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APPROXIMATION BY OSCILLATING GENERALIZED POLYNOMIALS

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1. Introduction. Oscillating generalized polynomials extend to generalized polynomials the concept of oscillating polynomials (defined below) which were studied first by Bernstein ([3]; [4]). The subjects of oscillating generalized polynomials (abbreviated hereafter as *OGP*'s), and uniform approximations, by polynomials with real coefficients, to real powers of x are closely related. Indeed if r_i is a positive real number for $i = 1, 2, \cdots, k$ such that for some integer n_0, $n_0 < r_1 < \cdots < r_k < n_0 + 1$, then $q(x)$ is the best approximation on $[0, 1]$ to $\sum_{i=1}^{k} x^{r_i}$ by a polynomial of degree n if and only if $\sum_{i=1}^{k} x^{r_i} - q(x)$ is an *OGP*.

In Section 2 we develop the theory of *OGP*'s. We prove an existence and uniqueness theorem. Further we derive properties of *OGP*'s useful in approximations to real powers of x. In particular we show that if $p(x) = \sum_{k=0}^{n} A_k g_{\alpha_k}(x)$ and $q(x) = \sum_{k=0}^{n} B_k g_{\alpha_k}(x)$ are distinct generalized polynomials (abbreviated hereafter as *GP*'s) where $A_k = B_k$ for at least one k with g_{α_k} not a constant function and p is an *OGP*, then $\max_{0\leq x\leq 1} |q(x)| \equiv \|q\| > \max_{0\leq x\leq 1} |p(x)| \equiv \|p\|$.

In Section 3 we study, by use of the theory of *OGP*'s, the uniform approximation in $[0, 1]$ of real powers of x by polynomials with real coefficients. Here we derive lower bounds for the best approximation error in $[0, 1]$ to x^{α}, where α is a real number lying in $(0, 1/3)$, by polynomials of a given degree. Further, we give in Examples 4 and 5 the polynomials which provide the best uniform approximation to $x^{1/\pi}$ and $1/2(x^{1/3} + x^{1/2})$, respectively, by polynomials of degree not exceeding n.

2. Oscillating generalized polynomials. Throughout this paper n, $\alpha_0, \cdots, \alpha_n$ will denote integers such that $n \geqq 1$ and $0 \leqq \alpha_0 < \alpha_1 < \cdots < \alpha_n$. We now define *OGP*'s.

DEFINITION 2.1. Let $\{g_{\alpha}\}_{\alpha=0}^{\infty}$ be a sequence of functions, real valued, non-negative and continuous on $[0, 1]$ and analytic on $(0, 1]$. Further suppose that g_{α} is not a constant function if $\alpha \geqq 1$, g_0 is not identically zero and $g_{\alpha}(0) = 0$ unless g_{α} is a constant. Then $\{g_{\alpha}\}_{\alpha=0}^{\infty}$ is said to have property $\mathscr{D}$ if and only if the following hold:

(i) For every set of non-zero real numbers $\{c_0, c_1, \cdots, c_n\}$ and for every choice of integers $\{\alpha_0, \alpha_1, \cdots, \alpha_n\}$ the number of zeros, counted with due regard to multiplicity in $(0, 1]$, of the $GP \sum_{k=0}^{n} c_k g_{\alpha_k}$ is at most equal to the number of variations of sign in the sequence $\{c_0, c_1, \cdots, c_n\}$.

(ii) For every set of non-zero real numbers $\{c_0, c_1, \cdots, c_n\}$ and for every choice of integers $\{\alpha_0, \alpha_1 \cdots, \alpha_n\}$ the number of zeros (counted with due regard to multiplicity) in $(0, 1]$ of the $GP \sum_{k=0}^{n} c_k g'_{\alpha_k}$ is at most equal to the number of variations of sign in the sequence $\{c_0, c_1, \cdots, c_n\}$. (Here g' denotes the derivative of g.)

Clearly, by Descartes rule of signs, the sequence of functions $\{x^j\}_{j=0}^{\infty}$ has property $\mathscr{D}$. Moreover, by a familiar argument (cf. [7], pp. 118–120) we obtain the following example of a sequence of functions with property $\mathscr{D}$.

