# $\alpha$-INVARIANTS OF SOME DIFFERENTIABLE $S^{1}$-ACTIONS 

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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1. Introduction and results. We mean by differentiable manifolds $C^{\infty}$ manifolds and by differentiable maps $C^{\infty}$ maps. Let $S^{1}=\{z| | z \mid=1\}$ be the unit circle. For a differentiable $S^{1}$-manifold $X$, we denote by $X^{s^{1}}$ the set of points which are left fixed by all elements of $S^{1}$. Let $Y$ and $Y^{\prime}$ be compact connected oriented $S^{1}$ manifolds of dimension $2 n-1$, such that $Y^{S^{1}}=Y^{\prime S^{1}}=\varnothing$. Some multiples of $Y, Y^{\prime}$ bound compact oriented $S^{1}$-manifolds $X, X^{\prime}$ respectively (see E. Ossa [5]) and $\alpha$-invariants of $Y$, $Y^{\prime}$ are defined by M. F. Atiyah and I. M. Singer [2] and D. Zagier [9]. Let $W$ be a compact oriented manifold $W$ of dimension $2 m$. In the subgroup $\hat{H}^{m}(W)=\operatorname{Im}\left\{j^{*}: H^{m}(W, \partial W ; R) \rightarrow H^{m}(W ; R)\right\}$, we have a bilinear form defined by $B\left(j^{*}(a), g^{*}(b)\right)=a \cdot b[W]$, where [ $W$ ] is the fundamental homology class of ( $W, \partial W$ ), and we denote the signature of $B$ by $\operatorname{Sign}(W)$. We say that $Y$ and $Y^{\prime}$ are $\sigma$-equivalent with respect to the $S^{1}$-actions if there exists a compact oriented $S^{1}$-manifold $W$ such that $\partial W=Y \cup\left(-Y^{\prime}\right)$, the $S^{1}$-action on $Y \cup\left(-Y^{\prime}\right)$ is the restriction of the $S^{1}$-action of $W$ to the boundary and $\operatorname{Sign}(W)=0$. If $Y$ and $Y^{\prime}$ are $\sigma$-equivalent with respect to $S^{1}$-actions, then they have same $\alpha$-invariants: $\alpha(z, Y)=\alpha\left(z, Y^{\prime}\right)$ [2, pp. 589-590].

Let $M^{2 p}$ be a compact connected oriented $2 p$ dimensional manifold and $\xi=\left(E, \pi, M^{2 p}\right)$ be differentiable $n$ dimensional complex vector bundles over $M^{2 p}$, where $p \geqq 1$ and $n \geqq 1$. We denote by $X_{\xi}$ the differentiable manifold of the $2 n$-disk bundle associated to $\xi$ and by $Y_{\xi}$ the differentiable manifold of the $(2 n-1)$-sphere bundle associated to $\xi . \quad X_{\xi}$ and $Y_{\xi}$ are compact connected oriented manifolds of dimensions $2(n+p)$ and $2(n+p)-$ 1 respectively. $\quad Y_{\xi}$ is the boundary manifold of $X_{\xi} ; \partial X_{\xi}=Y_{\xi}$. We denote the total space of $\xi$ by $E(\xi)$. Let

$$
\begin{equation*}
F: S^{1} \times E(\xi) \rightarrow E(\xi) \tag{1}
\end{equation*}
$$

be a differentiable $S^{1}$-action such that $F(z):, E(\xi) \rightarrow E(\xi)$ is a differentiable vector space bundle map for each $z \in S^{1}$, and for a local product structure $U \times C^{n}$ on a neighborhood $U$ of each point of $M, F(z$, ) induces
a unitary transformation from $C^{n}$ onto itself with characteristic roots $\boldsymbol{z}^{m_{i}}(1 \leqq i \leqq s)$ where $m_{i}$ are all different positive integers. Let $k_{i}$ be the dimension of the characteristic space for $\boldsymbol{z}^{m_{i}}$. We note that $s \leqq n$ and $\sum_{i=1}^{s} k_{i}=n . \quad F$ defines compatible $S^{1}$-actions on the manifolds $X_{\xi}$ and $Y_{\xi}$. The set of positive integers $T=\left\{m_{1}, \cdots, m_{s} ; k_{1}, \cdots, k_{s}\right\}$ is called a type of the $S^{1}$-action defined by $F$. We denote $Y_{\xi}$ with the $S^{1}$-action of type $T$ by $Y_{\xi}(T)$. By $F, \xi$ decomposes into the Whitney sum $\xi=\bigoplus_{i=1}^{s} \xi_{i}$ of differentiable complex $k_{i}$-vector bundle $\xi_{i}$ (see Section 2, Lemma (2.1)). We denote $k$-th Chern classes of $\xi$ and $\xi_{i}$ by $c_{k}(\xi)$ and $c_{k}\left(\xi_{i}\right)$. Let $\delta_{i j}(\xi)$ be the kronecker index for $i$, $j$, if the Euler class with real coefficients, $e(\xi)$ is not zero and let $\delta_{i j}(\xi)=0$ if $e(\xi)=0$.

Theorem (1.1). Let $\xi$ be a differentiable complex n-vector bundle ( $n \geqq 1$ ) over a compact connected oriented differentiable 2 dimensional manifold $M^{2}$. Suppose $Y_{\xi}$ has an $S^{1}$-action of the type $T=\left\{m_{1}, \cdots, m_{s}\right.$; $\left.k_{1}, \cdots, k_{s}\right\}$ and the Whitney sum decomposition of $\xi$ by the $S^{1}$-action is $\xi=\bigoplus_{i=1}^{s} \xi_{i}$. Then we have the $\alpha$-invariant of $Y_{\xi}(T)$,

$$
\alpha\left(z, Y_{\xi}(T)\right)=4 \prod_{i=1}^{s}\left(\frac{z^{m_{i}}+1}{z^{m_{i}}-1}\right)^{k_{i}}\left(\sum_{i=1}^{s} \frac{z^{m_{i}}}{z^{2 m_{i}}-1} c_{1}\left(\xi_{i}\right)\right)\left[M^{2}\right]
$$

where $z \in S^{1}$ and $\left[M^{2}\right]$ is the fundamental homology class of $M^{2}$, and $z \in S^{1}$ are not $m_{i}$-th roots of unity.

