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# POLAR DECOMPOSITION FOR ISOMORPHISMS OF $C^*$-ALGEBRAS

Dedicated to Professor Masanori Fukamiya on his 60th birthday

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**1. Introduction and preliminaries.** In [8], it is shown in essence that any isomorphism of von Neumann algebras is decomposed uniquely as the product of a *-isomorphism and an automorphism implemented by an invertible positive element. This statement bears a remarkable resemblance to the polar decomposition theorem for operators on Hilbert spaces, so we prefer to call it the polar decomposition theorem for isomorphisms of von Neumann algebras.

But, the author thinks that *this* polar decomposition for isomorphisms must be stated more tidily and more generaly; and, the main purposes of this paper are to state it in a satisfactory fashion and to give an application.

After introducing a concept of positivity along with that of self-duality for automorphisms of $C^*$-algebras, we shall state in § 7 the main result: *Any isomorphism of $C^*$-algebras is decomposed uniquely as the product of a *-isomorphism and a positive automorphism, and this decomposition is norm-continuous.* This seems to give an answer to Kaplansky's proposal in [6] and must be of large practical importance in investigations of automorphisms of $C^*$-algebras and their groups.

Several interesting secondary results will be given, also. Particularly, in the last section, characterizations of positive automorphisms will be discussed.

When we study automorphisms of $C^*$-algebras, we must take care of their derivations. A *derivation* $\delta$ of a $C^*$-algebra $A$ means an operator on $A$ which satisfies the condition

$$\delta(xy) = \delta(x)y + x\delta(y) \text{ for } x \in A \text{ and } y \in A ;$$

an *inner* derivation of $A$ means a derivation of the form

$$\mathrm{ad}\, a(x) = ax - xa \quad \text{for} \quad x \in A ,$$

with $a$ an element of $A$; and, if $A$ acts on a Hilbert space $H$, the *spatial* derivation implemented by an operator $s$ on $H$ means the derivation $\mathrm{ad}\, s \mid A$ obtained by restricting to $A$ the derivation $\mathrm{ad}\, s$ of $B(H)$, the $C^*$-algebra of all bounded operators on $H$, where $\mathrm{ad}\, s(A) \subseteqq A$.

It is shown by Sakai in [11], and by Kadison in [5], that any derivation of a von Neumann algebra is inner. Moreover, it has been known that many other $C^*$-algebras than von Neumann algebras has this property. In fact, in [12] Sakai showed this property for simple $C^*$-algebras with identities (see also [13] and [14]) and recently in [9] Olesen for $AW^*$-algebras. This property for $C^*$-algebras is so favorable for us, then we agree to employ for convenience' sake

DEFINITION 1.1. If any derivation of a $C^*$-algebra is inner, then we say that it has the property $(D)$.

It also is shown that any derivation of a $C^*$-algebra acting on a Hilbert space is extended to a derivation of the von Neumann algebra generated by the $C^*$-algebra and hence it is spatial (see [5] and [11]).

The subsequent contents are: Spatial isomorphisms (§ 2), Canonical representations (§ 3), Dual isomorphisms (§ 4), Inner automorphisms (§ 5), Self-dual automorphisms and positive automorphisms (§ 6), Polar decomposition for isomorphisms (§ 7) and More on positive automorphisms (§ 8).

2. **Spatial isomorphisms.** Suppose that $H$ and $K$ are Hilbert spaces, $s$ an invertible bounded operator from $H$ onto $K$. Then the mapping

$$\mathrm{Ad}\, s(x) = sxs^{-1} \quad \text{for} \quad x \in B(H)$$

is an isomorphism of $B(H)$ onto $B(K)$. Suppose in addition that $A$ is a $C^*$-algebra acting on $H$, $B$ a $C^*$-algebra acting on $K$ which is just the image of $A$ under $\mathrm{Ad}\, s$. Then the restriction $\mathrm{Ad}\, s \mid A$ of $\mathrm{Ad}\, s$ to $A$ is an isomorphism of $A$ onto $B$. If an isomorphism has this form, we say that it is *spatial.*

The following theorem is to the effect that every spatial *-isomorphism is implemented by some isometric operator:

THEOREM 2.1 (cf. Th. 1 in [8]). *Let $A$ be a $C^*$-algebra acting on a Hilbert space $H$, $B$ a $C^*$-algebra acting on a Hilbert space $K$ and $\rho$ a spatial isomorphism of $A$ onto $B$. Then, $\rho$ is a *-isomorphism if and only if it is of the form*

$$\rho = \mathrm{Ad}\, u \mid A\ ,$$

*where $u$ is an isometric operator from $H$ onto $K$.*

PROOF. Only the necessity is important. Suppose that $\rho$ is a spatial *-isomorphism implemented by an invertible bounded operator $s$ from $H$

onto $K$. Then,

$$sx^*s^{-1} = \mathrm{Ad}\, s(x^*) = (\mathrm{Ad}\, s(x))^* = (sxs^{-1})^* = (s^*)^{-1}x^*s^* ,$$

and hence,

$$s^*sx = xs^*s$$

for each $x$ in $A$. Thus, $h = (s^*s)^{1/2}$ commutes with each element of $A$ and therefore, putting $u = sh^{-1}$, we have

$$\mathrm{Ad}\, u(x) = uxu^{-1} = sh^{-1}xhs^{-1} = sxs^{-1} = \mathrm{Ad}\, s(x)$$

for each $x$ in $A$. It is easy to see that $u$ is an isometric operator and the proof is completed.