EXAMPLE 2.2. Let $\{r_\alpha\}_{\alpha=0}^{\infty}$ denote a sequence of strictly increasing non-negative real numbers with $r_\alpha > 0$ if $\alpha \geqq 1$. Define $g_\alpha(x) = x^{r_\alpha}$, where for each α we take the principal branch of $\log z$ in $z^{r_\alpha} = \exp(r_\alpha \log z)$. Then $\{g_\alpha\}_{\alpha=0}^{\infty}$ has property $\mathscr{D}$.

DEFINITION 2.3. Let $\{A_0, \cdots, A_n\}$ be a set of non-zero real numbers. Then $p(x) = \sum_{k=0}^{n} A_k x^{\alpha_k}$ is said to be an oscillating polynomial (OP) if $|p(x)| = \|p\|$ for $n+1$ values of x in $[0, 1]$.

DEFINITION 2.4. Let $\{g_\alpha\}_{\alpha=0}^{\infty}$ be a sequence of functions with property $\mathscr{D}$. Suppose that $\{A_0, A_1, \cdots, A_n\}$ is a set of non-zero real numbers. Then $p(x) = \sum_{k=0}^{n} A_k g_{\alpha_k}(x)$ is said to be an OGP if and only if $|p(x)| = \|p\|$ for at least $n+1$ values of x in $[0, 1]$.

It is easy to verify that the functions given in the following example are OGP's.

EXAMPLE 2.5. Let α be a positive real number. For each non-negative integer k, define $g_k(x) = x^{k\alpha}$ where we take the principal branch of $\log z$ in $z^{k\alpha} = \exp(k\alpha \log z)$. Then the following are examples of OGP's:

(i) $T_{2n}(x^{\alpha/2})$, a linear combination of the form $\sum_{k=0}^{n} c_k x^{k\alpha}$. (Here and in what follows $T_n(x)$ denotes Chebyshev polynomial of degree n [5, pp. 62–63], [6, pp. 30–31].)

(ii) $T_{2n}(x^\alpha)$, a linear combination of the form $\sum_{k=0}^{n} c_k x^{2k\alpha}$.

(iii) $T_{2n+1}(x^\alpha)$, a linear combination of the form $\sum_{k=0}^{n} c_k x^{(2k+1)\alpha}$.

From Definitions 2.1 and 2.4 the following two properties of OGP's can be easily derived.

THEOREM 2.6. *Let $\{g_\alpha\}_{\alpha=0}^{\infty}$ be a sequence of functions with property $\mathscr{D}$. If $p(x) = \sum_{k=0}^{n} A_k g_{\alpha_k}(x)$ is an OGP, then $|p(x)| = \|p\|$ exactly $(n+1)$*

times in $[0, 1]$. *In particular, if* g_{α_0} *is a constant function, then* $|p(x)| = ||p||$ *at* $0, x_1, \cdots, x_{n-1}, 1$ *where* $0 < x_1 < \cdots < x_{n-1} < 1$. *On the other hand, if* g_{α_0} *is not constant, then* $|p(x)| = ||p||$ *at* $x_1, x_2, \cdots, x_n, 1$ *where* $0 < x_1 < x_2 < \cdots < x_n < 1$.

THEOREM 2.7. *Let* $\{g_\alpha\}_{\alpha=0}^{\infty}$ *be a sequence of functions with property* $\mathscr{D}$. *The GP* $q(x) = \sum_{k=0}^{n} A_k g_{\alpha_k}(x)$, *not necessarily an OGP, has at most* n *distinct zeros in* $(0, 1]$. *If it has* n, *the coefficients alternate in sign.*

THEOREM 2.8. *The coefficients of an OGP* $p(x) = \sum_{k=0}^{n} A_k g_{\alpha_k}(x)$ *alternate in sign.*

PROOF. Since $p'(x)$ cannot equal zero in $(0, 1]$ except at the points where $|p(x)| = ||p||$, it follows that $p(x)$ has n distinct zeros in $(0, 1]$. By Theorem 2.7, the coefficients of $p(x)$ alternate in sign.

COROLLARY 2.9. *Let* $p(x) = \sum_{k=0}^{n} A_k g_{\alpha_k}(x)$ *be an OGP. Then every zero of* $p(x)$ *on* $(0, 1]$ *is simple.*

PROOF. This follows immediately if we observe that the zeros of $p'(x)$ in $(0, 1]$ are points where $|p(x)| = ||p||$.

We now give the most interesting property of *OGP*'s.