Theorem (1.2). Let $\xi$ be a differentiable complex $n$-vector bundle ( $n \geqq 1$ ) over the complex projective plane $C P^{2}$. Suppose $Y_{\xi}$ has an $S^{1}$ action of the type $T=\left\{m_{1}, \cdots, m_{s} ; k_{1}, \cdots, k_{s}\right\}$ and the Whitney sum decomposition of $\xi$ by the $S^{1}$-action is $\xi=\bigoplus_{i=1}^{\varepsilon} \xi_{i}$. Then we have the $\alpha$ invariant of $Y_{\xi}(T)$.

$$
\begin{aligned}
\alpha\left(z, Y_{\xi}(T)\right)= & -\prod_{i=1}^{s}\left(\frac{z^{m_{i}}+1}{z^{m_{i}}-1}\right)^{k_{i}}\left\{1+\left[16 \sum_{i<j} \frac{z^{m_{i}+m_{j}}}{\left(z^{2 m_{i}}-1\right)\left(z^{2 m_{j}}-1\right)} c_{1}\left(\xi_{i}\right) c_{1}\left(\xi_{j}\right)\right.\right. \\
& \left.\left.+4 \sum_{i=1}^{s} \frac{z^{m_{i}}}{\left(z^{2 m_{i}}-1\right)^{2}}\left(-2\left(z^{2 m_{i}}+1\right) c_{2}\left(\xi_{i}\right)+\left(z^{m_{i}}+1\right)^{2} c_{1}\left(\xi_{i}\right)^{2}\right)\right]\left[C P^{2}\right]\right\} \\
& +\delta_{2, n}(\xi),
\end{aligned}
$$

where $z \in S^{1}$ are not $m_{i}$-th roots of unity.
As applications of these theorems, we obtain, from $\sigma$-equivalences, criteria on bundle isomorphisms of $Y_{\xi}$ associated to some differentiable complex vector bundle $\xi$ over compact connected oriented differentiable 2-dimensional manifold $M^{2}$ and over $C P^{2}$. These results are analogies of bundle isomorphism theorems obtained from the spin-invariants $\rho$ (cf. [6] and [7]). Let $\xi_{1}$ be a differentiable complex $(n+1)$-vector bundle over
$M^{2}$ and $\xi_{2}$ a differentiable complex $n$-vector bundle over $C P^{2}$. Let $T^{(i)}$ be types of $S^{1}$-actions on $\xi_{i}$ for $i=1,2$. By Theorem (1.1) and (1.2), we show that the $S^{1}$-manifolds $Y_{\xi_{1}}\left(T^{(1)}\right)$ and $Y_{\xi_{2}}\left(T^{(2)}\right)$ can not be $\sigma$-equivalent, for $n>2$.

Let $Y_{\zeta}(m)$ be the differentiable manifold of differentiable $S^{1}$-bundle associated to a differentiable complex line bundle $\zeta$ over the complex $p$ dimensional projective space $C P^{p}$, having the natural $S^{1}$-action defined by complex scalar multiplication of $z^{m}$ in fibres, i.e., the $S^{1}$-action of the type $\{m ; 1\}$. Let the first Chern class of $\zeta$ be $c_{1}(\zeta)=a x \in H^{2}\left(C P^{p} ; Z\right)$, where $x$ is the canonical generator of $H^{2}\left(C P^{2} ; Z\right)$.

Theorem (1.3). Let $m$ be a positive integers and $z \in S^{1}$. The $\alpha$ invariant $\alpha\left(z, Y_{\zeta}(m)\right.$ ) with respect to the $S^{1}$-action of $Y_{\zeta}(m)$ is given by the following formulas: If the integer a is zero, then we have

$$
\alpha\left(z, Y_{\zeta}(m)\right)=\left(\frac{(-1)^{p+1}}{2}-\frac{1}{2}\right)\left(\frac{Z^{m}+1}{Z^{m}-1}\right) .
$$

If $a$ is not zero, then we have

$$
\begin{aligned}
& \alpha\left(z, Y_{\zeta}(m)\right)=\frac{1}{a} \sum_{\omega^{a}=z^{m}}\left(\frac{1+\omega}{1-\omega}\right)^{p+1} \\
& =\frac{-(-\operatorname{sgn} a)^{p}}{a}(p+1) \sum_{(\mu)}(-1)^{\mu_{1}+\cdots+\mu_{i a} \mid} \frac{\left(\mu_{1}+\cdots+\mu_{|a|}-1\right)!}{\mu_{1}!\cdots \mu_{|a|}!} \\
& \quad \times\left(\frac{z^{m}+1}{1-z^{m}}\binom{|a|}{1}\right)^{\mu_{1}}\binom{|a|}{2}^{\mu_{2}}\left(\frac{z^{m}+1}{1-z^{m}}\binom{|a|}{3}\right)^{\mu_{3}} \cdots(*)^{\mu_{|a|} \mid},
\end{aligned}
$$

where $\omega$ are the formal a-th roots of $z^{m}$, ( $\mu$ ) is the set of $a$-tuples of integers, $\left\{\left(\mu_{1}, \cdots, \mu_{|a|}\right)\left|\mu_{i} \geqq 0, \mu_{1}+2 \mu_{2}+\cdots+|a| \mu_{|a|}=p+1\right\}\right.$, (*) means 1 or $\left(z^{m}+1\right) /\left(1-z^{m}\right)$ according to that $a$ is even or odd and $z \in S^{1}$ are not $m$-th roots of unity.

By Theorem (1.3), we see that under a condition the $S^{1}$-manifold $Y_{\zeta}(m)$ can not be $\sigma$-equivalent to the $S^{1}$-manifolds of the type $T\left(\sum k_{i}=p\right)$ in Theorem (1.1) and to the $S^{1}$-manifolds of the type $T\left(\sum k_{i}=p-1\right)$ in Theorem (1.2).