**3. Canonical representations.** It is known that the quotient space $A/I$ of a $C^*$-algebra $A$ by a maximal left ideal $I$ of $A$ is a Hilbert space and the left regular representation $\varphi_I$ of $A$ on $A/I$ is a *-representation (see [4] and [15]). We denote by $H_A$ the Hilbert space $\sum \bigoplus_I A/I$ and by $\Phi_A$ the *-representation $\sum \bigoplus_I \varphi_I$ which is necessarily faithful, where $I$ runs over the set $\mathscr{L}_A$ of all maximal left ideals of $A$. We call $\Phi_A$ the *canonical representation* of $A$ and $H_A$ the *canonical representation space.*

Suppose that $\rho$ is an isomorphism of a $C^*$-algebra $A$ onto a $C^*$-algebra $B$. We then call an isomorphism $\Phi_B\rho\Phi_A^{-1}$ of $\Phi_A(A)$ onto $\Phi_B(B)$ the *canonical representation* of $\rho$. We are now going to see that it must be spatial.

With $I \in \mathscr{L}_A$, let us denote in the following by $x \mapsto x_I$ the quotient mapping of $A$ onto $A/I = A_I$.

LEMMA 3.1. (a) *$\rho(I) \in \mathscr{L}_B$ for each $I \in \mathscr{L}_A$ and the mapping $\mathscr{L}_A \ni I \mapsto \rho(I) \in \mathscr{L}_B$ is one-to-one and onto; and*

(b) *$\rho(x_1)_{\rho(I)} = \rho(x_2)_{\rho(I)}$ if $x_1 \in A$, $x_2 \in A$, $I \in \mathscr{L}_A$ and $(x_1)_I = (x_2)_I$.*

This lemma enables us to define for each $I \in \mathscr{L}_A$ the operator $s_I$ of $A_I$ onto $B_{\rho(I)}$ by

$$s_I(x_I) = \rho(x)_{\rho(I)}$$

with $x$ in $A$. From

$$\| s_I(x_I) \| = \inf_{x_1 \in I} \| \rho(x) + \rho(x_1) \| = \| \rho \| \inf_{x_1 \in I} \| x + x_1 \| = \| \rho \| \| x_I \| ,$$

it follows that $s_I$ is bounded with norm less than or equal to $\| \rho \|$. Therefore, so does the sum $s$ of all $s_I$'s. Moreover, it turns out to be invertible and satisfy the relation

$$\Phi_B\rho = \mathrm{Ad}\, s\Phi_A .$$

In fact,

$$\begin{aligned}\varphi_{\rho(I)}(\rho(a))s_I(x_I) &= \varphi_{\rho(I)}(\rho(a))(\rho(x)_{\rho(I)}) \\ &= (\rho(a)\rho(x))_{\rho(I)} \\ &= \rho(ax)_{\rho(I)} \\ &= s_I((ax)_I) \\ &= s_I\varphi_I(a)(x_I)\end{aligned}$$

and hence for each $a$ in $A$,

$$\varphi_{\rho(I)}(\rho(a))s_I = s_I\varphi_I(a) .$$

Now we can state

THEOREM 3.2 (Gardner's theorem). *The canonical representation of an isomorphism of $C^*$-algebras is necessarily spatial. In fact, under the foregoing notations,*

$$\Phi_B\rho\Phi_A^{-1} = \mathrm{Ad}\, s \mid \Phi_A(A) .$$

4. **Dual isomorphisms.** Given an isomorphism of $C^*$-algebras, we define its dual isomorphism as follows:

DEFINITION 4.1. Suppose that $\rho$ is an isomorphism of a $C^*$-algebra $A$ onto a $C^*$-algebra $B$. Then we call the isomorphism $\rho'$ of $B$ onto $A$ defined by the relation

$$\rho'(y)^* = \rho^{-1}(y^*) \quad \text{for} \quad y \in B$$

the dual isomorphism of $\rho$.

It is easy to see the relations $(\rho')^{-1} = (\rho^{-1})'$, $\rho'' = \rho$, $\|\rho'\| = \|\rho^{-1}\|$ and $(\sigma\rho)' = \rho'\sigma'$ with $\sigma$ an isomorphism of $B$ onto a $C^*$-algebra.

It is meaningful for us to remark that $\rho$ is a $*$-isomorphism if and only if $\rho'\rho$ is the identity automorphisms of $A$. This fact seems to suggest us that $*$-isomorphisms behave like isometric operators.