THEOREM 2.10. *Let* $\{g_\alpha\}_{\alpha=0}^{\infty}$ *be a sequence of functions with property* $\mathscr{D}$ *and suppose that* $p(x) = \sum_{k=0}^{n} A_k g_{\alpha_k}(x)$ *is an OGP in* $[0, 1]$. *If* $q(x) = \sum_{k=0}^{n} B_k g_{\alpha_k}(x)$ *is another GP such that* $A_k = B_k$ *for at least one* k *where* g_{α_k} *is not a constant function then* $||q|| > ||p||$.

PROOF. Suppose if possible that $||q|| < ||p||$. Then by considering separately when g_{α_0} is a constant function and when it is not, we find that $p(x) - q(x) = \sum_{j=0}^{n} (A_j - B_j) g_{\alpha_j}(x)$ has at least n zeros in $(0, 1]$ but $p - q$ cannot have more than $(n - 1)$ zeros in $(0, 1]$. Hence we have a contradiction and so $||q|| \geqq ||p||$.

Suppose now that $||p|| = ||q||$. If $(p - q)(0) \neq 0$ then g_{α_0} is a constant function and by Theorem 2.6, $|p(x)| = ||p||$ at $x = 0, x_1, \cdots, x_{n-1}, 1$. With the convention that if $(p - q)(x)$ has a zero of order at least two at x_i, we count one zero for the interval $[x_{i-1}, x_i]$ and another zero for the interval $[x_i, x_{i+1}]$, then we can count at least one zero of $(p - q)(x)$ for each interval $[0, x_1], [x_1, x_2], \cdots, [x_{n-1}, 1]$. But property $\mathscr{D}$ shows that $(p - q)$ cannot have n zeros in $(0, 1]$ and we have a contradiction.

Next suppose $(p - q)(0) = 0$. If $p(0) = 0$, then g_{α_0} is not a constant function. By an argument similar to the one given just now, $(p - q)(x)$ has zeros in each of the intervals $[x_1, x_2], \cdots, [x_n, 1]$, where $|p(x_i)| = ||p||$ for $i = 1, 2, \cdots, n$. But $(p - q)$ cannot have n zeros in $(0, 1]$. However if $p(0) \neq 0$, then g_{α_0} is a constant function and $|p(0)| = \|p\|$. It follows that $A_0 = B_0$ as well as $A_k = B_k$, and since $(p-q)(x) = \sum_{j=0}^{n} (A_j - B_j) g_{\alpha_j}(x)$, $p - q$ has at most $(n-2)$ zeros in $(0, 1]$. By the above argument $(p-q)(x)$ has at least $(n-1)$ zeros in $(0, 1]$ and we have a contradiction. The theorem is proved.

We now state the converse to Theorem 2.10.

THEOREM 2.11. *Suppose that* $\{g_\alpha\}_{\alpha=0}^{\infty}$ *is a sequence of functions with property* $\mathscr{D}$. *Let* $p(x) = \sum_{j=0}^{n} A_j g_{\alpha_j}(x)$, $q(x) = \sum_{j=0}^{n} B_j g_{\alpha_j}(x)$ *be two GP's with* $A_0, \cdots, A_n$ *all non-zero and at least one coefficient* $A_k = B_k$, *where* g_{α_k} *is not a constant function. If* $\|p\| < \|q\|$ *for every such* $q \neq p$, *then* p *is an OGP.*

For the proof of this theorem, we require the following

LEMMA. *Suppose that* $\{g_\alpha\}_{\alpha=0}^{\infty}$ *is a sequence of functions with property* $\mathscr{D}$ *and that*

(i) $0 \leqq x_1 < x_2 < \cdots < x_n \leqq 1$, *and*

(ii) $0 \leqq \alpha_1 < \alpha_2 < \cdots < \alpha_n$, *where each* α_j *is an integer.*

(iii) *Suppose further that when* $x_1 = 0$, g_{α_1} *is a non-zero constant function. Then the determinant*

$$\begin{vmatrix} g_{\alpha_1}(x_1), & \cdots, & g_{\alpha_n}(x_1) \\ \cdot\ \cdot\ \cdot & & \cdot\ \cdot\ \cdot \\ g_{\alpha_1}(x_n), & \cdots, & g_{\alpha_n}(x_n) \end{vmatrix} \neq 0 .$$

To prove this lemma we have to consider two cases $x_1 = 0$ and $x_1 > 0$. We omit the details of the proof.

