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ON THE TANGENT SPHERE BUNDLE OF A 2-SPHERE

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Introduction. Let S^2 be the unit sphere in a Euclidean space E^3 with the induced metric g. Then, the set of all unit tangent vectors $T_1(S^2)$ with the natural topology is the total space of the tangent sphere bundle $p: T_1(S^2) \to S^2$. $T_1(S^2)$ has a natural Riemannian metric. In this paper, we prove first that $T_1(S^2)$ with this metric is isometric with the elliptic space of constant curvature 1/4 (Theorem 1). Then, we give two proofs of a theorem which characterizes each geodesic on $T_1(S^2)$ as a vector field along a circle in S^2 (Theorem 2 and § 4). Finally, we give a theorem on the set of tangent vectors of a one parameter family of circles, the set corresponds to a Clifford surface in $T_1(S^2)$ regarded as an elliptic space (Theorem 4).

1. $T_1(S^2)$ as a Riemannian manifold.

First we shall show

LEMMA 1. *$T_1(S^2)$ is diffeomorphic with the real projective space P^3.*

PROOF. For $y \in T_1(S^2)$, we consider the unit vector $e_1(y)$ which issues from the center O of S^2 and ends at the point $p(y)$. Then, the map $\psi: T_1(S^2) \to SO(3)$ defined by $y \to (e_1(y), e_2(y), e_1(y) \times e_2(y))$, where $e_2(y) \equiv y$ and $\times$ means vector product in E^3, is a diffeomorphism. On the other hand, it is well known that $SO(3)$ is diffeomorphic with P^3 (cf. for example [3] p. 115). Hence, $T_1(S^2)$ is diffeomorphic with P^3.

Now, let U be an arbitrary coordinate neighborhood with local coordinates $x^a (a, b, c = 1, 2)$ and y^a be components of a tangent vector y in U with respect to the natural frame $\partial/\partial x^a$. Then, $p^{-1}(U)$ gives a coordinate neighborhood of $T_1(S^2)$ with local coordinates (x^a, y^a). By virtue of the induced metric g on S^2 in E^3, the natural Riemannian metric $\hat{g}$ on $T_1(S^2)$ is given by the following line element:

$$d\sigma^2 = g_{bc}(x)dx^b dx^c + g_{bc}(x)\delta y^b \delta y^c \, , \tag{1.1}$$

([2]) where we have put

$$g_{bc}(x)y^b y^c = 1 \, , \qquad \delta y^b = dy^b + \left\{ {b \atop ef} \right\} y^e dx^f \, . \tag{1.2}$$

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First, let us prove the following

LEMMA 2. *$(T_1(S^2), \hat{g})$ is a Riemannian manifold of constant positive curvature* $1/4$.

PROOF. Let $e_1(r, \theta)$ be the point on S^2 with coordinates (r, θ) in geodesic polar coordinates with the north pole N as its center. Then, the unit tangent vectors for the r-curve and the θ-curve at the point $e_1(r, \theta)$ are given by

$$f_2 = \frac{\partial}{\partial r}, \qquad f_3 = \frac{1}{\sin r}\frac{\partial}{\partial \theta}. \tag{1.3}$$

Now, let e_2 be an element of $T_1(S^2)$ at the point $e_1(r, \theta)$ of S^2. If we denote the angle between f_2 and e_2 by ω, then (r, θ, ω) can be considered as local coordinates for e_2 in $p^{-1}(S^2 - \{N, S\})$, S being the south pole. As

$$\begin{cases} e_2 = \cos\omega \cdot f_2 + \sin\omega \cdot f_3 , \\ e_3 = -\sin\omega \cdot f_2 + \cos\omega \cdot f_3 \end{cases} \tag{1.4}$$

and

$$\begin{cases} de_1 = dr \cdot f_2 + \sin r\, d\theta \cdot f_3 , \\ df_2 = -dr \cdot e_1 + \cos r\, d\theta \cdot f_3 , \\ df_3 = -\sin\theta d\theta \cdot e_1 - \cos r\, d\theta \cdot f_3 , \end{cases} \tag{1.5}$$

