Tôhoku Math. Journ.
25 (1973), 461-467.

A PINCHING PROBLEM ON THE SECOND FUNDAMENTAL TENSORS AND SUBMANIFOLDS OF A SPHERE

Dedicated to Professor Shigeo Sasaki on his 60th birthday

MASAFUMI OKUMURA

(Received November 28, 1972; Revised March 12, 1973)

Introduction. In a previous paper [4] the present author proved the following

THEOREM A. *Let M be an n-dimensional compact, connected hypersurface with constant mean curvature immersed in an $(n+1)$-dimensional Riemannian manifold of non-negative constant curvature. If the second fundamental tensor H satisfies*

$$\text{trace } H^2 < \frac{1}{n-1} (\text{trace } H)^2 , \tag{0.1}$$

then M is a totally umbilical hypersurface and consequently a sphere.

Then in [5] we generalized Theorem A to a submanifold of any codimension and proved.

THEOREM B. *Let M be a compact, connected submanifold of dimension n immersed in an $(n+p)$-dimensional Riemannian manifold of non-negative constant curvature and suppose that the connection of the normal bundle is flat. If the mean curvature vector field is parallel with respect to the connection of the normal bundle and the inequality*

$$\sum_{A=1}^{p} \text{trace } H_A^2 < \frac{1}{n-1} \sum_{A=1}^{p} (\text{trace } H_A)^2 \tag{0.2}$$

is satisfied, then M is a totally umbilical submanifold, where H_A's are the second fundamental tensors with respect to unit normals N_A.

The purpose of the present paper is to prove the following

THEOREM. *Let M be a complete, connected submanifold of dimension n $(\geqq 3)$ immersed in an $(n+p)$-dimensional Riemannian manifold of positive constant curvature whose mean curvature vector field is parallel with respect to the connection of the normal bundle. If the second fundamental tensors H_A satisfy* (0.2), *then M is umbilical with respect to the* *mean curvature normal direction. Furthermore, if the ambient manifold is an $(n+p)$-dimensional sphere $(n \geqq 3)$, M is a minimal submanifold of a small sphere.*

1. Preliminaries. Let M be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold $\bar{M}$ of constant curvature c. The Riemannian connections of M and $\bar{M}$ are denoted by ∇ and $\bar{\nabla}$ respectively, whereas the connection in the normal bundle of M in $\bar{M}$ is denoted by D. Let $N_1, \cdots, N_p$ be mutually orthogonal unit normal vectors at a point $p \in M$ and extend them to local vector fields in a neighborhood of p. We define $-H_A X\,(A = 1, 2, \cdots, p)$ to be the tangential components of $\bar{\nabla}_X N_A$ for $X \in T_p(M)$ and call H_A the second fundamental tensor with respect to N_A. We know that the H_A's are symmetric linear transformations on $T_p(M)$. Then we have the following equations of Gauss and Weingarten:

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{A=1}^{p} g(H_A X, Y) N_A\,, \tag{1.1}$$

$$\bar{\nabla}_X N_A = -H_A X + D_X N_A\,, \tag{1.2}$$

where g is the Riemannian metric of M. Since $D_X N_A$ is normal to M, it is expressed as a linear combination of N_A, that is,

$$D_X N_A = \sum_{B=1}^{p} S_{AB}(X) N_B\,. \tag{1.3}$$

The ambient manifold being of constant curvature c, the curvature tensor $R(X, Y)Z$, scalar curvature K, and the normal curvature R^N are respectively given by

$$\begin{aligned} R(X, Y)Z = {} & c\{g(Y, Z)X - g(X, Z)Y\} \\ & + \sum_{A=1}^{p} \{g(H_A Y, Z)H_A X - g(H_A X, Z)H_A Y\}\,, \end{aligned} \tag{1.4}$$

$$K = n(n-1)c + \sum_{A=1}^{p} (\operatorname{trace} H_A)^2 - \sum_{A=1}^{p} \operatorname{trace} H_A^2\,, \tag{1.5}$$

$$\begin{aligned} R^N(X, Y)N_A &= \sum_{B=1}^{p} g([H_A, H_B]X, Y)N_B \\ &= \sum_{B=1}^{p} \{(\nabla_X S_{AB})Y - (\nabla_Y S_{AB})X \\ &\quad + \sum_{C=1}^{p} (S_{AC}(Y)S_{CB}(X) - S_{AC}(X)S_{CB}(Y))\}N_B\,, \end{aligned} \tag{1.6}$$

where we put

$$[H_A, H_B]X = H_A H_B X - H_B H_A X\,.$$

The mean curvature vector N is defined by

$$N = \sum_{A=1}^{p} (\text{trace } H_A) N_A , \tag{1.7}$$

and it is well known that N is independent of the choice of unit normal vector to M.