PROOF OF THEOREM 2.11. Suppose that p is not an OGP. Let $S = \{x \in [0, 1] \mid |p(x)| = \|p\|\}$. Then $S = \{x_1, x_2, \cdots, x_h\}$ where $h < n + 1$ and $0 \leqq x_1 < \cdots < x_h \leqq 1$.

Since g_{α_0} must be a constant if $x_1 = 0$, there exists a vector $(d_0, \cdots, d_{k-1}, d_{k+1}, \cdots, d_n)$ such that

$$d_0 g_{\alpha_0}(x_i) + \cdots + d_{k-1} g_{\alpha_{k-1}}(x_i) + d_{k+1} g_{\alpha_{k+1}}(x_i) + \cdots + d_n g_{\alpha_n}(x_i) = p(x_i)$$

for $i = 1, 2, \cdots, h$. Define $r(x) = \sum_{i=0}^{n} d_i g_{\alpha_i}(x)$, where $d_k = 0$, and let U be an open set containing S such that $p(x)$ and $r(x)$ are of the same sign in U. Then there exists a real number ε such that $0 < \varepsilon < \|p\|$ and $\max_{[0,1]-U} |p(x)| = \|p\| - \varepsilon$.

Select a real number λ so that $0 < \lambda < (\varepsilon / \|r\|)$. Then $\|p - \lambda r\| < \|p\|$. But $p(x) - \lambda r(x) = \sum_{j=0}^{n} (A_j - \lambda d_j) g_{\alpha_j}(x)$ satisfies the hypothesis (on q) of this theorem and we have reached a contradiction. The proof is complete.

COROLLARY 2.12. *Suppose that* $\{g_\alpha\}_{\alpha=0}^{\infty}$ *is a sequence of functions with property* $\mathscr{D}$. *Suppose* $\prod_{k=0}^{n} t_k \neq 0$ *and* $p(x) = \sum_{k=0}^{n} t_k g_k(x)$ *is an OGP with* $\|p\| = 1$. *Then if* $q(x) = \sum_{k=0}^{n} a_k g_k(x)$ *is a GP with real coefficients, we have*

$$|a_k| \leqq \|q\|\,|t_k| \qquad \text{for } k = 0, 1, \cdots, n .$$

PROOF. The result holds for $k = 0$, when g_0 is constant, since $|q(0)| = |a_0 g_0| \leqq \|q\| = \|q\|\,|t_0 g_0|$. Suppose then $k \geqq 0$ and g_k is not a constant function. Consider the GP

$$p_1(x) = \frac{a_k}{t_k} \sum_{j=0}^{n} t_j g_j(x) .$$

Since p is an OGP, we have, by Theorem 2.10, that $\|q\| > |a_k/t_k|$.

From Example 2.5 and Corollary 2.12 we have the following results.

COROLLARY 2.13. *If* α *is a positive real number and if* $q(x) = \sum_{k=0}^{n} a_k x^{k\alpha}$, *then* $|a_k| \leqq \|q\|\,|t_k|$, $k = 0, 1, \cdots, n$ *where* t_k *is the coefficient of* $x^{k\alpha}$ *in* $T_{2n}(x^{\alpha/2})$.

COROLLARY 2.14. *If* α *is a positive real number and if* $q(x) = \sum_{k=0}^{n} a_k x^{(2k+1)\alpha}$, *then* $|a_k| \leqq \|q\|\,|t_k|$, $k = 0, 1, \cdots, n$, *where* t_k *is the coefficient of* $x^{(2k+1)\alpha}$ *in* $T_{2n+1}(x^{\alpha})$.

COROLLARY 2.15. *If* α *is a positive real number and if* $q(x) = \sum_{k=0}^{n} a_k x^{2k\alpha}$, *then* $|a_k| \leqq \|q\|\,|t_k|$, $k = 0, 1, \cdots, n$, *where* t_k *is the coefficient of* $x^{2\alpha k}$ *in* $T_{2n}(x^{\alpha})$.

We now give an existence and uniqueness theorem for OGP's.