we see that

$$\langle de_1, de_1 \rangle = dr^2 + \sin^2 r\, d\theta^2 \tag{1.6}$$

and

$$\begin{aligned} &\langle de_2, e_3 \rangle \\ &= \langle (*) \cdot e_1 - \sin\omega \cdot \Phi \cdot f_2 + \cos\omega \cdot \Phi \cdot f_3 , \quad - \sin\omega \cdot f_2 + \cos\omega \cdot f_3 \rangle = \Phi , \end{aligned} \tag{1.7}$$

where $(*)$ means the term which we do not need to know and

$$\Phi = d\omega + \cos r\, d\theta . \tag{1.8}$$

On the other hand, we see easily that

$$d\sigma^2 = \langle de_1, de_1 \rangle + \langle de_2, e_3 \rangle^2 . \tag{1.9}$$

So, we get by (1.6) and (1.7)

$$d\sigma^2 = dr^2 + d\theta^2 + 2\cos r\, d\theta d\omega + d\omega^2 . \tag{1.10}$$

As the right hand side of (1.10) is of very simple form we can calculate its curvature tensor by a routine method. A little long but simple calculation shows us that the Riemannian metric (1.10) is of constant curvature 1/4.

From Lemmas 1 and 2, we get the following

THEOREM 1. *The Riemannian manifold* $(T_1(S^2), \hat{g})$ *is isometric with the elliptic space* $\mathscr{E}^3 = (P^3, k)$, *where* k *is the Riemannian metric of constant curvature* 1/4.

2. **Geodesics on** $T_1(S^2)$. Now, we shall prove the following theorem.

THEOREM 2. *Any geodesic on* $T_1(S^2)$ *is interpreted as a unit vector field along a circle* C *on* S^2 *which makes constant angle with* C.

REMARK 1. C may reduce to a point. Thus, each fibre of the bundle $p: T_1(S^2) \to S^2$ is a geodesic of $T_1(S^2)$.

REMARK 2. Both of Theorems 1 and 2 tell us that all geodesics are closed. Moreover, Theorem 1 tells us that every geodesic has of length 2π. This can be also proved directly by virtue of Theorem 2.

PROOF. If we denote a geodesic Γ in $T_1(S^2)$ parametrically by $(x^a(\sigma), y^a(\sigma))$, where σ is the arc length of Γ, then $x^a(\sigma)$ and $y^a(\sigma)$ satisfy the following set of differential equations (cf. [2]*) II, p. 152):

$$\begin{cases} x'' = -by + ay' , \\ y'' = \rho y , \end{cases} \tag{2.1}$$

where x' means the tangent vector $dx^a/d\sigma$, and dashes on the shoulders of y's mean covariant derivatives along the curve $C = p(\Gamma)$ and

$$a = \langle x', y\rangle , \qquad b = \langle x', y'\rangle \tag{2.2}$$

are inner products on S^2. Of course, we have

$$\langle y, y\rangle = 1 , \qquad \langle y, y'\rangle = 0 . \tag{2.3}$$

If we put

$$c^2 = \langle y', y'\rangle \equiv |y'|^2 , \qquad c \geqq 0 \tag{2.4}$$

then, we see easily that a, b, c are constants. For example, we shall prove the constancy of b. We get first

$$b' = \langle x', y'\rangle' = \langle -by + ay', y'\rangle + \rho\langle x', y\rangle = a(c^2 + \rho) .$$

However, by $(2.3)_2$ we have $\rho = -c^2$. So, we see that b is a constant.

Now, the horizontal component and the vertical component of the tangent vector T of Γ are given by x'^h and y'^v respectively, where x'^h

*) K in [2] I p. 353 ↑1 and p. 354 ↓1 should be replaced by $-K$.

is the horizontal lift of x' and y'^v is the vertical lift of y'. So, if we denote the norm of a tangent vector of $T_1(S^2)$ by $\| \ \|$, then we have

$$\| x'^h \|^2 = \| T \|^2 - \| y'^v \|^2 = 1 - | y' |^2 , \qquad \| x'^h \|^2 = | x' |^2 .$$

So, we get

$$| x' |^2 = 1 - c^2 . \tag{2.5}$$

The last equation shows that $0 \leqq c \leqq 1$ and (i) C reduces to a point if $c = 1$ and Γ is a fibre over the point, (ii) C reduces to a geodesic on S^2 if $c = 0$ and Γ is a trajectory of the geodesic flow.

