \in S_\varepsilon$. This implies that

$$\eta_k^{-1}(w_k, z_k, s) \in X_i$$

for $z_k \in U_i^e \cap U_k$, $|w_k| < \varepsilon$ and $s \in S_\varepsilon$. q.e.d.

PROOF OF LEMMA 3.6. For each integer $\nu > 1/\varepsilon_o$, we put

$$A(\nu) = \{P \in \overline{X_i^{e'}} \cap \overline{X_k} \,|\, |w_k| \leqq 1/\nu \quad \text{and} \quad |s| \leqq 1/\nu\} ,$$

where $\eta_k(P) = (w_k, z_k, s)$. Then each $A(\nu)$ is compact. It is easy to see that $\{A(\nu)\}$ is a decreasing sequence of compact sets and

$$\bigcap_\nu A(\nu) = \overline{U_i^{e'}} \cap \overline{U_k} .$$

By Lemma 7.1, there is ν such that

$$A(\nu) \subset \eta_k^{-1}(\tilde{D}_k \times (U_i^e \cap \tilde{U}_k) \times S) .$$

Thus, if $P = \eta_k^{-1}(w_k, z_k, s) \in X_i^{e'} \cap X_k$, then

$$P \in A(\nu) \subset \eta_k^{-1}(\tilde{D}_k \times (U_i^e \cap \tilde{U}_k) \times S) ,$$

provided that $|w_k| < 1/\nu$ and $s \in S_{1/\nu}$. This implies that $z_k \in U_i^e \cap \tilde{U}_k$. Of course $P \in X_k$ implies that $z_k \in U_k$. Hence $z_k \in U_i^e \cap U_k$. q.e.d.

PROOF OF LEMMA 3.7. We first prove 1). Let ν_0 be an integer greater than $1/\varepsilon_o$. For any integer ν greater than or equal to ν_0, we put

$$\Omega_{1/\nu} = \{s \in \Omega_o \,|\, |s| < 1/\nu\}$$

and

$$D_{1/\nu} = \{w_k \in D_k \,|\, |w_k| < 1/\nu\} .$$

Then

$$\tilde{W}_k \times \Omega_o \supset \overline{D_{1/\nu_0}} \times (\overline{U_i} \cap \overline{U_j} \cap \overline{U_k}) \times \overline{\Omega_{1/\nu_0}} \supset \overline{D_{1/(\nu_0+1)}} \times (\overline{U_i} \cap \overline{U_j} \cap \overline{U_k}) \times \overline{\Omega_{1/(\nu_0+1)}} \supset \cdots \supset \overline{U_i} \cap \overline{U_j} \cap \overline{U_k}$$

and

$$\bigcap_{\nu \geqq \nu_0} (\overline{D_{1/\nu}} \times (\overline{U_i} \cap \overline{U_j} \cap \overline{U_k}) \times \overline{\Omega_{1/\nu}}) = \overline{U_i} \cap \overline{U_j} \cap \overline{U_k} .$$

By Lemma 7.1, there is ν such that

$$\eta_{jk}^{-1}(W_{o(ij)} \times \Omega_o) \supset \overline{D_{1/\nu}} \times (\overline{U_i} \cap \overline{U_j} \cap \overline{U_k}) \times \overline{\Omega_{1/\nu}} .$$

Thus $\varepsilon = 1/\nu$ satisfies the requirement.

Next we prove 2). We have

$$g_{jk}(0, z_k, o) = z_j \in \overline{U_i^e} \cap \overline{U_j^e} \subset U_i^{e/2} \cap U_j^{e/2}$$

for all $z_k \in \overline{U_i^e} \cap \overline{U_j^e} \cap \overline{U_k}$. For any point $z_k \in \overline{U_i^e} \cap \overline{U_j^e} \cap \overline{U_k}$, there are a neighborhood $U(z_k)$ of z_k in $\tilde{U}_k$ and a positive number $\varepsilon(z_k)$, $0 < \varepsilon(z_k) < \varepsilon_o$, such that if $|w_k| < \varepsilon(z_k)$, $|s| < \varepsilon(z_k)$ and $z'_k \in U(z_k)$, then $g_{jk}(w_k, z'_k, s)$ is defined and is a point of $U_i^{e/2} \cap U_j^{e/2}$. We cover $\overline{U_i^e} \cap \overline{U_j^e} \cap \overline{U_k}$ by a finite number of such $U(z_k^1), \cdots, U(z_k^a)$. We put

$$\varepsilon = \min\{\varepsilon(z_k^1), \cdots, \varepsilon(z_k^a)\} .$$

Then ε satisfies the requirement. q.e.d.

PROOF OF LEMMA 5.1. Let

$$\pi_i \colon \tilde{X}_i \to \tilde{U}_i$$

be the projection map defined by

$$\pi_i \eta_i^{-1}(w_i, z_i, s) = z_i .$$

For each positive integer ν, we set

$$A_i(\nu) = \{\eta_i^{-1}(w_i, z_i, s) \in \tilde{X}_i \,|\, |w_i| \leqq 1/\nu, |z_i| \leqq 1 \quad \text{and} \quad |s| \leqq 1/\nu\} ,$$

and

$$A(\nu) = \bigcup_{i \in I} A_i(\nu) \subset X .$$

Since $A_i(\nu)$ is compact for each $i \in I$, $A(\nu)$ is also compact. It is clear that

$$V \subset A(\nu) .$$

We show that

$$\bigcap_{\nu} A(\nu) = V .$$

Let $b \in \bigcap_{\nu} A(\nu)$. Then there are an index $i \in I$ and a subsequence

$$\nu_1 < \nu_2 < \cdots$$

such that $b \in A_i(\nu_\alpha)$, $\alpha = 1, 2, \cdots$. Then

$$|w_i(b)| \leqq 1/\nu_\alpha ,$$
$$|z_i(b)| \leqq 1$$

and

$$|s(b)| \leqq 1/\nu_\alpha$$

for $\alpha = 1, 2, \cdots$, where $\eta_i(b) = (w_i(b), z_i(b), s(b))$. Thus $w_i(b) = 0$, $|z_i(b)| \leqq 1$ and $s(b) = o$. Hence $b \in \overline{U_i} \subset V$. By Lemma 7.1, there is ν such that

$$A(\nu) \subset \bigcup_i X_i^\varepsilon .$$

Then $\varepsilon = 1/\nu$ satisfies the requirement. q.e.d.

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