For some H_A, if there exists a function ρ_A such that

$$H_A X = \rho_A X , \tag{1.8}$$

at each point of M, we call M is umbilical with respect to normal N_A at p.

2. Lemmas. First we state the following

LEMMA 1. [5]. *Let $a_1, a_2, \cdots, a_n$ and k be $n+1$ $(n \geqq 2)$ real numbers satisfying the inequality*

$$\sum_{i=1}^{n} a_i^2 + k < \frac{1}{n-1} (\sum_{i=1}^{n} a_i)^2 , \tag{2.1}$$

then for any pair of distinct i and $j = 1, 2, \cdots, n$, we have

$$k < 2a_i a_j . \tag{2.2}$$

PROOF. Since $(\sum_{i=1}^n a_i)^2 = \sum_{i=1}^n a_i^2 + 2\sum_{i<j}^n a_i a_j$, we have from (2.1),

$$(n-2)\sum_{i=1}^{n} a_i^2 - 2\sum_{i<j}^{n} a_i a_j + (n-1)k < 0 ,$$

that is,

$$(n-2)a_n^2 - 2a_n\Big(\sum_{i=1}^{n-1} a_i\Big) + (n-2)\sum_{i=1}^{n-1} a_i^2 - 2\sum_{i<j\leqq n-1} a_i a_j + (n-1)k < 0 . \tag{2.3}$$

We regard that (2.3) is a quadratic inequality with respect to a_n. Then, a_n being a real number, the discriminant of (2.3) must be positive. Thus we get

$$\Big(\sum_{i=1}^{n-1} a_i\Big)^2 > (n-2)\Big\{(n-1)\Big(\sum_{i=1}^{n-1} a_i^2 + k\Big) - \Big(\sum_{i=1}^{n-1} a_i\Big)^2\Big\} ,$$

from which

$$\sum_{i=1}^{n-1} a_i^2 + k < \frac{1}{n-2}\Big(\sum_{i=1}^{n-1} a_i\Big)^2 . \tag{2.4}$$

Continuing the same process $(n-2)$-times, we have (2.2).

Next we prove the

LEMMA 2. *Let M be an n-dimensional submanifold of a Riemannian manifold M of constant curvature c. If the second fundamental tensors*

H_A satisfy (0.2) *at a point $p \in M$, then the sectional curvature $R(i, j)$ for the plane section spanned by E_i and E_j is greater than c at p.*

PROOF. From (0.2) it follows that M has no minimal point. So we can choose the first unit normal vector N_1 to M in the direction of the mean curvature vector N. Then by the definition of the mean curvature vector we can easily see that

$$\text{trace}\, H_A = 0\,, \qquad A = 2, 3, \cdots, p\,. \tag{2.5}$$

Let $E_1, E_2, \cdots, E_n$ be orthonormal eigenvectors of the second fundamental tensor H_1 and $a_1, a_2, \cdots, a_n$ corresponding eigenvalues to $E_1, E_2, \cdots, E_n$. Then denoting components of H_A $(A = 2, \cdots, p)$ by λ^A_{ji}, we have, from (0.2) and (2.5),

$$\frac{1}{n-1}\Big(\sum_{i=1}^{n} a_i\Big)^2 > \sum_{i=1}^{n} a_i^2 + \sum_{A=2}^{p}\sum_{i,k=1}^{n} (\lambda^A_{ik}\lambda^A_{ik})\,. \tag{2.6}$$

Applying Lemma 1 to (2.6), we have

$$2a_i a_j > \sum_{A=2}^{p}\sum_{i,k=1}^{n} (\lambda^A_{ik}\lambda^A_{ik}) \geqq \sum_{A=2}^{p} \{(\lambda^A_{ii})^2 + 2(\lambda^A_{ij})^2 + (\lambda^A_{jj})^2\} \geqq 2\sum_{A=2}^{p} \{|\lambda^A_{ii}\lambda^A_{jj}| + (\lambda^A_{ij})^2\}\,.$$

Thus we have

$$a_i a_j > \sum_{A=2}^{p} \{|\lambda^A_{ii}\lambda^A_{jj}| + (\lambda^A_{ij})^2\}\,. \tag{2.7}$$

On the other hand, by (1.4), the sectional curvature $R(i, j)$ for the plane section spanned by E_i and E_j is given by

$$R(i, j) = g(R(E_i, E_j)E_j, E_i) = c + a_i a_j + \sum_{A=2}^{p} \{\lambda^A_{ii}\lambda^A_{jj} - (\lambda^A_{ij})^2\}\,. \tag{2.8}$$

Combining (2.7) and (2.8) we have

$$R(i, j) > c + \sum_{A=2}^{p} \{|\lambda^A_{ii}\lambda^A_{jj}| + \lambda^A_{ii}\lambda^A_{jj}\} \geqq c\,.$$

This completes the proof.