When C does not reduce to a point, let us denote its arc length by s. Then, (2.5) shows us that

$$\frac{ds}{d\sigma} = \sqrt{1 - c^2} = \text{const.} . \tag{2.6}$$

Then, the relation

$$| x'' |^2 = b^2 + a^2c^2$$

and (2.6) tell us that the geodesic curvature κ of C is given by

$$\kappa^2(1 - c^2)^2 = b^2 + a^2c^2 . \tag{2.7}$$

Thus, κ is constant along C and so C is a circle on S^2.

The angle $\alpha(\sigma)$ between the tangent vector $x'(\sigma)$ and $y(\sigma)$ along C is given by

$$\cos \alpha(\sigma) = a/| x' |^2 .$$

So, by (2.5) $\alpha(\sigma)$ is constant along C. This completes the proof.

3. **The isometry ψ: $T_1(S^2) \to SO(3)$.** In § 1, we showed that the map ψ: $T_1(S^2) \to SO(3)$ is a diffeomorphism. Now, as $SO(3)$ is a compact connected Lie group, it admits a natural symmetric Riemannian structure. Although it is a well-known fact, we shall explain a little which seems necessary for our purpose.

For simplicity, we put $G = SO(3)$ and denote its Lie algebra by $\mathfrak{g}$. $\mathfrak{g}$ is identified with the tangent space of G at the unit element e. Denoting the rectangular coordinates in E^3 by (x, y, z), the basis of $\mathfrak{g}$ is given by

$$B_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} , \qquad B_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} ,$$
$$B_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

and the structural equations are given by

$$[B_2, B_3] = -B_1,\ [B_3, B_1] = -B_2,\ [B_1, B_2] = -B_3\ . \tag{3.1}$$

So, if we express the components of elements X_e and Y_e of $\mathfrak{g}$ with respect to the above basis by $(\lambda_1, \lambda_2, \lambda_3)$ and (μ_1, μ_2, μ_3), then we see that the Killing form B of G is given by

$$B(X_e, Y_e) = -2(\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3)\ . \tag{3.2}$$

If we define a Riemannian metric h on G by

$$h(X, Y) = -\frac{1}{2}B(L'_{a^{-1}}X, L'_{a^{-1}}Y) \tag{3.3}$$

for $X, Y \in G_a$, where $L'_{a^{-1}}$ is the differential of the left translation $L_{a^{-1}}$ and G_a is the tangent space at $a \in G$, then h is biinvariant and (G, h) is a globally symmetric Riemannian space. Moreover, as $G = SO(3)$ is semi-simple, G is an Einstein space (cf. [1] p. 206). So, the vanishing of Weyl's conformal curvature tensor of every Riemannian 3-space tells us that (G, h) is a globally symmetric Riemannian space of constant curvature.

Now, we shall prove the following

THEOREM 3. *The map* ψ: $T_1(S^2) \to SO(3)$ *is an isometry of* $(T_1(S^2), \hat{g})$ *with* $(SO(3), h)$.

PROOF. $G = SO(3)$ acts on G from the left as a simply transitive group of isometries. It acts also on $T_1(S^2)$ as a simply transitive group of isometries considered to act from the left. So, to show the isometry of the map ψ of $(T_1(S^2), \hat{g})$ with (G, h), it is sufficient to show the isometry of the differential of the map ψ of the tangent space $(T_1(S^2))_{y_0}$ at the point $y_0 = \psi^{-1}(e)$ with the one G_e at the unit element e of G. We see that y_0 is the tangent vector $e_2^0 = (0, 1, 0)$ at the point $e_1^0 = (1, 0, 0)$.

