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ON THE CATEGORY OF THE DOUBLE MAPPING CYLINDER

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Abstract. If $a: X \to A$ is a cofibration and if R is the adjunction space obtained by attaching A to B by means of $b: X \to B$ then $\operatorname{cat} R \leqslant \min(1 + \operatorname{cat} A + \operatorname{cat} B, \operatorname{cat} X + \max(\operatorname{cat} A, \operatorname{cat} B))$, where $\operatorname{cat} Y$ denotes the Lusternik-Schnirelmann category of Y as redefined by G. W. Whitehead, renormalised to take the value 0 on contractible spaces.

Let $a: X \to A$, $b: X \to B$ be maps in the category of pointed connected CW-complexes and let Z be the associated (reduced) double mapping cylinder. If $\operatorname{cat} Y$ denotes the Lusternik-Schnirelmann category of Y as redefined by G. W. Whitehead [10], renormalised to take the value 0 on contractible spaces, then Tsuchida [9; 3.4] has proved that

$$\operatorname{cat} Z \leqslant \operatorname{cat} A + \operatorname{cat} B + 1 . \tag{1}$$

The chief purpose of this paper is to show that the results of [6] combined with a simple-minded homotopy argument yield:

$$\operatorname{cat} Z \leqslant \operatorname{cat} X + \max(\operatorname{cat} A, \operatorname{cat} B) . \tag{2}$$

If a is a cofibration then it is well-known [3; p. 247] that Z has the same homotopy type as the adjunction space R obtained by attaching A to B by means of b. Thus (1) and (2) together imply

$$\operatorname{cat} R \leqslant \min(1 + \operatorname{cat} A + \operatorname{cat} B, \operatorname{cat} X + \max(\operatorname{cat} A, \operatorname{cat} B)) . \tag{3}$$

In the sequel we shall consider briefly what may be said in the presence of a "primitivity" condition [2; p. 441], [9; 3.7]. We recall that Z is the space obtained from the (pointed) sum $A + (X \times I) + B$ by factoring out by the relations

$$(x, 0) \sim a(x),\ (x, 1) \sim b(x),\ (*, t) \sim (*, t') \qquad (x \in X;\ t, t' \in I) .$$

Let $f: A \to W$, $g: B \to W$ and let $H: f \circ a \simeq g \circ b: X \to W$. (Homotopies

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are of course required to respect base-points.) Then we say that the square

$$(4)\qquad \begin{array}{ccc} X & \xrightarrow{a} & A \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{g} & W \end{array}$$

commutes via the homotopy H. We remark that the square

$$\begin{array}{ccc} X & \xrightarrow{a} & A \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle b'} \\ B & \xrightarrow{a'} & Z \end{array}$$

commutes via F, where $F(x, t) = \{(x, t)\}$ $(x \in X, t \in I)$ and a' and b' are the obvious maps. If (4) commutes via H it is easy to see that there exists a unique map $\theta = \theta(f, H, g)\colon Z \to W$ such that $\theta \circ b' = f$, $\theta \circ a' = g$ and $\theta \circ F = H$. Let $K\colon f \circ a \simeq g \circ b$ be another homotopy. Then the *reverse* homotopy $rK\colon g \circ b \simeq f \circ a$ and the *conjunction* $H \bigoplus rK\colon f \circ a \simeq f \circ a$ are defined. (For details see [5; p. 338].) The homotopy class $\{H \bigoplus rK\}$ is an element of the $(f \circ a)$-*based track group* $\pi_1^X(W; f \circ a)$. (See [8], [1].) We omit the proof of the following.

LEMMA 1. *$\theta(f, H, g) \simeq \theta(f, K, g)$ if and only if*

$$\{H \bigoplus rK\} = 0 \in \pi_1^X(W; f \circ a).$$

Finally, given maps $\alpha\colon A \to V$, $\beta\colon B \to V$, $\gamma\colon W \to V$ and homotopies $G\colon \alpha \simeq \gamma \circ f$, $G'\colon \gamma \circ g \simeq \beta$, it is clear (omitting unnecessary brackets) that

$$H' = G \circ (a \times id_I) \bigoplus \gamma \circ H \bigoplus G' \circ (b \times id_I)\colon \alpha \circ a \simeq \beta \circ b\,.$$

The routine proof of the following lemma is also omitted.

LEMMA 2. *$\theta(\alpha, H', \beta) \simeq \gamma \circ \theta(f, H, g)\colon Z \to V$.*

PROOF OF (2). Let $p = 1 + \operatorname{cat} X$. We may assume that $n = p + \max(\operatorname{cat} A, \operatorname{cat} B)$ is finite. Let TY denote the product of n copies of Y and let $T^r Y = \{(y_1, y_2, \cdots, y_n) \in TY \mid \text{at least } r \text{ coordinates are at } *\}$.

Then, by [5; Theorem], there exist maps $\phi_A\colon A \to T^p A$, $\phi_B\colon B \to T^p B$ and homotopies $G_A\colon \Delta \simeq j \circ \phi_A$, $G_B\colon j \circ \phi_B \simeq \Delta$, where $\Delta\colon Y \to TY$ is the diagonal transformation and $j\colon T^p Y \to TY$ the injection. Let k denote the injection $T^p Y \to T^1 Y$. We postpone the proof of the following lemma which refers to the diagram

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "b"'] & A \arrow[r, "\Delta"] \arrow[d, "k\circ T^p b'\circ\phi_A"'] & TA \arrow[d, "Tb'"'] \\
B \arrow[r, "k\circ T^p a'\circ\phi_B"] \arrow[d, "\Delta"'] & T^1Z \arrow[r, "j"] & TZ \\
TB \arrow[urr, "Ta'"'] & &
\end{tikzcd}$$

LEMMA 3. *The top-left square commutes via a homotopy H.*

Thus $\theta = \theta(k\circ T^p b'\circ\phi_A, H, k\circ T^p a'\circ\phi_B)\colon Z \to T^1Z$ is well-defined and we must prove that $j\circ\theta \simeq \Delta\colon Z \to TZ$.