If $A$ acts on a Hilbert space $H$, $B$ on a Hilbert space $K$ and $\rho$ is a spatial isomorphism of $A$ onto $B$ implemented by an invertible bounded operator $s$ from $H$ onto $K$, then we know that the dual isomorphism $\rho'$ of $\rho$ is spatial implemented by $s^*$, because for each $y$ in $B$ we have

$$\mathrm{Ad}\, s^*(y) = s^*y(s^*)^{-1} = (s^{-1}y^*s)^* = (\mathrm{Ad}\, s)'(y) .$$

5. **Inner automorphisms.** Suppose that $A$ is a $C^*$-algebra with identity and $a$ an invertible element of $A$. Then the mapping

$$\mathrm{Ad}\, a(x) = axa^{-1} \quad \text{for} \quad x \in A$$

is an automorphism of $A$. We say that an automorphism of $A$ is *inner* if it is written in this form.

The following lemmas are proved as in [8] by considering the left and the right regular representations of $A$. $Sp(\ )$ denotes the spectrum of ( ).

LEMMA 5.1. *Let $a$ be an invertible element of a $C^*$-algebra with identity, then*

$$Sp\,(\mathrm{Ad}\,a) \subseteqq Sp(a)Sp(a)^{-1} = \{\lambda\mu^{-1} \colon \lambda \in Sp(a) \text{ and } \mu \in Sp(a)\}\ .$$

LEMMA 5.2. *Let $a$ be an element of a $C^*$-algebra, then*

$$Sp(\mathrm{ad}\,a) \subseteqq Sp(a) - Sp(a) = \{\lambda - \mu \colon \lambda \in Sp(a) \text{ and } \mu \in Sp(a)\}\ .$$

In what follows, Log denotes the principal branch of the logarithm on the plane slit along the negative half-axis.

LEMMA 5.3 (cf. [16]). *Suppose that $A$ is a $C^*$-algebra with identity and $a$ an element of $A$. Then,*

(a) $$\mathrm{Ad}\exp a = \exp\mathrm{ad}\,a\ ; \quad \textit{and}$$

(b) *the negative half-axis does not pass through the spectrum of* $\mathrm{Ad}\,a$ *and*

$$\mathrm{Log}\,\mathrm{Ad}\,a = \mathrm{ad}\log a$$

*provided that the spectrum of $a$ lies in an open half-plane $\Omega$ whose boundary is a line through the origin, where* log *is any branch of the logarithm on $\Omega$.*

PROOF. (a) Define $\rho(t)$ for each real $t$ by

$$\rho(t) = \mathrm{Ad}\exp(ta)\ ,$$

then $\{\rho(t)\}$ is a norm-continuous one-parameter group of automorphisms of $A$ and by the standard computations it is seen that its generator is certainly $\mathrm{ad}\,a$. Therefore we have for each $t$,

$$\mathrm{Ad}\exp(ta) = \rho(t) = \exp(t\,\mathrm{ad}\,a)$$

(see e.g. Chap. 9 of [3]) and in particular,

$$\mathrm{Ad}\exp a = \rho(1) = \exp\mathrm{ad}\,a\ .$$

(b) We can choose a suitable real number $\theta$ such that $|\theta + \mathrm{Im}\,\lambda| < \pi$ whenever $\lambda \in Sp(\log a)$. Then the negative half-axis does not pass through the spectrum of $b = (\exp i\theta)a$. We can find by Lorch's theorem (see e.g. [7]) a finite number of orthogonal idempotents $e_k$'s in the closed subalgebra $B$ of $A$ generated by $a$ and the identity 1 of $A$, satisfying the relation

$$\operatorname{Log} b = i\theta 1 + \log a + 2\pi i \sum_k n_k e_k ,$$

where $n_k$'s are non-zero integers. Suppose that $f(e_k) \neq 0$ for some multiplicative functional $f$ on $B$. Then $f(e_k) = 1$ since

$$f(e_k)(f(e_k) - 1) = f(e_k^2) - f(e_k) = f(e_k) - f(e_k) = 0 .$$

Thus for $l \neq k$,

$$f(e_k e_l) = f(e_k)f(e_l) = 0$$

and hence $f(e_l) = 0$. Therefore,

$$f(\operatorname{Log} b) = i\theta + f(\log a) + 2\pi i n_k .$$

But this is impossible because both $Sp(\operatorname{Log} b)$ and $Sp(i\theta 1 + \log a)$ lie in the strip $\{\lambda: |\operatorname{Im} \lambda| < \pi\}$. It follows that $f(e_k) = 0$ for every multiplicative functional $f$ on $B$. Since any idempotent in the radical must be zero, $e_k = 0$ and we conclude that

$$\operatorname{Log} b = i\theta 1 + \log a .$$

Put next $c = \operatorname{Log} b$. Then $b = \exp c$ and by (a),

$$\operatorname{Ad} a = \operatorname{Ad} b = \operatorname{Ad} \exp c = \exp \operatorname{ad} c .$$

Thus by Lorch's theorem again, we have

$$\operatorname{Log} \operatorname{Ad} a = \operatorname{ad} c + 2\pi i \sum_k m_k \varepsilon_k ,$$

where $\varepsilon_k$'s are orthogonal idempotents in the norm-closed algebra of operators on $A$ generated by $\operatorname{ad} c$ and the identity operator on $A$ and $m_k$'s non-zero integers.