Now, take an element $X_e = \lambda_1 B_1 + \lambda_2 B_2 + \lambda_3 B_3$. Then, it corresponds by ψ^{-1} to

$$\begin{cases} e'_1 = \lambda_3 e_2^0 - \lambda_2 e_3^0\ , \qquad e'_2 = -\lambda_3 e_1^0 + \lambda_1 e_3^0\ , \\ e'_3 = e'_1 \times e_2^0 + e_1^0 \times e'_2\ . \end{cases} \tag{3.4}$$

So, by (1.9), we have

$$\hat{g}((\psi^{-1})'X_e, (\psi^{-1})'X_e) = \langle e'_1, e'_1\rangle + \langle e'_2, e_3^0\rangle^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = h(X_e, X_e)\ .$$

This completes the proof.

4. **Another proof of Theorem 2.** By virtue of Theorem 3, $(T_1(S^2), \hat{g})$ can be identified with the globally symmetric space $(SO(3), h)$. Geodesics of the latter through the unit element e are 1-parameter subgroups of $SO(3)$ and other geodesics are cosets of these 1-parameter subgroups.

Now, let H be a 1-parameter subgroup of $SO(3)$. Then, H is a group of rotations around a fixed axis l through the origin O.

We identify e with $(e_1^0, e_2^0, e_1^0 \times e_2^0)$ and denote elements of H by f_σ $\sigma \in R$ mod 2π. If we put $e_1(\sigma) = f_\sigma(e_1^0)$, $e_2(\sigma) = f_\sigma(e_2^0)$, then $(e_1(\sigma), e_2(\sigma), e_1(\sigma) \times e_2(\sigma))$ draws a geodesic on $(SO(3), h)$ as σ varies. This shows that $e_2(\sigma)$ draws a geodesic Γ on $(T_1(S^2), \hat{g})$. When l does not have the direction e_1^0, the initial point of $e_2(\sigma)$, i.e. the end point of $e_1(\sigma)$, draws a circle C on S^2 and $e_2(\sigma)$ makes a constant angle with C as σ varies. When l has the direction e_1^0, $e_1(\sigma)$ coincides with the fixed vector e_1^0. We denote the end point of e_1^0 by x_0. Then, $e_2(\sigma)$ draws a fibre $p^{-1}(x_0)$. Thus the assertion of Theorem 2 is true for geodesics of $T_1(S^2)$ which correspond to 1-parameter subgroups of $SO(3)$ by the map ψ^{-1}.

Any geodesic of $(SO(3), h)$ which does not pass through e is given as a left coset of a 1-parameter subgroup H, i.e. as a family of frames $f(e_1(\sigma), e_2(\sigma), e_1(\sigma) \times e_2(\sigma))$ where $f \in SO(3)$ and $e_1(\sigma) = f_\sigma(e_1^0)$, $e_2(\sigma) = f_\sigma(e_2^0)$, $f_\sigma \in H$ $(\sigma \in R)$. By ψ^{-1} this corresponds to a vector field $f(e_2(\sigma))$ on $T_1(S^2)$. Thus the geodesic on $T_1(S^2)$ which corresponds to a left coset of a 1-parameter subgroup H of $SO(3)$ is either a unit vector field along a circle $f(C)$ which makes a constant angle with $f(C)$ or a fibre $p^{-1}(f(x_0))$. This completes the proof.

5. A family of tori in $T_1(S^2)$. Let us consider two parallel small circles C_{ϕ_0} and $C_{-\phi_0}$ on S^2 which are defined by $\phi = \phi_0$ and $\phi = -\phi_0$ $(\phi = \pi/2 - r)$ and lie equidistant from the equator. We consider a point (ϕ_0, θ) on C_{ϕ_0} and denote it by the unit vector $f_1(\theta)$ and the unit tangent vector at the point to the circle C_{ϕ_0} with the orientation coherent with its parameter θ by $f_2(\theta)$. Then, the great circle K_θ which passes through the point $f_1(\theta)$ and has the direction $f_2(\theta)$ is expressed by the field of unit vectors

$$e_1(\theta, t) = \cos t \cdot f_1(\theta) + \sin t \cdot f_2(\theta) \tag{5.1}$$

with the origin O as its initial point. The unit tangent vector to K_θ at the point $e_1(\theta, t)$ is given by

$$e_2(\theta, t) = -\sin t \cdot f_1(\theta) + \cos t \cdot f_2(\theta) . \tag{5.2}$$