In Section 2, we prove Theorem (1.1) and (1.2). In Section 3, we show isomorphism theorems of the manifolds $Y_{\xi}$ for the differentiable complex $n$-vector bundle $\xi$ over the compact connected oriented differentiable 2 dimensional manifold $M^{2}$ and over $C P^{2}$. We consider also $\sigma$-nonequivalences between $S^{1}$-manifolds constructed from complex vector bundles over $M^{2}$ and those constructed from complex vector bundles over $C P^{2}$. In the last section, we prove Theorem (1.3) by using the residue theorem. We then obtain, as applications of Theorems (1.1), (1.2), and (1.3), results
on $\sigma$-non-equivalences of the $S^{1}$-manifolds $Y_{\zeta}(m)$ of $S^{1}$-bundles associated to differentiable complex line bundles $\zeta$ over the $p$ dimensional complex projective space $C P^{p}$.
2. The invariant $\alpha\left(z, Y_{\xi}\right)$. First of all, we consider the differentiable complex $n$ dimensional vector bundle over a compact connected oriented $2 p$ dimensional manifold $M^{2 p}, \xi=\left(E, \pi, M^{2 p}\right)$, with the differentiable $S^{1}$-action $F$ defined by (1) and obtain the following lemma on splitting of $\xi$ :

Lemma (2.1). Let $m_{1}, \cdots, m_{s}$ be all different positive integers. If the $S^{1}$-action $F$ on $\xi=\left(E, \pi, M^{2 p}\right)$ is of the type $\left\{m_{1}, \cdots, m_{s} ; k_{1}, \cdots, k_{s}\right\}$ then $\xi$ splits into a Whitney sum,

$$
\xi=\bigoplus_{i=1}^{n} \xi_{i}
$$

of differentiable complex $k_{i}$-vector bundle $\xi_{i}$ and the restrictions of the actions of $z \in S^{1}$ on $\xi_{i}$ are the multiplications by $z^{m_{i}}$.

Proof. The actions of $z \in S^{1}$ are commutative with the coordinate transformations $g_{\alpha \beta}(x)$ of $\xi$ for $x \in U_{\alpha} \cap U_{\beta}$, where $U_{\alpha}, U_{\beta}$ are coordinate neighborhoods of $\xi$ and hence, for the characteristic vectors $v_{i}$ of characteristic values $\boldsymbol{z}^{m_{i}}$, we have

$$
z \circ\left(g_{\alpha \beta}(x) v_{i}\right)=g_{\alpha \beta}(x)\left(z \circ v_{i}\right)=g_{\alpha \beta}(x)\left(\boldsymbol{z}^{m_{i}} v_{i}\right)=\boldsymbol{z}^{m_{i}}\left(g_{\alpha \beta}(x) v_{i}\right),
$$

that is, $g_{\alpha \beta}(x) v_{i}$ are also characteristic vectors for $z^{m_{i}}$. Since $m_{1}, \cdots, m_{s}$ are all different positive integers, the set of all characteristic vectors for $\boldsymbol{z}^{m_{i}}$ in each fibre of $\xi$ makes a complex $k_{i}$-vector subbundle $\xi_{i}$ and gives a decomposition of $\xi$ into the Whitney sum of $\xi_{i}: \xi=\bigoplus_{i=1}^{n} \xi_{i}$. The actions of $z \in S^{1}$ on $\xi_{i}$ are clearly the multiplications by $z^{m_{i}}$ and we complete the proof of the lemma.

Let $X_{\xi}$ be the manifold with the $2 n$-disk bundle structure over $M^{2 p}$ associated to the differentiable complex $n$-vector bundle which has the $S^{1}$-action $F$ of the type $T=\left\{m_{1}, \cdots, m_{s} ; k_{1}, \cdots, k_{s}\right\}$, stated in Section 1. Let $Y_{\xi}(T)$ be the boundary manifold of $X_{\xi}$ with the $S^{1}$-action of the type $T$, which has the $(2 n-1)$-sphere bundle structure. By [2] or [9], we have the $\alpha$-invariant,

$$
\begin{equation*}
\alpha\left(z, Y_{\xi}(T)\right)=-\left\{2^{p} \prod_{i=1}^{s}\left(\prod_{j=1}^{k_{i}} \frac{z^{m_{i}} e^{x_{j}\left(\xi_{i}\right)}+1}{z^{m_{i}} e^{x_{j}\left(\xi_{i}\right)}-1}\right) \mathscr{L}(M)\right\}[\mathrm{M}]+\operatorname{Sign}\left(X_{\xi}\right) \tag{2}
\end{equation*}
$$

where, as usual, the elementary symmetric functions of the $x_{j}\left(\xi_{i}\right)$ are the Chern classes of $\xi_{i}$, and $\mathscr{L}$ is the multiplicative sequence with the characteristic series $(x / 2) /(\tanh x / 2) . \quad \mathscr{L}$ is determined by $\mathscr{L}=\sum \mathscr{L}_{r}(p)=$
$\Pi\left(x_{i}(M) / 2\right) /\left(\tanh x_{i}(M) / 2\right)$ and $\mathscr{L}_{r}(p)$ are homogeneous polynomials of degree $r$ in Pontrjagin classes. Sign $\left(X_{\xi}\right)$ is the signature of the bilinear form $B$ on $\hat{H}^{p+n}\left(X_{\xi}\right)=\operatorname{Im}\left\{j^{*}: H^{p+n}\left(X_{\xi}, Y_{\xi} ; R\right) \rightarrow H^{p+n}\left(X_{\xi} ; R\right)\right\}$ defined by

$$
B\left(j^{*}(a), j^{*}(b)\right)=a \cdot b\left[X_{\xi}\right],
$$

which is non-degenerate because of the Poincare duality between $H^{p+n}\left(X_{\xi} ; R\right)$ and $H^{p+n}\left(X_{\xi}, Y_{\xi} ; R\right)$, where $R$ denotes the real number field. The homomorphism $j^{*}: H^{p+n}\left(X_{\xi}, Y_{\xi} ; R\right) \rightarrow H^{p+n}\left(X_{\xi} ; R\right)$ can be identified with the homomorphism $H^{p-n}(M ; R) \rightarrow H^{p+n}(M ; R)$ given by multiplication of the Euler class $e(\xi) \in H^{2 n}(M ; R) . \operatorname{Sign}\left(X_{\xi}\right)$ is just the signature of the degenerate form on $H^{p-n}(M ; R)$ given by

$$
\begin{equation*}
(u, v) \rightarrow e(\xi) \cdot u \cdot v[M] \tag{3}
\end{equation*}
$$