THEOREM 2.16. *Let* $\{g_\alpha\}_{\alpha=0}^{\infty}$ *be a sequence of functions with property* $\mathscr{D}$. *Further suppose that for each positive integer n, there is an OGP* $p_n(x) = \sum_{k=0}^{n} A_k g_k(x)$ *with* $\prod_{k=0}^{n} A_k \neq 0$. *Then to a given set of integers* $\{\alpha_0, \cdots, \alpha_n\}$ *there corresponds an OGP* $p(x) = \sum_{k=0}^{n} a_k g_{\alpha_k}(x)$ *with* $\prod_{k=0}^{n} a_k \neq 0$, *which is unique except for a constant factor.*

REMARK. We have mentioned that the product $\prod_{k=0}^{n} A_k \neq 0$ to emphasize the precise set of subscripts with respect to which p_n is an OGP. A similar remark applies to the condition following the definition of p.

PROOF. Let $\{\alpha_0, \cdots, \alpha_n\}$ be the given set of integers, let c be a nonzero real constant, and let k be an integer less than or equal to n such that $g_{\alpha_k}(x)$ is not a constant function. We will show that there exists a unique OGP $p(x) = \sum_{j=0}^{n} a_j g_{\alpha_j}(x)$ with $a_k = c$.

Let R^n denote Euclidean n-space and define $Q\colon R^n \to R^1$ so that

$$Q(B) = \max_{0 \leq x \leq 1} | B_0 g_{\alpha_0}(x) + \cdots + B_{k-1} g_{\alpha_{k-1}}(x) + c g_{\alpha_k}(x) + B_{k+1} g_{\alpha_{k+1}}(x) + \cdots + B_n g_{\alpha_n}(x) |$$

for all $B = (B_0, \cdots, B_{k-1}, B_{k+1}, \cdots, B_n) \in R^n$. Then there exists $C = (C_0, \cdots, C_{k-1}, C_{k+1}, \cdots, C_n) \in R^n$ such that $\inf_{B \in R^n} Q(B) = Q(C)$. Moreover, by an argument, identical to the one employed in the proof of Theorem 2.10, we have that C is unique and

$$C(x) = \sum_{\substack{j=0 \\ j \neq k}}^{n} C_j g_{\alpha_j}(x) + c g_{\alpha_k}(x)$$

is an *OGP*.

THEOREM 2.17. *Let $\{g_\alpha\}_{\alpha=0}^{\infty}$ be a sequence of functions with property $\mathscr{D}$ such that if $\alpha > \beta$ then $g_\alpha(x) = o(g_\beta(x))$ as $x \to 0$. If*

$$p(x) = \sum_{\substack{k=0 \\ k \neq m}}^{n} A_k g_{\alpha_k}(x) + g_{\alpha_m}(x)$$

and

$$q(x) = \sum_{\substack{k=0 \\ k \neq m}}^{n} B_k g_{\beta_k}(x) + g_{\alpha_m}(x)$$

are both OGP's in $[0, 1]$ *where* $0 \leq \alpha_0 < \beta_0 < \cdots < \beta_{i-1} < \alpha_m < \beta_{i+1} < \alpha_{i+1} < \cdots < \beta_n < \alpha_n$ *then* $\| p \| > \| q \|$.

PROOF. Suppose that $\| p \| \leq \| q \|$. Since $g_{\beta_0}(x)$ is not a constant function we have, by Theorem 2.6, $| q(x) | = \| q \|$ at $x_1, \cdots, x_n, 1$ where $0 < x_1 < \cdots < x_n < 1$. Since p and q are both *OGP*'s, by property $\mathscr{D}$ and Theorem 2.8, we have that $(q - p)$ has at most n zeros in $(0, 1]$. It follows by an easy argument that $(q - p)(x) \neq 0$ in $(0, x_1]$. As $x \to 0$,

$$(q - p)(x) = g_{\alpha_0}(x) \left\{ -A_0 + \frac{o(g_{\alpha_0}(x))}{g_{\alpha_0}(x)} \right\}$$

and so $(q - p)$ takes the sign of $-A_0$ in $(0, x_1]$. But

$$q(x) = g_{\beta_0}(x) \left\{ B_0 + \frac{o(g_{\beta_0}(x))}{g_{\beta_0}(x)} \right\} \qquad (x \to 0)$$

and so q and hence $q - p$ takes the sign of B_0 in $(0, x_1]$. Since A_0 and B_0 are of the same sign, we have a contradiction, and the theorem is proved.

Finally we note the following special case of Theorem 2.17.