LEMMA 3. *Let M be a complete, connected submanifold of dimension $n > 2$ immersed in an $(n + p)$-dimensional Riemannian manifold of positive constant curvature c. If the second fundamental tensors H_A satisfy* (0.2) *on M, then M is compact.*

PROOF. Let X be a unit vector and E_i be a unit eigenvector of H_1 which corresponds to the eigenvalue a_i. Then, putting $X = \sum_{j=1}^{n} x^j E_j$, the sectional curvature for the plane section spanned by X and E_i is

$$g(R(X, E_i)E_i, X) = c\Big\{\sum_{j=1}^{n} (x^j)^2 - (x^i)^2\Big\} + a_i \sum_{j=1}^{n} a_j(x^j)^2 - (a_i x^i)^2$$
$$+ \sum_{A=2}^{p} \Big\{\lambda_{ii}^A \sum_{j,k=1}^{n} \lambda_{jk}^A x^j x^k - \Big(\sum_{j=1}^{n} \lambda_{ji}^A x^j\Big)^2\Big\} .$$

Thus Ricci tensor Ric (X, X) becomes

$$\begin{aligned} \mathrm{Ric}\,(X, X) &= \sum_{i=1}^{n} g(R(X, E_i)E_i, X) \\ &= (n-1)c + \sum_{i=1}^{n} \{a_i a_1 (x^1)^2 + \cdots + \widehat{a_i{}^2(x^i)^2} + \cdots + a_i a_n (x^n)^2\} \\ &\quad - \sum_{A=2}^{p} \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} \lambda_{ji}^A x^j\Big)^2 , \end{aligned}$$

because of $\sum_{i=1}^{n} \lambda_{ii}^A = 0$, where the roof "$\wedge$" denotes a term which will be omitted.

Substituting

$$2a_i a_h > \sum_{A=2}^{p} \sum_{j,k=1}^{n} \lambda_{jk}^A \lambda_{jk}^A$$

into the last equation and making use of $\sum_{i=1}^{n} (x^i)^2 = 1$, we have

$$\begin{aligned} \mathrm{Ric}\,(X, X) &> (n-1)c + \sum_{A=2}^{p} \Big\{\frac{n-1}{2} \sum_{j,k=1}^{n} \lambda_{jk}^A \lambda_{jk}^A - \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} \lambda_{ji}^A x^j\Big)^2\Big\} \\ &\geqq (n-1)c + \sum_{A=2}^{p} \Big\{\frac{n-1}{2} \sum_{j,k=1}^{n} \lambda_{jk}^A \lambda_{jk}^A - \sum_{i=1}^{n} (\lambda_{ji}^A \lambda_{ji}^A) \sum_{k=1}^{n} (x^k)^2\Big\} \\ &= (n-1)c + \frac{n-3}{2} \sum_{A=2}^{p} \sum_{j,k=1}^{n} \lambda_{jk}^A \lambda_{jk}^A \geqq (n-1)c > 0 , \end{aligned}$$

where we have used Cauchy-Schwarz inequality. Thus, from Myers' theorem [3], M is compact.

3. Proof of Theorem. Let f be the square of the length of the second fundamental tensor with respect to N_1, that is,

$$f = \mathrm{trace}\, H_1^2 . \tag{3.1}$$

The Laplacian for f is given by

$$\frac{1}{2}\Delta f = \mathrm{trace}\,(\Delta' H_1)H_1 + g(\nabla H_1, \nabla H_1) , \tag{3.2}$$

where

$$(\Delta' H_1)_X = \sum_{i=1}^{n} \{\nabla_{E_i}(\nabla_{E_i} H_1) - \nabla_{\nabla_{E_i} E_i} H_1\} ,$$

and we extend the metric g to the tensor space in the standard fashion.