Proof of Theorem (1.1). Since $\operatorname{dim} M=2$, we have $\mathscr{L}(M)=1$ and by direct calculations we obtain

$$
\begin{align*}
\prod_{j=1}^{k_{i}} & \left(\frac{z^{m_{i}} e^{x_{j}\left(\xi_{i}\right)}+1}{\boldsymbol{z}^{m_{i}} e^{x_{j}\left(\xi_{i}\right)}-1}\right) \\
= & \left(\frac{\boldsymbol{z}^{m_{i}}+1}{\boldsymbol{z}^{m_{i}}-1}\right)^{k_{i}}-2 \frac{\left(\boldsymbol{z}^{m_{i}}+1\right)^{k_{i}-1} z^{m_{i}}}{\left(z^{m_{i}}-1\right)^{k_{i}+1}} c_{1}\left(\xi_{i}\right)  \tag{4}\\
& \quad+\frac{\left(\boldsymbol{z}^{m_{i}}+1\right)^{k_{i}-2} z^{m_{i}}}{\left(\boldsymbol{z}^{m_{i}}-1\right)^{k_{i}+2}}\left(-2\left(\boldsymbol{z}^{2 m_{i}}+1\right) c_{2}\left(\xi_{i}\right)+\left(\boldsymbol{z}^{m_{i}}+1\right)^{2} c_{1}\left(\xi_{i}\right)^{2}\right)+\cdots
\end{align*}
$$

By the bilinear form (3), we have easily that

$$
\operatorname{Sign}\left(X_{\xi}\right)=\left\{\begin{array}{l}
1 \text { if } n=1 \text { and } e(\xi) \neq 0,  \tag{5}\\
0 \text { if } n>0 \text { or } e(\xi)=0
\end{array}\right.
$$

Hence it follows from Lemma (2.1) and from (2), (4), and (5) that

$$
\alpha\left(z, Y_{\xi}(T)\right)=4 \prod_{i=1}^{s}\left(\frac{z^{m_{i}}+1}{z^{m_{i}}-1}\right)^{k_{i}}\left(\sum_{i=1}^{s} \frac{z^{m_{i}}}{z^{2_{i}}-1} c_{1}\left(\xi_{i}\right)\right)[M]+\delta_{1, n}(\xi),
$$

and we complete the proof of Theorem (1.1).
Proof of Theorem (1.2). Let $x \in H^{2}\left(C P^{2} ; Z\right)$ be the canonical generator. By definition of $\mathscr{L}$, we have

$$
\begin{equation*}
\mathscr{L}\left(C P^{2}\right)=1+\frac{1}{4} x^{2} . \tag{6}
\end{equation*}
$$

By the formula (4) and direct calculations we have
(7)

$$
\begin{aligned}
& \prod_{i=1}^{s} \prod_{j=1}^{k_{i}}\left(\frac{z^{m_{i}} e^{x_{j}\left(\xi_{i}\right)}+1}{\boldsymbol{z}^{m_{i}} e^{x_{j}\left(\xi_{i}\right)}-1}\right) \\
&=\prod_{i=1}^{s}\left\{\left(\frac{z^{m_{i}}+1}{\boldsymbol{z}^{m_{i}}-1}\right)^{k_{i}}-\frac{2\left(z^{m_{i}}+1\right)^{k_{i}-1} z^{m_{i}}}{\left(\boldsymbol{z}^{m_{i}}-1\right)^{k_{i}+1}} c_{1}\left(\xi_{i}\right)\right. \\
&\left.+\frac{\left(\boldsymbol{z}^{m_{i}}+1\right)^{k_{i}-2} z^{m_{i}}}{\left(\boldsymbol{z}^{m_{i}}-1\right)^{k_{i}+2}}\left(-2\left(z^{2 m_{i}}+1\right) c_{2}\left(\xi_{i}\right)+\left(z^{m_{i}}+1\right)^{2} c_{1}\left(\xi_{i}\right)^{2}\right)+\cdots\right\} \\
&=\prod_{i=1}^{s}\left(\frac{z^{m_{i}}+1}{\boldsymbol{z}^{m_{i}}-1}\right)^{k_{i}}-2 \prod_{i=1}^{s}\left(\frac{z^{m_{i}}+1}{\boldsymbol{z}^{m_{i}}-1}\right)^{k_{i}}\left(\sum_{i=1}^{s} \frac{z^{m_{i}}}{z^{2 m_{i}}-1} c_{1}\left(\xi_{i}\right)\right) \\
&+4 \prod_{i=1}^{s}\left(\frac{z^{m_{i}}+1}{\boldsymbol{z}^{m_{i}}-1}\right)^{k_{i}} \sum_{i<j} \frac{z^{m_{i}+m_{j}}}{\left(z^{2 m_{i}}-1\right)\left(z^{m_{j}}-1\right)} c_{1}\left(\xi_{i}\right) c_{1}\left(\xi_{j}\right)+\prod_{i=1}^{s}\left(\frac{z^{m_{i}}+1}{z^{m_{i}}-1}\right)^{k_{i}} \\
& \times\left(\sum_{i=1}^{s} \frac{\boldsymbol{z}^{m_{i}}}{\left(z^{2 m_{i}}-1\right)^{2}}\left(-2\left(z^{2_{i}}+1\right) c_{2}\left(\xi_{i}\right)+\left(z^{m_{i}}+1\right)^{2} c_{1}\left(\xi_{i}\right)^{2}\right)+\cdots .\right.
\end{aligned}
$$