From Lemma 5.1 we know

$$Sp(\operatorname{Ad} a) = Sp(\operatorname{Ad} b) \subseteqq Sp(b)Sp(b)^{-1} \subseteqq \boldsymbol{C}\backslash\text{“the negative half-axis”} ,$$

where $\boldsymbol{C}$ is the plane, and hence,

$$Sp(\operatorname{Log} \operatorname{Ad} a) \subseteqq \{\lambda \in \boldsymbol{C}: |\operatorname{Im} \lambda| < \pi\} .$$

On the other hand, from Lemma 5.2 we know

$$Sp(\operatorname{ad} c) \subseteqq Sp(c) - Sp(c) \subseteqq \{\lambda \in \boldsymbol{C}: |\operatorname{Im} \lambda| < \pi\} .$$

Thus the arguments analogous to the foregoing apply to show

$$\operatorname{Log} \operatorname{Ad} a = \operatorname{ad} c = \operatorname{ad} \log a .$$

Now the proof is completed.

A consequence of Lemma 5.3 is the following

THEOREM 5.4 (cf. Th. 3.5 in [2] and Th. 5 in [8]). *Let $A$ be a $C^*$-*

*algebra with identity, $a$ an invertible element of $A$ of which spectrum lies in an open half-plane $\Omega$ whose boundary is a line through the origin and $S$ a closed subspace of $A$. If the inner automorphism implemented by $a$ leaves $S$ invariant, then* $\mathrm{ad}\log a$ *and, for any real $t$,* $\mathrm{Ad}\exp(t\log a)$ *leave $S$ invariant, where* log *denotes any branch of the logarithm on $\Omega$.*

PROOF. By Runge's theorem there exists a sequence $\{p_n\}$ of polynomials which converges to Log uniformly on each compact set containing $Sp(\mathrm{Ad}\,a)$ and being contained in $C\backslash$"the negative half-axis". Hence $\{p_n(\mathrm{Ad}\,a)\}$ norm-converges as $n \to \infty$ to $\mathrm{Log\,Ad}\,a$ which coincides with $\mathrm{ad}\log a$ by Lemma 5.3. Therefore, $\mathrm{ad}\log a$ leaves $S$ invariant because so does each $p_n(\mathrm{Ad}\,a)$.

We have for any $t$,

$$\mathrm{Ad}\exp(t\log a) = \exp(\mathrm{ad}\,t\log a) = \exp(t\,\mathrm{ad}\log a)$$

by Lemma 5.3 again, so we conclude that $\mathrm{Ad}\exp(t\log a)$ leaves $S$ invariant, completing the proof.

Another consequence of Lemma 5.3 is the next theorem which gives a mild sufficient condition for a spatial automorphism of a $C^*$-algebra with property $(D)$ to be inner.

LEMMA 5.5. *A spatial derivation $\delta$ of a $C^*$-algebra acting on a Hilbert space is skew-adjoint, that is, satisfies*

$$\delta(x^*) = -\delta(x)^* \text{ for each element } x$$

*if and only if it is implemented by a self-adjoint operator; and this operator can be found in the $C^*$-algebra provided $\delta$ is inner.*

The proof of this lemma is easy and omitted (see [8]).

THEOREM 5.6 (cf. Cor. 7 in [8]). *Let $A$ be a $C^*$-subalgebra of a $C^*$-algebra $B$ with identity, which contains the identity of $B$ and has the property $(D)$. Let moreover $a$ be an invertible element of $B$ with spectrum in an open half-plane whose boundary is a line through the origin. If the inner automorphism implemented by $a$ leaves $A$ invariant, then* $\mathrm{Ad}\,a \mid A$ *is an inner automorphism of $A$; and it is implemented by an invertible positive element of $A$ provided $a$ is positive.*

PROOF. It follows from Theorem 5.4 that $\mathrm{Ad}\,a \mid A$ is an automorphism of $A$ and $\mathrm{ad}\log a$ leaves $A$ invariant, where log is a branch of the logarithm on a domain containing $Sp(a)$. Since $A$ has the property $(D)$, there exists an element $b$ in $A$ such that $\mathrm{ad}\log a \mid A = \mathrm{ad}\,b$. Then Lemma 5.3 applies to obtain

$$\mathrm{Ad}\,a \mid A = \exp \mathrm{ad}\,b = \mathrm{Ad}\exp b$$

and $\exp b$ is of course in $A$.