Using recent results of J. A. Erbacher [2], we have

$$(3.3)\quad \begin{aligned} \Delta' H_1 = {} & ncH_1 - c(\operatorname{trace} H_1)I + \sum_{A=1}^{p} (\operatorname{trace} H_A)H_1H_A \\ & - \sum_{A=1}^{p} (\operatorname{trace} H_AH_1)H_A + \sum_{A=1}^{p} [H_A, H_1H_A] + \sum_{A=1}^{p} H_A[H_1, H_A] \\ & + \sum_{i=1}^{n}\sum_{A=1}^{p} (\nabla_{E_i}S_{1A})(E_i)H_A + 2\sum_{i=1}^{n}\sum_{A=1}^{p} S_{1A}(E_i)\nabla_{E_i}H_A \\ & - \sum_{i=1}^{n}\sum_{A,B=1}^{p} S_{1A}(E_i)S_{AB}(E_i)H_B\,. \end{aligned}$$

By the assumption of Theorem, N_1 is parallel with respect to the connection of the normal bundle and so we have

$$D_X N_1 = \sum_{A=1}^{p} S_{1A}(X)N_A = 0\ ,$$

from which $S_{1A} = 0$. Consequently we get $[H_1, H_A] = 0$.

Substituting these into (3.3) and making use of (3.2), we have

$$(3.4)\quad \begin{aligned} \frac{1}{2}\Delta \operatorname{trace} H_1^2 = {} & nc \operatorname{trace} H_1^2 - c(\operatorname{trace} H_1)^2 + (\operatorname{trace} H_1)(\operatorname{trace} H_1^3) \\ & + \sum_{A=2}^{p} \operatorname{trace}(H_AH_1)^2 - (\operatorname{trace} H_1^2)^2 - \sum_{A=2}^{p} (\operatorname{trace} H_A^2H_1^2) \\ & - \sum_{A=2}^{p} (\operatorname{trace} H_AH_1)^2 + g(\nabla H_1, \nabla H_1)\,. \end{aligned}$$

Thus at a point $p \in M$, we have

$$(3.5)\quad \begin{aligned} \frac{1}{2}\Delta \operatorname{trace} H_1^2 = {} & nc\Big(\sum_{i=1}^{n} a_i^2 - \frac{1}{n}\Big(\sum_{i=1}^{n} a_i\Big)^2\Big) \\ & + \sum_{i<j} (a_ia_j + \sum_{A=2}^{p} (\lambda_{ii}^A\lambda_{jj}^A - (\lambda_{ij}^A)^2))(a_i - a_j)^2 + g(\nabla H_1, \nabla H_1) \\ \geqq {} & nc\Big(\sum_{i=1}^{n} a_i^2 - \frac{1}{n}\Big(\sum_{i=1}^{n} a_i\Big)^2\Big) + (R(i,j) - c)(a_i - a_j)^2 + g(\nabla H_1, \nabla H_1)\,, \end{aligned}$$

because of (2.7), (2.8) and the fact that $\sum_{j>i} a_i^2\lambda_{ii}^A\lambda_{ij}^A = 0$ for $A \geqq 2$.

From Lemma 3, M is compact and so by Hopf's theorem and Lemma 2, we see that $\nabla H_1 = 0$ and

$$(3.6)\qquad a_1 = a_2 = \cdots = a_n \neq 0\,.$$

This shows that M is umbilical with respect to the mean curvature vector H_1 and H_1 is parallel. Furthermore, if the ambient manifold $\bar{M}$ is an $(n + p)$-dimensional sphere in Euclidean $n + p + 1$ space E^{n+p+1}, then, using $D_X H_1 = 0$, we have, for example by [1] or [7], that M is a minimal submanifold of a sphere S^{n+p-1}. This completes the proof.

REMARK. From (1.5), the condition (0.2) can be written as

$$K > n(n-1)c + (n-2)\sum_{A=1}^{n} \operatorname{trace} H_A^2 . \tag{3.7}$$

BIBLIOGRAPHY

[1] B. Y. CHEN AND K. YANO, Pseudo-umbilical submanifolds in a Riemannian manifold of constant curvature, Differential Geometry, in honor of K. Yano, Kinokuniya Tokyo, 1972, 61–71.

[2] J. ERBACHER, Isometric immersions of constant mean curvature and triviality of the normal connection, Nagoya Math. J., 45 (1971), 139–165.

[3] S. MYERS, Riemannian manifolds with positive mean curvature, Duke Math. J., 8 (1941), 401–404.

[4] M. OKUMURA, Hypersurfaces and a pinching problem on the second fundamental tensor, to appear.

[5] M. OKUMURA, Submanifolds and a pinching problem on the second fundamental tensor, Transactions of A.M.S. 178 (1973), 285-291.

[6] J. SIMONS, Minimal varieties in riemannian manifolds, Ann. of Math., 88 (1968), 62–105.

[7] B. SMYTH, Submanifolds of constant mean curvature, Mathematics Annalen 205 (1973), 265-280.

MICHIGAN STATE UNIVERSITY
AND
SAITAMA UNIVERSITY