By the bilinear form (3), we obtain easily that

$$
\operatorname{Sign}\left(X_{\xi}\right)=\left\{\begin{array}{l}
1 \text { if } n=2 \text { and } e(\xi) \neq 0,  \tag{8}\\
0 \text { if } n \neq 2 \text { or } e(\xi)=0 .
\end{array}\right.
$$

Hence it follows from Lemma (2.1) and from (2), (6), (7), and (8) that

$$
\begin{aligned}
\alpha\left(z, Y_{\xi}(T)\right)= & -\prod_{i=1}^{s}\left(\frac{z^{m_{i}}+1}{z^{m_{i}}-1}\right)^{k_{i}}\left\{1+\left[16 \sum_{i<j} \frac{z^{m_{i}+m_{j}}}{\left(z^{2 m_{i}}-1\right)\left(z^{2 m_{j}}-1\right)} c_{1}\left(\xi_{i}\right) c_{1}\left(\xi_{j}\right)\right.\right. \\
& \left.\left.+4 \sum_{i=1}^{s} \frac{z^{m_{i}}}{\left(z^{2 m_{i}}-1\right)^{2}}\left(-2\left(z^{2 m_{i}}+1\right) c_{2}\left(\xi_{i}\right)+\left(z^{m_{i}}+1\right)^{2} c_{1}\left(\xi_{i}\right)^{2}\right)\right]\left[C P^{2}\right]\right\} \\
& +\delta_{2, n}(\xi),
\end{aligned}
$$

and we complete the proof of theorem (1.2).
3. $\alpha$-equivalence of $Y_{\xi}$. Let $\xi$ and $\xi^{\prime}$ be differentiable complex $n$-vector bundles over a compact connected oriented 2 dimensional manifold $M^{2}$ or over the complex projective plane $C P^{2}$, which have differentiable $S^{1}$-actions $F$ and $F^{\prime}$ defined by (1). Let $X_{\xi}$ and $X_{\xi}^{\prime}$ be associated $2 n$-disk bundles which are compact connected oriented manifolds with boundaries, and let $Y_{\xi}$ and $Y_{\xi}^{\prime}$ be their respective boundary manifolds. Suppose that $F$ and $F^{\prime}$ have the type $T$. Then the $\alpha$-invariants $\alpha\left(z, Y_{\xi}(T)\right)$ and $\alpha\left(z, Y_{\xi}^{\prime}(T)\right)$ are computed by Theorems (1.1) and (1.2), and we obtain at first following results on the structures of $\xi$ and $\xi^{\prime}$.

Proposition (3.1). Let $\xi$ and $\xi^{\prime}$ be differentiable complex $n$-vector bundles over a compact connected oriented 2 dimensional differentiable manifold $M^{2}$. Suppose that they have differentiable $S^{1}$-actions of the type $T=\left\{m_{1}, m_{2} ; k_{1}, k_{2}\right\}$. If we have

$$
\alpha\left(z, Y_{\xi}(T)\right)=\alpha\left(z, Y_{\xi^{\prime}}(T)\right)
$$

for any $z \in S^{1}\left(z^{m_{i}} \neq 1, i=1,2\right)$ then there is a differentiable isomorphism of complex vector bundles between $\xi$ and $\xi^{\prime}$, including $S^{1}$-actions.

Proof. Let $\xi=\xi_{1} \oplus \xi_{2}$ and $\xi^{\prime}=\xi_{1}^{\prime} \oplus \xi_{2}^{\prime}$ be the decompositions of $\xi$ and $\xi^{\prime}$ into the Whitney sums of complex $k_{i}$-vector bundles ( $i=1,2$ ) given by Lemma (1.2). Let $u \in H^{2}(M ; Z) \cong Z$ be the standard generator. For $c_{1}\left(\xi_{i}\right)$ and $c_{1}\left(\xi_{i}^{\prime}\right)$, there are unique integers $d_{i}$ and $d_{i}^{\prime}$ such that $c_{1}\left(\xi_{i}\right)=$ $d_{i} u$ and $c_{1}\left(\xi_{i}^{\prime}\right)=d_{i}^{\prime} u$. By Theorem (1.1) and the equality $\alpha\left(z, Y_{\xi}(T)\right)=$ $\alpha\left(z, Y_{\xi}(T)\right)$, we have

$$
\left(z^{2 m_{2}}-1\right) z^{m_{i}} d_{1}+\left(z^{2 m_{1}}-1\right) z^{m_{2}} d_{2}=\left(z^{2 m_{2}}-1\right) z^{m_{1}} d_{1}^{\prime}+\left(z^{2 m_{1}}-1\right) z^{m_{2}} d_{2}^{\prime}
$$

for any $z \in S^{1}$. Since we have $m_{1} \neq m_{2}$, it follows that

$$
d_{i}=d_{i}^{\prime}, \quad i=1,2 .
$$

By the definition of $d_{i}$ and $d_{i}^{\prime}$, the above equalities yield

$$
c_{1}\left(\xi_{i}\right)=c_{1}\left(\xi_{i}^{\prime}\right), \quad i=1,2
$$

Since $M^{2}$ is a 2 dimensional manifold, we have $c_{j}\left(\xi_{i}\right)=c_{j}\left(\xi_{i}^{\prime}\right)=0$ for $j>1$. Since $M$ has no 2 -torsion, complex $k_{i}$-vector bundles $\xi_{i}$, $\xi_{i}^{\prime}$ are stably isomorphic by [ $1,2.5$ Corollary (i)]. Since we have $k_{i} \geqq 1$, by the argument of [9, pp. 893-894] we obtain isomorphisms of complex $k_{i}$-vector bundles,

$$
\xi_{i} \cong \xi_{i}^{\prime}, \quad i=1,2
$$

Moreover, by differentiable approximations [3] of homotopies of classifying maps and by the method of parallelisms for connections in principal fibre bundles [4], one obtains differentiable isomorphisms between $\xi_{i}$ and $\xi_{i}^{\prime}$. Since we have $\xi=\xi_{1} \oplus \xi_{2}$ and $\xi^{\prime}=\xi_{1}^{\prime} \oplus \xi_{2}^{\prime}$, it follows that there is a differentiable isomorphism of complex vector bundles between $\xi$ and $\xi^{\prime}$, including $S^{1}$-actions. Thus we complete the proof of the Proposition (3.1).