When $a$ is positive, by Lemma 5.5, we can make the above $b$ self-adjoint; and hence $\exp b$ positive. Then the proof is completed.

**6. Self-dual automorphisms and positive automorphisms.** Here we introduce the following

DEFINITION 6.1. An automorphism $\eta$ of a $C^*$-algebra is said to be self-dual if $\eta = \eta'$, and positive if it is self-dual and its spectrum consists of non-negative real numbers.

It is obvious from the comment in the last paragraph of § 4 that a spatial automorphism implemented by an invertible self-adjoint operator is self-dual, and from Lemma 5.1 that an inner automorphism implemented by an invertible positive element is positive.

LEMMA 6.2. *Let $\tau$ be a bounded operator on a Banach space $X$ and $S$ a closed subspace of $X$ invariant under $\tau$. If the spectrum of $\tau$ does not separate the plane, then the spectrum of the restriction $\tau \mid S$ of $\tau$ to $S$ is contained in the spectrum of $\tau$.*

PROOF. Suppose $\mathfrak{A}$ is the norm-closed subalgebra of $B(X)$, the algebra of all bounded operators on $X$, generated by $\tau$ and the identity operator $\iota$. Since any operator in $\mathfrak{A}$ leaves $S$ invariant and the restriction $(\ )\mid S$ of operators in $\mathfrak{A}$ to $S$ is a homomorphism, the spectrum $Sp_{\mathfrak{A}}(\tau)$ of $\tau$ with respect to $\mathfrak{A}$ contains the spectrum $Sp(\tau \mid S)$ of $\tau \mid S$. On the other hand, the spectrum $Sp(\tau)$ of $\tau$ coincides with $Sp_{\mathfrak{A}}(\tau)$ because $Sp(\tau)$ does not separate the plane (see Th. (1.6.13) in [10]). Then the proof is completed.

Now the following is obvious:

LEMMA 6.3. *Let $A$ be a $C^*$-subalgebra of a $C^*$-algebra $B$. If $\eta$ is a self-dual automorphism of $B$ which leaves $A$ invariant and the restriction $\eta \mid A$ of $\eta$ to $A$ is an automorphism of $A$, then $\eta \mid A$ is self-dual; and it is positive provided $\eta$ is positive.*

Suppose that $\rho$ is an isomorphism of a $C^*$-algebra $A$ onto a $C^*$-algebra $B$. Then, there is an invertible bounded operator $s$ from $H_A$ onto $H_B$ which satisfies

$$\Phi_B \rho \Phi_A^{-1} = \operatorname{Ad} s \mid \Phi_A(A) .$$

Since

$$\Phi_A \rho' \Phi_B^{-1} = (\Phi_B \rho \Phi_A^{-1})' = \operatorname{Ad} s^* \mid \Phi_B(B) ,$$

we have

$$\begin{aligned}\Phi_A\rho'\rho\Phi_A^{-1} &= (\Phi_A\rho'\Phi_B^{-1})(\Phi_B\rho\Phi_A^{-1}) \\ &= (\mathrm{Ad}\, s^* \mid \Phi_B(B))(\mathrm{Ad}\, s \mid \Phi_A(A)) \\ &= \mathrm{Ad}\, s^*s \mid \Phi_A(A)\ .\end{aligned}$$

Thus, $\Phi_A\rho'\rho\Phi_A^{-1}$ is a positive automorphism of $\Phi_A(A)$ from Lemma 6.3, and $\rho'\rho$ is a positive automorphism of $A$.

From Theorem 5.4, Ad $|s|$ leaves $\Phi_A(A)$ invariant, where $|s| = (s^*s)^{1/2}$; and from Lemma 6.3 its restriction to $\Phi_A(A)$ is a positive automorphism of $\Phi_A(A)$. Hence $\Phi_A^{-1}(\mathrm{Ad}\,|s|\mid\Phi_A(A))\Phi_A$ is a positive automorphism of $A$. Since the square of this automorphism is $\rho'\rho$ and $(\rho'\rho)^{1/2}$, the square root of $\rho'\rho$ with spectrum in the open right half-plane, is uniquely determined by $\rho'\rho$, we know that $(\rho'\rho)^{1/2}$ is $\Phi_A^{-1}(\mathrm{Ad}\,|s|\mid\Phi_A(A))\Phi_A$.

DEFINITION 6.4. Let $\rho$ be an isomorphism of $C^*$-algebras, then we denote the automorphism $(\rho'\rho)^{1/2}$ by $|\rho|$ and it is called the absolute value of $\rho$.

Through a lemma which is easily verified, we show that the square root of a positive automorphism is also a positive automorphism.