THEOREM 2.18. *Let $\{g_\alpha\}_{\alpha=0}^{\infty}$ be a sequence of functions with property $\mathscr{D}$ such that if $\alpha > \beta$, $g_\alpha(x) = o(g_\beta(x))$ as $x \to 0$. If $p(x) = g_{\alpha_0}(x) + \sum_{k=1}^{n} A_k g_{\alpha_k}(x)$ and $q(x) = g_{\alpha_0}(x) + \sum_{k=1}^{n} B_k g_{\beta_k}(x)$ are both OGP's with $0 \leq \alpha_0 < \beta_1 < \alpha_1 < \cdots < \alpha_n$, where $g_{\alpha_0}(x)$ is not a constant function, then $\| q \| < \| p \|$.*

COROLLARY 2.19. *If* $p(x) = x + \sum_{k=1}^{n} a_k x^{r_k}$ *is an OGP with* $r_i \in (2i-1, 2i+1)$ *for* $i = 1, 2, \cdots, n$, *then* $\|p\| < 1/(2n+1)$ *for* $n \geqq 1$.

PROOF. By Example 2.5, $T_{2n+1}(x) = \sum_{k=0}^{n} A_k x^{2k+1}$ is an *OGP* with $\|T_{2n+1}\| = 1$ and $|A_1| = 2n+1$. Since $0 < 1 < r_1 < 3 < r_2 < \cdots < 2n-1 < r_n < 2n+1$, by Theorem 2.18, we have $\|p\| < 1/(2n+1)$.

3. Approximation to Real Powers of x. For a given set $\{r_1, \cdots, r_k\}$ of positive non-integral real numbers and for each positive integer n, define

$$E_n(\textstyle\sum_{i=1}^{k} x^{r_i}) = \min_{c_\lambda} \max_{0 \leqq x \leqq 1} \left| \sum_{\lambda=0}^{n} c_\lambda x^\lambda - \sum_{i=1}^{k} x^{r_i} \right| .$$

Here c_i are all real numbers.

We now relate the study of *OGP*'s to the discussion of $E_n(\sum_{i=1}^{n} x^{r_i})$.

THEOREM 3.1. *Let* $r_1, \cdots, r_k$ *be real numbers with* $r_1 < r_2 < \cdots < r_k$. *Suppose there exists an integer* n_0 *such that* $n_0 < r_1 < \cdots < r_k < n_0 + 1$. *Define for each non-negative integer* α

$$g_\alpha(x) = \begin{cases} x^\alpha & \text{if } \alpha \leqq n_0 , \\ \sum_{i=1}^{k} x^{r_i} & \text{if } \alpha = n_0 + 1 , \\ x^{\alpha-1} & \text{if } \alpha \geqq n_0 + 2 . \end{cases}$$

Then to a given set of integers $\{\alpha_0, \alpha_1, \cdots, \alpha_n\}$ *there corresponds an OGP* $p(x) = \sum_{k=0}^{n} a_k g_{\alpha_k}(x)$ *with* $\prod_{k=0}^{n} a_k \neq 0$.

PROOF. By comparison with Example 2.2, it is easily verified that $\{g_\alpha\}_{\alpha=0}^{\infty}$ has property $\mathscr{D}$. We now use Theorem 2.16. Let n be a positive integer. If $n \leqq n_0$, then, as noted in Example 2.5, $T_{2n}(\sqrt{x}) = \sum_{k=0}^{n} A_k g_k(x)$ is an *OGP* with $\prod_{k=0}^{n} A_k \neq 0$. If $n \geqq n_0 + 1$, since $\{1, x, \cdots, x^n\}$ satisfies the Haar condition, there exists a unique polynomial $q(x) = \sum_{i=0}^{n} c_i x^i$ of best approximation to $g_{n_0+1}(x)$ (see [5, p. 81]). Again since $\{1, x, \cdots, x^n\}$ satisfies the Haar condition, there exist at least $n+1$ points $x_0, \cdots, x_n$ with $0 \leqq x_0 < \cdots < x_n \leqq 1$ with $q(x_i) - g_{n_0+1}(x_i) = \pm\|q - g_{n_0+1}\|$. Hence the *GP* $p_n(x) = q(x) - g_{n_0+1}(x) = \sum_{k=0}^{n} A_k g_k(x)$ is an *OGP*. By property $\mathscr{D}$, $\prod_{k=0}^{n} A_k \neq 0$. Theorem 2.16 enables us now to complete the proof.