Corollary (3.2). Let $\xi$ and $\xi^{\prime}$ be differentiable complex n-vector bundles over a compact connected oriented 2 dimensional differentiable manifold $M^{2}$. Suppose that they have differentiable $S^{1}$-actions of the type $T=\left\{m_{1}, m_{2} ; k_{1}, k_{2}\right\}$. There is a differentiable bundle isomorphism between $Y_{\xi}$ and $Y_{\xi}$, including $S^{1}$-actions if and only if they are $\sigma$-equivalent with respect to the $S^{1}$-actions.

Proof. If $Y_{\xi}$ and $Y_{\xi}$, are $\sigma$-equivalent with respect to $S^{1}$-actions $T$, we have

$$
\alpha\left(z, Y_{\xi}(T)\right)=\alpha\left(z, Y_{\xi},(T)\right)
$$

by [1, pp. 589-590]. The differentiable isomorphism between $Y_{\xi}$ and $Y_{\xi}$ including $S^{1}$-actions follows from Proposition (3.1). The converse is obvious and one completes the proof of the corollary.

Proposition (3.3). Let $\xi$ and $\xi^{\prime}$ differentiable complex $n$-vector bundle over the complex projective plane $C P^{2}$. Suppose that they have differentiable $S^{1}$-actions of the type $T=\left\{m_{1}, m_{2} ; k_{1}, k_{2}\right\}$, and suppose that $m_{1} \neq m_{2}$, $m_{1} \neq 2 m_{2}$ and $2 m_{1} \neq m_{2}$. If we have

$$
\alpha\left(z, Y_{\xi}(T)\right)=\alpha\left(z, Y_{\xi^{\prime}}(T)\right)
$$

for any $z \in S^{1}\left(z^{m_{i}} \neq 1, i=1,2\right)$, then there is a complex or complex conjugate isomorphism of complex vector bundles between $\xi$ and $\xi^{\prime}$ including $S^{1}$-actions given by their complex structures.

Proof. As in the proof of Proposition (3.1), let $\xi=\xi_{1} \oplus \xi_{2}$ and $\xi^{\prime}=$ $\xi_{1}^{\prime} \oplus \xi_{2}^{\prime}$ be the decompositions of $\xi$ and $\xi^{\prime}$ into the Whitney sums of complex $k_{i}$-vector bundles given by Lemma (2.1). Let $x \in H^{2}\left(C P^{2} ; Z\right)$ be the standard generators. For $c_{1}\left(\xi_{i}\right), c_{1}\left(\xi_{i}^{\prime}\right), c_{2}\left(\xi_{i}\right)$, and $c_{2}\left(\xi_{i}^{\prime}\right)$, there are unique integers $d_{i}, d_{i}^{\prime}, e_{i}$, and $e_{i}^{\prime}$ such that $c_{1}\left(\xi_{i}\right)=d_{i} x, c_{1}\left(\xi_{i}^{\prime}\right)=d_{i}^{\prime} x, c_{2}\left(\xi_{i}\right)=e_{i} x^{2}$, and $c_{2}\left(\xi_{i}^{\prime}\right)=e_{i}^{\prime} x^{2}$. One can assume without loss of generality that $m_{1}>m_{2}$. By Theorem (1.2) and the equality $\alpha\left(z, Y_{\xi}(T)\right)=\alpha\left(z, Y_{\xi^{\prime}}(T)\right)$, we have

$$
\begin{aligned}
d_{2}^{2}-2 e_{2}= & d_{2}^{\prime 2}-2 e_{2}^{\prime}, \quad d_{1}^{2}-2 e_{1}=d_{1}^{\prime 2}-2 e_{1}^{\prime}, \quad d_{2}^{2}=d_{2}^{\prime 2}, \\
& d_{1}^{2}+d_{2}^{2}=d_{1}^{\prime 2}+d_{2}^{\prime 2} \quad \text { and } d_{1} d_{2}=d_{1}^{\prime} d_{2}^{\prime}
\end{aligned}
$$

for the case $m_{1} \neq 2 m_{2}$ and $m_{1} \neq 3 m_{2}$, and also we have

$$
\begin{gathered}
d_{2}^{2}-2 e_{2}=d_{2}^{\prime 2}-2 e_{2}^{\prime}, \quad d_{2}^{2}=d_{2}^{\prime 2} \\
d_{1}^{2}-2 e_{1}+d_{2}^{2}-2 e_{2}=d_{1}^{\prime \prime}-2 e_{1}^{\prime}+d_{2}^{\prime 2}-2 e_{2}^{\prime} \\
d_{1} d_{2}=d_{1}^{\prime} d_{2}^{\prime} \quad \text { and } \quad d_{1}^{2}-2 d_{1} d_{2}=d_{1}^{\prime 2}-2 d_{1}^{\prime} d_{2}^{\prime}
\end{gathered}
$$

for the case $m_{1}=3 m_{2}$. Therefore, if $m_{1}>m_{2}$ and $m_{1} \neq 2 m_{2}$, one obtains

$$
\begin{array}{cc}
d_{i}=d_{i}^{\prime} \quad \text { or } & d_{i}=-d_{i}^{\prime} \\
e_{i}=e_{i}^{\prime}, & i=1,2
\end{array}
$$

By the definition of $d_{i}, d_{i}^{\prime}, e_{i}$, and $e_{i}^{\prime}$, the above relations yield

$$
\begin{gathered}
c_{1}\left(\xi_{i}\right)=c_{1}\left(\xi_{i}^{\prime}\right) \text { or } \quad c_{1}\left(\xi_{i}\right)=-c_{1}\left(\xi_{i}^{\prime}\right), \\
c_{2}\left(\xi_{i}\right)=c_{2}\left(\xi_{i}^{\prime}\right) \\
i=1,2 .
\end{gathered}
$$