LEMMA 6.5. *Let $\rho$ be an automorphism of a $C^*$-algebra $A$. Then, $\rho$ is positive if and only if $\rho = |\rho|$, and if and only if its canonical representation is implemented by an invertible positive operator on the canonical representation space of $A$.*

THEOREM 6.6. *Let $\eta$ be a positive automorphism of a $C^*$-algebra $A$. Then $\eta^{1/2}$ is a unique positive automorphism of $A$ whose square is $\eta$.*

PROOF. With an invertible positive operator $h$ on $H_A$,

$$\Phi\eta\Phi^{-1} = \mathrm{Ad}\, h \mid \Phi(A)\ ,$$

where $\Phi = \Phi_A$. From Theorem 5.4 and Lemma 6.3, Ad $(h^{1/2})$ leaves $\Phi(A)$ invariant and its restriction of $\Phi(A)$ is a positive automorphism of $\Phi(A)$. Hence we have

$$\Phi\eta^{1/2}\Phi^{-1} = \mathrm{Ad}\,(h^{1/2}) \mid \Phi(A)\ ,$$

proving the theorem.

**7. Polar decomposition for isomorphisms.** We prove here for isomorphisms of $C^*$-algebras the polar decomposition theorem:

THEOREM 7.1 (cf. Th. 8 in [8]). *Any isomorphism $\rho$ of a $C^*$-algebra $A$ onto a $C^*$-algebra $B$ is written uniquely as*

$$\rho = \pi\eta\ ,$$

*where $\pi$ is a *-isomorphism of $A$ onto $B$ and $\eta$ a positive automorphism of $A$; and $\rho$ is written uniquely as*

$$\rho = \eta_1 \pi_1 ,$$

*where $\eta_1$ is a positive automorphism of $B$ and $\pi_1$ a *-isomorphism of $A$ onto $B$.*

PROOF. Put $\eta = |\rho|$ and $\pi = \rho\eta^{-1}$. Then the former is a positive automorphism of $A$ and the latter a *-isomorphism of $A$ onto $B$ because

$$\pi'\pi = \eta^{-1}\rho'\rho\eta^{-1} = \eta^{-1}\eta^2\eta^{-1} = \iota_A ,$$

where $\iota_A$ is the identity automorphism of $A$; and $\rho = \pi\eta$. Suppose next that $\rho = \pi_0\eta_0$ with $\pi_0$ a *-isomorphism of $A$ onto $B$ and $\eta_0$ a positive automorphism of $A$. Then,

$$\eta_0' = (\pi_0^{-1}\rho)' = \rho'\pi_0 .$$

Therefore,

$$\eta_0 = (\eta_0'\eta_0)^{1/2} = (\rho'\pi_0\pi_0^{-1}\rho)^{1/2} = (\rho'\rho)^{1/2} = \eta$$

and hence

$$\pi_0 = \rho\eta_0^{-1} = \rho\eta^{-1} = \pi .$$

Now the first half of the conclusion is proved. By applying it to $\rho^{-1}$ we show the rest and the proof is completed.

In what follows, we examine the norm-continuity of the decomposition.

LEMMA 7.2. *Let $X$ and $Y$ be Banach spaces, then the inversion $\tau \mapsto \tau^{-1}$ under which each invertible bounded operator of $X$ onto $Y$ corresponds its inverse is norm-bicontinuous.*

The proof follows along the same line as a proof of the bicontinuity of the inversion in Banach algebras (see [10]).

PROOF. Suppose that $\tau$ and $\sigma$ are invertible bounded operators of $X$ onto $Y$ and $\varepsilon > 0$. Since

$$\tau^{-1} - \sigma^{-1} = -\tau^{-1}(\tau - \sigma)\tau^{-1} + (\tau^{-1} - \sigma^{-1})(\tau - \sigma)\tau^{-1} ,$$

we have

$$\|\tau^{-1} - \sigma^{-1}\| \leqq \|\tau^{-1}\|\,\|\tau - \sigma\|\,\|\tau^{-1}\| + \|\tau^{-1} - \sigma^{-1}\|\,\|\tau - \sigma\|\,\|\tau^{-1}\|$$

and hence

$$(1 - \|\tau^{-1}\|\,\|\tau - \sigma\|)\|\tau^{-1} - \sigma^{-1}\| \leqq \|\tau^{-1}\|^2\,\|\tau - \sigma\| .$$

Therefore, if $\|\tau - \sigma\| < \delta$ with

$$\delta = \min\left(\frac{1}{2\|\tau^{-1}\|}, \frac{\varepsilon}{2\|\tau^{-1}\|^2}\right),$$

then,

$$\begin{aligned}\|\tau^{-1} - \sigma^{-1}\| &= 2\left(1 - \|\tau^{-1}\|\cdot\frac{1}{2\|\tau^{-1}\|}\right)\|\tau^{-1} - \sigma^{-1}\| \\ &\leqq 2(1 - \|\tau^{-1}\|\,\|\tau - \sigma\|)\|\tau^{-1} - \sigma^{-1}\| \\ &\leqq 2\|\tau^{-1}\|^2\|\tau - \sigma\| \\ &< 2\|\tau^{-1}\|^2\cdot\frac{\varepsilon}{2\|\tau^{-1}\|^2} \\ &= \varepsilon\,.\end{aligned}$$

So, it follows that the mapping $\tau \mapsto \tau^{-1}$ is norm-continuous. Thus it is norm-bicontinuous and the proof is completed.