Let $H' = Tb'\circ G_A\circ(a \times id_I) \oplus j\circ H \oplus Ta'\circ G_B\circ(b \times id_I)$. Then Lemma 2 yields $\theta(Tb'\circ\Delta, H', Ta'\circ\Delta) \simeq j\circ\theta$. Hence it will suffice to prove, for each $s=1, 2, \cdots, n$, that $\pi\circ\theta(Tb'\circ\Delta, H', Ta'\circ\Delta) \simeq id_Z$, where $\pi = \pi_s$ is the projection given by $\pi_s(x_1, x_2, \cdots, x_n) = x_s$. But $\pi\circ\theta(Tb'\circ\Delta, H', Ta'\circ\Delta) = \theta(b', \pi\circ H', a')$ and, since $id_Z = \theta(b', F, a')$, it will be enough in view of Lemma 1 to prove that

$$\{\pi\circ H' \oplus rF\} = 0 \in \pi_1^X(Z; b'\circ a)\ . \tag{5}$$

This is certainly the case if X is contractible, for then the group is trivial [8; p. 338]. If X is not contractible then $p > 1$. We shall show that we may change H so that (5) is satisfied. Let $M\colon b'\circ a \simeq b'\circ a$ be such that $\{M\} + \{\pi\circ H' \oplus rF\} = 0 \in \pi_1^X(Z; b'\circ a)$, and let $N\colon k\circ T^p b'\circ\phi_A \simeq k\circ T^p b'\circ\phi_A\colon X \to T^1Z$ be such that

$$\begin{cases} \pi\circ j\circ N = r(b'\circ\pi\circ G_A\circ(a \times id_I)) \oplus M \oplus b'\circ\pi\circ G_A\circ(a \times id_I) \\ \pi_i\circ j\circ N = \text{constant homotopy at } b'\circ\pi_i\circ\phi_A (i \neq s)\ . \end{cases}$$

Then, since $p > 1$, N is indeed a homotopy $X \to T^1Z$. If we replace H by $N \oplus H$ an easy computation now shows that (5) is satisfied. We may thus add a correcting homotopy to H for each $s = 1, 2, \cdots, n$.

PROOF OF LEMMA 3. The assertion clearly holds if X is contractible. If not then, since $\operatorname{cat} X = p - 1$ is finite, it follows [4] that X is dominated by a space of the form $\sum \Omega X \cup C(\Omega X * \Omega X) \cdots \cup C$(join of $p - 1$ copies of ΩX). Hence we can assume without loss of generality that $X = X_{p-1}$, where X_0 is the base-point and X_r is obtained by attaching a reduced cone $C\sum Y_r$ by means of a map $\sum Y_r \to X_{r-1}$ $(1 \leqslant r \leqslant p-1)$. Let $h\colon T^pZ \to T^{p-r+1}Z$ be the inclusion and suppose (inductively) that a homotopy

$$H_{r-1}\colon h\circ T^p a'\circ\phi_B\circ b \mid X_{r-1} \simeq h\circ T^p b'\circ\phi_A\circ a \mid X_{r-1}\colon X_{r-1} \times I \to T^{p-r+1}Z$$

exists. The obstruction to extending H_{r-1} over X_r is a class $\sigma \in [\sum Y_r,$

$T^{p-r+1}Z]$ and, since the outside of the diagram and the remaining rectangles are homotopy commutative, H_{r-1} can be corrected so that σ vanishes after injection into $[\sum Y_r, TZ]$. But if F_q is the fibre of $j\colon T^qZ \to TZ$ then Porter [7] has shown that F_q is contractible in F_{q-1}. It follows that σ vanishes after injection into $[\sum Y_r, T^{p-r}Z]$ and thus $H_r\colon X_r \times I \to T^{p-r}Z$ may be defined $(1 \leqslant r \leqslant p-1)$. This completes the proof of Lemma 3 and of (2). It is clear that the homotopy commutativity of the square referred to in Lemma 3 is a kind of primitivity condition. We offer the following tentative formulation. Suppose that cat $Y < n-1$. A map $\phi\colon Y \to T^rY$ is a *structure map* if $r > 0$ and $j \circ \phi \simeq \Delta$. Let $\phi_A\colon A \to T^rA$, $\phi_B\colon B \to T^sB$ be structure maps. The cotriad (a, b) is *primitive* if $k \circ T^rb' \circ \phi_A \circ a \simeq k' \circ T^sa' \circ \phi_B \circ b\colon X \to T^1Z$, where $k\colon T^rZ \to T^1Z$, $k'\colon T^sZ \to T^1Z$ denote inclusions (cf. [9; 3.7].) A proof analysis yields without difficulty the following corollary.

COROLLARY. *If (a, b) is primitive (relative to ϕ_A, ϕ_B) and if* $\max(r,s) > 1$ *then* cat $Z < n-1$.

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