We have obviously $c_{j}\left(\xi_{i}\right)=c_{j}\left(\xi_{i}^{\prime}\right)=0$ for $j>2$. Since $C P^{2}$ has no 2-torsion, one can apply [1, 2.5 Corollary (i)] to stable isomorphisms of complex vector bundles. Since we have $k_{i} \geqq 2$, by the argument of
[4, pp. 893-894], we conclude that in the case $c_{1}\left(\xi_{i}\right)=c_{1}\left(\xi_{i}^{\prime}\right)$, there are isomorphisms of complex $k_{i}$-vector bundles between $\xi_{i}$ and $\xi_{i}^{\prime}$ and in the case $c_{1}\left(\xi_{i}\right)=-c_{1}\left(\xi_{i}^{\prime}\right)$, there are conjugate isomorphism of complex $k_{i}$-vector bundles between $\xi_{i}$ and $\xi_{i}^{\prime}$. Moreover, again as in the proof of Proposition (3.1), by differentiable approximations [3] of homotopies of classifying maps and by connection arguments in principal fibre bundles [4], one obtains differentiable complex or complex conjugate isomorphisms between $\xi_{i}$ and $\xi_{i}^{\prime}$ according to $c_{1}\left(\xi_{i}\right)=c_{1}\left(\xi_{i}^{\prime}\right)$ or $c_{1}\left(\xi_{i}\right)=-c_{1}\left(\xi_{i}^{\prime}\right)$, for $i=$ 1,2. Since we have $\xi=\xi_{1} \oplus \xi_{2}$ and $\xi^{\prime}=\xi_{1}^{\prime} \oplus \xi_{2}^{\prime}$ it follows that there is a differentiable complex or complex conjugate isomorphism between $\xi$ and $\xi^{\prime}$, including $S^{1}$-actions given by the actions $F$ in their complex structures, and we complete the proof of Proposition (3.3).

Corollary (3.4). Let $\xi$ and $\xi^{\prime}$ be differentiable complex $n$ dimensional vector bundles over the complex projective plane CP ${ }^{2}$. Suppose that they have differentiable $S^{1}$-actions of the type $T=\left\{m_{1}, m_{2} ; k_{1}, k_{2}\right\}$ and suppose that $m_{1} \neq m_{2}, m_{1} \neq 2 m_{2}$ and $2 m_{1} \neq m_{2}$. There is a differentiable bundle isomorphism between $Y_{\xi}$ and $Y_{\xi^{\prime}}$ including $S^{1}$-actions, modulo conjugation, if and only if $Y_{\xi}$ and $Y_{\xi}$, are $\sigma$-equivalent with respect to the $S^{1}$-actions, modulo conjugation.

The proof of the corollary is similar to that of Corollary (3.2) and is made by using Proposition (3.3) instead of Proposition (3.1).

Let $\xi_{1}$ be a differentiable complex $(n+1)$-vector bundle over $M^{2}$ and $\xi_{2}$ a differentiable complex $n$-vector bundle over $C P^{2}$. Let $T^{(i)}=\left\{m_{1}^{(i)}, \cdots\right.$, $\left.m_{s^{(i)}}^{(i)} ; k_{1}^{(i)}, \cdots, k_{s^{(i)}}^{(i)}\right\}$ be types of $S^{1}$-actions on $\xi_{i}$ for $i=1,2$.

Proposition (3.5). If we have $n>2$, then the $S^{1}$-manifolds $Y_{\xi_{1}}\left(T^{(1)}\right)$ and $Y_{\xi_{2}}\left(T^{(2)}\right)$ are not $\sigma$-equivalent.

Proof. $\alpha$-invariants are rational functions of $z \in S^{1}$ and invariant under $\sigma$-equivalences. From Theorems (1.1) and (1.2), and from the assumption $n>2$, it follows that $\left.\alpha\left(z, Y_{\xi_{1}}\left(T^{(1)}\right)\right)\right|_{z=0}=0$ and $\left.\alpha\left(z, Y_{\xi_{2}}\left(T^{(2)}\right)\right)\right|_{z=0}=$ $(-1)^{n+1}$. Thus one sees that $\alpha\left(z, Y_{\xi_{1}}\left(T^{(1)}\right)\right)$ and $\alpha\left(z, Y_{\xi_{2}}\left(T^{(2)}\right)\right)$ are different functions of $z \in S^{1}$ and the proof of the proposition is completed.
4. $S^{1}$-bundles over complex projective spaces. In this section, we prove Theorem (1.3) and consider $\sigma$-non-equivalences of the natural $S^{1}$-actions for manifolds of differentiable $S^{1}$-bundles over the complex $p$ dimensional projective space $C P^{p}$.

Let $Y_{\zeta}$ be the manifold of differentiable $S^{1}$-bundle associated to a differentiable complex line bundle $\zeta$ over $C P^{p}$ with the first Chern class $a x \in H^{2}\left(C P^{p} ; Z\right)$, where $x$ is the canonical generator $H^{2}\left(C P^{p} ; Z\right)$. We
denote the manifold $Y_{\zeta}$ with the $S^{1}$-action of the type $\{m ; 1\}$ by $Y_{\zeta}(m)$ for the positive integer $m$.

Proof of Theorem (1.3). Let $\tau\left(C P^{p}\right)$ be the tangent bundle of $C P^{p}$ and we denote the trivial line bundle by 1 . By the definition of the multiplicative sequence $\mathscr{L}$, one obtains

$$
\begin{equation*}
\mathscr{L}\left(C P^{p}\right)=\mathscr{L}\left(\tau\left(C P^{p}\right) \oplus 1\right)=\left(\frac{x / 2}{\tanh x / 2}\right)^{p+1} \tag{9}
\end{equation*}
$$