Iso $(A, B)$ denotes the set of all isomorphisms of a $C^*$-algebra $A$ onto a $C^*$-algebra $B$, $\mathrm{Iso}_*(A, B)$ the set of all $*$-isomorphisms of $A$ onto $B$; Aut $(A)$ the set of all automorphisms of $A$ and $\mathrm{Aut}_+(A)$ the set of all positive automorphisms of $A$.

LEMMA 7.3. *The mapping* $\rho \mapsto \rho'$ *from* Iso $(A, B)$ *onto* Iso $(B, A)$, *where* $A$ *and* $B$ *are* $C^*$*-algebras, is norm-bicontinuous.*

The proof is immediate from Lemma 7.2 and the relation

$$\|\rho' - \sigma'\| \leqq \|\rho^{-1} - \sigma^{-1}\| \leqq \|\rho^{-1}\|\,\|\sigma - \rho\|\,\|\sigma^{-1}\|$$

for $\rho \in$ Iso $(A, B)$ and $\sigma \in$ Iso $(A, B)$.

LEMMA 7.4. *If an analytic function* $f$ *is applicable to an element* $a$ *of a Banach algebra* $B$ *with identity, then there exists a positive number* $\delta$ *such that* $f$ *is applicable to every* $x$ *in* $B$ *with* $\|x - a\| < \delta$ *and the mapping* $x \mapsto f(x)$ *is continuous at* $a$.

We say that an analytic function is *applicable* to an element of a Banach algebra with identity if it is analytic on some bounded domain containing the spectrum of the element.

PROOF. Suppose that $f$ is analytic on a bounded domain $\varDelta$ containing $Sp(a)$. Then the continuity of spectra (see Th. (1.6.16) in [10]) makes us find a positive number $\delta$ and a rectifiable Jordan contour $\varGamma$ in $\varDelta$ such that $Sp(x)$ is contained in the interior of $\varGamma$ for every $x$ in $B$ with $\|x - a\| < \delta$. It follows that $f$ is applicable to every such $x$.

Suppose next that a sequence $\{a_n\}_{n\geqq 1}$ converges to $a$ as $n \longrightarrow \infty$ and put $a_0 = a$. Then $\{a_m\}_{m\geqq 0} \times \varGamma$ is compact and hence

$$\sup_{m \geqq 0 \text{ and } \lambda \in \Gamma} \|(a_m - \lambda e)^{-1}\| \|(a - \lambda e)^{-1}\| |f(\lambda)| < \infty ,$$

where $e$ denotes the identity of $B$. On the other hand, by the second resolvent equation

$$(a - \lambda e)^{-1} - (a_n - \lambda e)^{-1} = -(a_n - \lambda e)^{-1}(a - a_n)(a - \lambda e)^{-1}, n \geqq 1 ,$$

we have the formula

$$f(a) - f(a_n) = -\frac{1}{2\pi i}\int_\Gamma (a_n - \lambda e)^{-1}(a - a_n)(a - \lambda e)^{-1} f(\lambda) d\lambda, n \geqq 1 ,$$

implying the relation

$$\|f(a) - f(a_n)\| \leqq \frac{L}{2\pi}\|a - a_n\| \sup_{m \geqq 1 \text{ and } \lambda \in \Gamma} \|(a_m - \lambda e)^{-1}\| \|(a - \lambda e)^{-1}\| |f(\lambda)|, n \geqq 1 ,$$

with $L$ the length of $\Gamma$. Thus we know that $\{f(a_n)\}$ converges to $f(a)$ as $n \longrightarrow \infty$. Now the proof is completed.

From Lemmas 7.3 and 7.4 we have the following

LEMMA 7.5. *The mapping* $\rho \mapsto |\rho|$ *from* Iso $(A, B)$ *into* Aut $(A)$, *where $A$ and $B$ are $C^*$-algebras, is norm-continuous.*

In the sequel, we can show a theorem which asserts that Iso $(A, B)$ is norm-homeomorphic to $\text{Iso}_* (A, B) \times \text{Aut}_+ (A)$ and to $\text{Aut}_+ (B) \times \text{Iso}_* (A, B)$.

THEOREM 7.6. *Let $A$ and $B$ be $C^*$-algebras. Then, the mapping* $\text{Iso}(A, B) \ni \rho \mapsto (\pi, \eta) \in \text{Iso}_* (A, B) \times \text{Aut}_+ (A)$ *such that* $\rho = \pi\eta$ *is norm-bicontinuous, and the mapping* $\text{Iso}(A, B) \ni \rho \mapsto (\eta_1, \pi_1) \in \text{Aut}_+ (B) \times \text{Iso}_* (A, B)$ *such that* $\rho = \eta_1\pi_1$ *is norm-bicontinuous.*

PROOF. The mapping $\rho \mapsto |\rho| = \eta$ is norm-continuous from Lemma 7.5 and the mapping $\rho \mapsto \pi = \rho\eta^{-1}$ is also norm-continuous from Lemma 7.2. So, the mapping $\rho \mapsto (\pi, \eta)$ is norm-continuous. Since it is trivial that its inverse is norm-continuous, it is norm-bicontinuous. The rest is similarly seen and the proof is completed.