Note that if the set of real numbers $\{r_1, \cdots, r_k\}$ is given satisfying the hypothesis of Theorem 3.1, then there exists an *OGP* $p(x) = \sum_{i=0}^{n} c_i x^i + \sum_{\lambda=1}^{k} x^{r_\lambda}$. By Theorem 2.10, $E_n(\sum_{i=1}^{k} x^{r_i}) = \|p\|$. On the other hand, if we are given $p(x) = \sum_{i=0}^{n} c_i x^i + \sum_{\lambda=1}^{k} x^{r_\lambda}$ such that $E_n(\sum_{i=1}^{k} x^{r_i}) = \|p\|$, then by Theorem 2.11, we have that p is an *OGP*.

We now give a lower bound for $E_n(x^r)$ where $r \leqq 1/3$.

THEOREM 3.2. *If* $r \in (0, 1/3)$, *then for each integer* $n \geqq 2$, *we have*

$nE_n(x^r) > r/2$.

PROOF. Write $E'_n(x^r) = \min_{c_i} \max_{0 \leq x \leq 1} | x^r - \sum_{i=1}^n c_i x^i |$. By Theorem 3.1, there exist OGP's $q(x) = x^r + \sum_{k=1}^n B_k x^k$ and $p(x) = x^r + \sum_{k=0}^n A_k x^k$ such that $E'_n(x^r) = || q ||$ and $E_n(x^r) = || p ||$. By Theorem 2.10, $E'_n(x^r) < || p - A_0 || \leqq || p || + | A_0 | = 2E_n(x^r)$. So we only need to show that $E'_n(x^r) > r/n$.

Take $\alpha_1 = 3$ and for each integer $\lambda = 2, \cdots, n$, let α_λ be an odd integer such that $\lambda - 1 < \alpha_\lambda r < \lambda$. This choice is always possible. Let $C(x) = x^r + c_1 x^{3r} + c_2 x^{\alpha_2 r} + \cdots + c_n x^{\alpha_n r}$ be an OGP. Since $0 < r < 3r < 1 < \alpha_2 r < 2 < \cdots < n - 1 < \alpha_n r < n$, by Theorem 2.18, we have $|| q || > || C ||$. But $T_{\alpha_n}(x^r)$ is an OGP with coefficient of x^r equal to $\pm \alpha_n$, so that by Theorem 2.10,

$$|| C || > \max_{0 \leqq x \leqq 1} \left| \frac{T_{\alpha_n}(x^r)}{\alpha_n} \right| = \frac{1}{\alpha_n} .$$

Hence $E'_n(x^r) = || q || > 1/\alpha_n > r/n$.

REMARK. It is shown in [2] that $E_n(x^{1/3}) > 1/6(3n - 1)$, $n \geqq 2$.

We now give examples of OGP's

1. Let h and k be positive real numbers with $h < k$. Then $p(x) = a_1 x^h + a_2 x^k$ is an OGP with $\lambda^{k/(k-h)}/(h - \lambda) = k^{k/(k-h)}/(k - h)$ where $\lambda = h\{1 - (1/a_2) || p ||\}$.

If we take here $h = 1$, $k = 3$ then $\lambda = 3/4$ and $p(x) = || p || \, T_3(x)$.

2. Let h and k be positive real numbers with $h < k$. Then

$$p(x) = a_0 + 2a_0 \left(\frac{k}{h - k} \right) \left(\frac{k}{h} \right)^{h/(k-h)} (x^h - x^k)$$

is an OGP. (See [2].)

If we take $k = 2h$, we get $p(x) = a_0 T_4(x^{h/2})$.

3. Let h be a positive real number. Then $p(x) = 1 + a_1 x^h + a_2 x^{2h} + a_3 x^{4h}$ is an OGP, where (See [2].)

$$a_1 = \frac{-4(1 + y)^2}{y(1 + 2y)}, \quad a_2 = \frac{2(3y^2 + 2y + 1)}{y^2(2y + 1)}, \quad a_3 = \frac{-2}{y^2(2y + 1)},$$

and

$$y = \frac{1}{9}(2\sqrt{3} - 3 + \sqrt{6}(\sqrt{3} - 1)^{1/2}) .$$

4. The following examples of OGP's were obtained on the computer by a method similar to the one described in [1]. Let

$$E_n \equiv E_n(x^{1/\pi}) = \min_c \max_{0 \leqq x \leqq 1} \left| x^{1/\pi} - \left(\sum_{k=0}^{n} c_k x^k \right) \right| .$$

We list below OGP's and E_n's corresponding to $n = 1, 2, \cdots, 7$.