We may change the value of θ arbitrarily in the interval $[0, 2\pi]$ too. It is clear that the locus of the point $e_2(\theta, t)$ in $T_1(S^2)$ is a surface F homeomorphic with a torus. t-curves on F are geodesics of $T_1(S^2)$ and any two of them do not intersect. They are trajectories of the geodesic flow of S^2. Thus, F is covered by a family of geodesics. In the same way θ-curves are also geodesics of $T_1(S^2)$, because any of them is a vector field along a circle $\phi = \text{const.}$ which makes a constant angle with the tangent vector to the circle. So, F is covered also by another family of geodesics, any two of them do not have common point. As $(T_1(S^2), \hat{g})$ is isometric with the elliptic space $\mathscr{E}^3$ by Theorem 1, F must be a surface which corresponds to a quadric with two families of real generators. This suggests us that F may be a surface which corresponds to a Clifford torus in $\mathscr{E}^3$. In fact, we get the following

THEOREM 4. *The Riemannian metric on the surface F induced from the one in $T_1(S^2)$ is flat. Thus F is a surface in $(T_1(S^2), \hat{g})$ corresponding to a Clifford torus in $\mathscr{E}^3$.*

PROOF. We may easily verify that

$$\begin{aligned} f_1'(\theta) &= \cos\phi_0 \cdot f_2(\theta) , \\ f_2'(\theta) &= -\cos\phi_0 \cdot f_1(\theta) + \sin\phi_0 \cdot f_3(\theta) , \\ f_3'(\theta) &= -\sin\phi_0 \cdot f_2(\theta) \end{aligned}$$

hold good. So, we get

$$e_{1\theta} \equiv \frac{\partial e_1}{\partial\theta} = -\cos\phi_0 \sin t \cdot f_1(\theta) + \cos\phi_0 \cos t \cdot f_2(\theta) + \sin\phi_0 \sin t \cdot f_3(\theta) ,$$

$$e_{1t} \equiv \frac{\partial e_1}{\partial t} = -\sin t \cdot f_1(\theta) + \cos t \cdot f_2(\theta)$$

and

$$\begin{aligned} \langle e_{1\theta}, e_{1\theta}\rangle &= \cos^2\phi_0 + \sin^2\phi_0 \sin^2 t , \\ \langle e_{1\theta}, e_{1t}\rangle &= \cos\phi_0 , \qquad \langle e_{1t}, e_{1t}\rangle = 1 . \end{aligned}$$

Therefore, we have

$$\langle de_1, de_1\rangle = (\cos^2\phi_0 + \sin^2\phi_0 \sin^2 t)d\theta^2 + 2\cos\phi_0\, d\theta dt + dt^2 . \tag{5.3}$$

On the other hand, we get

$$\begin{aligned} de_2 &= e_{2\theta}d\theta + e_{2t}dt \\ &= (-\sin t \cdot f_1'(\theta) + \cos t \cdot f_2'(\theta))d\theta - (\cos t \cdot f_1(\theta) + \sin t \cdot f_2(\theta))dt \\ &= (*)\cdot f_1(\theta) + (*)\cdot f_2(\theta) + \sin\phi_0 \cos t\, d\theta \cdot f_3(\theta) \end{aligned}$$

where $(*)$'s mean factors which we do not need to know their exact forms. So, we have

$$\langle de_2, e_3\rangle = \sin\phi_0 \cos t\, d\theta . \tag{5.4}$$

Hence, we get by (1.9), (5.3) and (5.4)

$$d\sigma^2 \mid F = d\theta^2 + 2\cos\phi_0\, d\theta dt + dt^2 , \tag{5.5}$$

where the left hand side means the restriction of $d\sigma^2$ to F i.e. the induced metric on F. Clearly, it is flat.

As we have seen before, t-curves and θ-curves are geodesics of $T_1(S^2)$. (5.5) tells us that any pair of geodesics from different families intersects at a constant angle ϕ_0. This completes the proof.

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