8. **More on positive automorphisms.** In this section we show two theorems which characterize positive automorphisms and clarify the relation of Theorem 7.1 to the original theorem in [8].

THEOREM 8.1. *Suppose that $A$ is a $C^*$-subalgebra, with property (D), of a $C^*$-algebra $B$ with identity and $\eta$ an automorphism of $A$. Then, in order that $\eta$ is positive it is necessary and sufficient that there exists an invertible positive element $h$ in $B$ such that*

$$\eta = \operatorname{Ad} h \mid A \ ;$$

*and such an $h$ can be found in $A$ provided that $A$ contains the identity of $B$.*

PROOF. Lemma 6.3 shows the sufficiency, so we show only the necessity. Assume first that $A$ contains the identity 1 of $B$. Then from Lemma 6.5, we can find an invertible positive operator $h$ on the canonical representation space $H_A$ of $A$ such that

$$\Phi\eta\Phi^{-1} = \operatorname{Ad} h \mid \Phi(A) \ ,$$

where $\Phi$ denotes the canonical representation of $A$. Since $\Phi(A)$ has the property $(D)$, it follows from Theorem 5.6 that $\Phi\eta\Phi^{-1}$ is inner implemented by an invertible positive element, and hence so does $\eta$.

Next assume that $A$ does not contain 1. Then $A + \{\lambda 1\}$ is a $C^*$-subalgebra of $B$. Suppose that $\delta$ is a derivation of $A + \{\lambda 1\}$. Then $\delta(1) = 0$ is seen by a simple computation. Therefore, there exists an element $a$ in $A$ such that

$$\delta(x + \lambda 1) = \delta(x) = \operatorname{ad} a(x) = \operatorname{ad} a(x + \lambda 1)$$

for each $x$ in $A$ and complex $\lambda$. We know thus that $\delta$ is inner and $A + \{\lambda 1\}$ has the property $(D)$.

Define an automorphism $\tilde{\eta}$ of $A + \{\lambda 1\}$ by

$$\tilde{\eta}(x + \lambda 1) = \eta(x) + \lambda 1 \text{ for } x \in A \text{ and } \lambda \in \boldsymbol{C} \ .$$

Then it is positive because $Sp(\tilde{\eta}) = Sp(\eta) \cup \{1\}$. Hence, we have $\tilde{\eta} = \operatorname{Ad} h \mid (A + \{\lambda 1\})$, with $h$ an invertible positive element of $A + \{\lambda 1\}$. It follows that

$$\eta = \tilde{\eta} \mid A = \operatorname{Ad} h \mid A \ ,$$

and the proof is completed.

LEMMA 8.2. *Any skew-adjoint derivation of a $C^*$-algebra $A$ acting on a Hilbert space $H$ is implemented by a self-adjoint operator on $H$ contained in the von Neumann algebra $\tilde{A}$ on $H$ generated by $A$.*

The proof is direct from Lemma 5.5.

THEOREM 8.3. *Let $\eta$ be an automorphism of a $C^*$-algebra $A$ and $\Phi$ a faithful $^*$-representation of $A$. Then, in order that $\eta$ is positive it is necessary and sufficient that there exists a positive automorphism $\eta_1$ of the von Neumann algebra $\widetilde{\Phi(A)}$ generated by $\Phi(A)$ such that*

$$\Phi\eta = \eta_1\Phi \ .$$

PROOF. First we show the sufficiency. Since

$$\eta_1(\Phi(A)) = \Phi(\eta(A)) = \Phi(A) ,$$

$\eta_1 | \Phi(A)$ is an automorphism of $\Phi(A)$ and hence it is positive. Thus $\eta = \Phi^{-1}\eta_1\Phi$ is a positive automorphism of $A$.

Next we show the necessity. We may assume that $A$ acts on the representation space of $\Phi$ and $\Phi$ is the identity representation. Put $\delta = \operatorname{Log} \eta$. Then, it is a skew-adjoint derivation of $A$. In fact, with an invertible positive operator $h$ on the canonical representation space,

$$\Phi_A \delta \Phi_A^{-1} = (\operatorname{Log} \operatorname{Ad} h) | \Phi_A(A) = (\operatorname{ad} \operatorname{Log} h) | \Phi_A(A) ;$$

so, from Lemma 8.2, $\delta$ is implemented by a self-adjoint element $k$ in $A$. Hence

$$\eta = \exp \delta = \exp \operatorname{ad} k | A = \operatorname{Ad} \exp k | A .$$

Now the proof is completed.

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