$n = 1,\ E_1 = 0.19972$

$$p(x) = 1 - 5.007064x^{1/\pi} + 5.007064x .$$

$n = 2,\ E_2 = 0.13409$

$$p(x) = 1 - 7.457358x^{1/\pi} + 18.203350x - 12.746000x^2 .$$

$n = 3,\ E_3 = 0.10460$

$$p(x) = 1 - 9.559751x^{1/\pi} + 40.04487x - 74.21441x^2 + 43.72927x^3 .$$

$n = 4,\ E_4 = 0.087416$

$$p(x) = 1 - 11.4396x^{1/\pi} + 70.5913x - 244.568x^2 + 345.287x^3 - 161.870x^4 .$$

$n = 5,\ E_5 = 0.075972$

$$p(x) = 1 - 13.1627x^{1/\pi} + 109.852x - 608.063x^2 + 1501.12x^3 - 1607.66x^4 + 617.917x^5 .$$

$n = 6,\ E_6 = 0.067707$

$$p(x) = 1 - 14.7695x^{1/\pi} + 157.840x - 1273.10x^2 + 4786.99x^3 - 8648.85x^4 + 7385.69x^5 - 2395.80x^6 .$$

$n = 7,\ E_7 = 0.061418$

$$p(x) = 1 - 16.2818x^{1/\pi} + 214.550x - 2372.75x^2 + 12546.2x^3 - 33583.3x^4 + 47322.1x^5 - 33494.0x^6 + 9383.42x^7 .$$

5. Write

$$t \equiv \frac{1}{2}(x^{1/3} + x^{1/2}) ,$$

$$E_n = E_n(t) = \min_c \max_{0 \leqq x \leqq 1} | t - \textstyle\sum_{k=0}^{n} c_k x^k | .$$

The OGP's and E_n's corresponding to $n = 1, 2, \cdots, 7$ are as follows.

$n = 1,\ E_1 = 0.15818$

$$p(x) = 1 - 6.32153t + 6.32153x .$$

$n = 2,\ E_2 = 0.096893$

$$p(x) = 1 - 10.32060t + 22.30406x - 13.98345x^2 .$$

$n = 3,\ E_3 = 0.071621$

$$p(x) = 1 - 13.96234t + 48.33399x - 80.66461x^2 + 46.29292x^3 .$$

$n = 4,\ E_4 = 0.057675$

$$p(x) = 1 - 17.33824t + 84.37561x - 264.4150x^2 + 363.9411x^3 - 168.5634x^4 .$$

$n = 5,\ E_5 = 0.048751$

$$p(x) = 1 - 20.51203t + 130.3663x - 655.0507x^2 + 1577.841x^3 - 1670.111x^4 + 637.4665x^5 .$$

$n = 6,\ E_6 = 0.042501$

$$p(x) = 1 - 23.52862t + 186.2697x - 1367.813x^2 + 5021.601x^3 - 8969.480x^4 + 7607.542x^5 - 2456.596x^6 .$$

$n = 7,\ E_7 = 0.037858$

$$p(x) = 1 - 26.41432t + 252.0125x - 2543.215x^2 + 13136.03x^3 - 34767.89x^4 + 48662.42x^5 - 34287.53x^6 + 9574.613x^7 .$$

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References

[1] R. A. Bell and S. M. Shah, Oscillating polynomials and approximations to $|x|$, The Publications of the Ramanujan Institute, 1 (1969), 167–177.

[2] R. A. Bell and S. M. Shah, Oscillating polynomials and approximations to fractional powers of x, J. of Approximation Theory, 1 (1968), 269–274.

[3] S. Bernstein, Sur la meilleure approximation de $|x|$ par des polynomes de dégrés donnes, Acta Math. 37 (1913), 1–57.

[4] J. C. Burkill, Lectures on approximation by polynomials, Tata Institute of Fundamental Research, Bombay, 1959.

[5] E. W. Cheney, Introduction to Approximation Theory, McGraw Hill, New York, 1966.

[6] G. G. Lorentz, Approximation of Functions, Athena Series, Holt Rinehart and Winston, New York, 1966.

[7] J. V. Uspensky, Theory of Equations, McGraw-Hill, New York, 1948.

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