Let $X_{\zeta}$ be the disk bundle associated to $\zeta$. By the formula (2), we have

$$
\begin{aligned}
\alpha\left(z, Y_{\zeta}(m)\right) & =-\left\{2^{p}\left(\frac{z^{m} e^{a x}+1}{z^{m} e^{a x}-1}\right) \mathscr{L}\left(C P^{p}\right)\right\}\left[C P^{p}\right]+\operatorname{Sign}\left(X_{\zeta}\right) \\
& =-\left\{2^{p}\left(\frac{z^{m} e^{a x}+1}{z^{m} e^{a x}-1}\right)\left(\frac{x / 2}{\tanh x / 2}\right)^{p+1}\right\}\left[C P^{p}\right]+\operatorname{Sign}\left(X_{\zeta}\right) \\
& =-\left\{\frac{x^{p+1}}{2}\left(\frac{z^{m} e^{a x}+1}{z^{m} e^{a x}-1}\right)\left(\frac{e^{x}+1}{e^{x}-1}\right)^{p+1}\right\}\left[C P^{p}\right]+\operatorname{Sign}\left(X_{\zeta}\right) .
\end{aligned}
$$

We calculate the first term by residues of complex functions. We denote the residue of a complex function $f(x)$ at $x_{0}$ by

$$
\operatorname{res}_{x=x_{0}} f d x=\frac{1}{2 \pi i} \int \frac{f(x)}{x-x_{0}} d x,
$$

and we obtain

$$
\begin{aligned}
\left\{\frac{x^{p+1}}{2}\left(\frac{z^{m} e^{a x}+1}{z^{m} e^{a x}-1}\right)\left(\frac{e^{x}+1}{e^{x}-1}\right)^{p+1}\right\}\left[C P^{p}\right] & =\operatorname{res}_{x=0}\left\{\frac{1}{2}\left(\frac{z^{m} e^{a x}+1}{z^{m} e^{a x}-1}\right)\left(\frac{e^{x}+1}{e^{x}-1}\right)^{p+1} d x\right\} \\
& =\operatorname{res}_{v=1}\left\{\frac{1}{2}\left(\frac{z^{m} v^{a}+1}{z^{m} v^{a}-1}\right)\left(\frac{v+1}{v-1}\right)^{p+1} \frac{d v}{v}\right\} \\
& =-\operatorname{res}_{v=0}-\operatorname{res}_{v=\infty}-\operatorname{res}_{v^{a}=z^{-m}}
\end{aligned}
$$

We see that $\operatorname{res}_{v=0}=(-1)^{p} / 2, \operatorname{res}_{v=\infty}=-1 / 2$ for $a \neq 0$ and

$$
\begin{aligned}
& \operatorname{res}_{v^{a}=z^{-m}}=\left\{\begin{array}{l}
0 \\
\frac{1}{a} \sum_{\omega^{a}=z^{m}}\left(\frac{1+\omega}{1-\omega}\right)^{p+1} \\
= \\
\frac{-(-\operatorname{sgn} a)^{p}}{a}(p+1) \sum_{(\mu)}(-1)^{\mu_{1}+\cdots+\mu_{|a|}} \frac{\left(\mu_{1}+\cdots+\mu_{|a|}-1\right)!}{\mu_{1}!\cdots \mu_{|a|}!} \\
\\
\end{array} \quad \times\left(\frac{z^{m}+1}{1-z^{m}}\binom{|a|}{1}\right)^{\mu_{1}}\binom{|a|}{2}^{\mu_{2}}\left(\frac{z^{m}+1}{1-z^{m}}\binom{a \mid}{ 3}\right)^{\mu_{3}} \cdots(*)^{\mu_{|a|}},\right. \\
& \quad \text { for } a \neq 0,
\end{aligned}
$$

where $(\mu)$ is the set $\left\{\left(\mu_{1}, \cdots, \mu_{|a|}\right)\left|\mu_{i} \geqq 0, \mu_{1}+2 \mu_{2}+\cdots+|a| \mu_{|a|}=\right.\right.$ $p+1\}$. On the other hand, we can easily see by (3) that $\operatorname{Sign}\left(X_{\zeta}\right)=$ $(-1)^{p+1} / 2+1 / 2$. Thus we complete the proof of Theorem (1.3).

Let $\xi$ be (i) a differentiable complex $p$-vector bundle over a compact connected oriented 2 dimensional manifold $M^{2}$ for $p>0$ or (ii) a differentiable complex ( $p-1$ )-vector bundle over the complex projective plane $C P^{2}$ for $p>1$. Let $Y_{\xi}(T)$ be the differentiable manifold of the sphere bundle associated to $\xi$, having the $S^{1}$-action of the type $T=\left\{m_{1}, \cdots, m_{s}\right.$; $\left.k_{1}, \cdots, k_{s}\right\}\left(\sum_{i=1}^{s} k_{i}=p\right)$ in the case (i) or of the type $T=\left\{m_{1}, \cdots, m_{s}\right.$; $\left.k_{1}, \cdots, k_{s}\right\}$, ( $\sum_{i=1}^{s} k_{i}=p-1$ ) in the case (ii). By comparing $\alpha$-invariants of the $S^{1}$-actions, we obtain the following proposition.

Proposition (4.1). Let $m, p$ be positive integers and $p>2$. Let $Y_{\xi}(T)$ and $Y_{\zeta}(m)$ be the $S^{1}$-manifolds stated in the above. Suppose that $a>0$ and in the case (ii) $p$ is odd. Then $Y_{\xi}(T)$ can not be $\sigma$-equivalent to $Y_{\zeta}(m)$.

Proof. From Theorems (1.1) and (1.2), and from assumption on $p$, it follows that $\left.\alpha\left(z, Y_{\xi}(T)\right)\right|_{z=0}=0$ or -1 . On the other hand, we have $\left.\alpha\left(z, Y_{\zeta}(m)\right)\right|_{z=0}=1$ by Theorem (1.3) and the assumption $a>0$. Thus one sees that $\alpha\left(z, Y_{\xi}(T)\right) \neq \alpha\left(z, Y_{\zeta}(m)\right)$. Since $\alpha$-invariants are rational functions of $z$, they are different on $S^{1}=\{z| | z \mid=1\} . \quad \alpha$-invariants are invariant under $\sigma$-equivalences and hence we completes the proof of the proposition.

Remark. In the case of Proposition (4.1), $Y_{\xi}(T)$ is not $\sigma$-equivalent to $Y_{\zeta}(m)$, even modulo conjugate isomorphism.

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