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NONHOMOGENEOUS ELLIPTIC SYSTEMS AND SCATTERING

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Abstract. We prove the main conclusions of scattering theory for a class of first order elliptic systems in which the free (unperturbed) system is not homogeneous. The perturbed system need not be essentially self adjoint, and the assumptions on the perturbation are mild. The Dirac operator is a special case of the systems considered.

1. Introduction. We examine scattering theory for elliptic systems of the form

$$H(x, D) = E(x)^{-1}(A_0(D) - B(x)) , \tag{1.1}$$

where

$$A_0(D) = \sum_{j=1}^{m} A_0^j D_j + B_0 , \tag{1.2}$$

$x = (x_1, \cdots, x_n)$, $E(x)$, $B(x)$, B_0 and the A_0^j are hermitian $m \times m$ matrices and $D_j = \partial/i\partial x_j$. This is compared with the free system

$$H_0(D) = E_0^{-1} A_0(D) , \tag{1.3}$$

where E_0 is also hermitian. The aim of the paper is to obtain the main conclusions of scattering theory under minimal conditions on the perturbation $B(x)$. Our assumptions on this matrix will not be sufficient to make $H(x, D)$ essentially self adjoint. This creates several technical difficulties. The situation is further complicated by the appearance of the matrix B_0 in the free system, which spoils homogeniety. The fact that $E(x) \risingdotseq E_0$ adds to the difficulty.

To obtain a self adjoint extension of $H(x, D)$ we need a criterion involving two Hilbert spaces. For this purpose we generalized a theorem due to Kato [1] for one Hilbert space. We obtain this theorem without restricting the size of the perturbation or requiring it to be compact (see Section 2). We use the factored perturbation technique for two Hilbert spaces developed in [12]. In order to get maximum benefit from this technique we introduced pseudo-differential operators of order 1/2 to make the factorization as even as possible. This introduces other technical difficulties which require special attention (see Section 3).

We assume that E_0 is positive definite and that $E(x)$ is bounded and uniformly positive definite. If we let $\mathscr{H}_0$ be $[L^2(E^n)]^m$ equipped with the scalar product

$$(u, v)_0 = \int v^* E_0 u dx$$

(here v^* denotes the complex conjugate transpose of v), then $H_0(D)$ defined on the test functions $C_0^\infty = [C_0^\infty]^m$ becomes a symmetric operator on $\mathscr{H}_0$. It is a simple matter to show via Fourier transforms that its closure H_0 is self adjoint. Similarly, if $\mathscr{H}$ is the same space equipped with the scalar product

$$(u, v) = \int v^* E u dx ,$$

then $H(x, D)$ with suitable domain becomes hermitian on $\mathscr{H}$.

Our assumptions are

1) $\sup_x \int_{|x-y|<\delta} |B(y)| \, |x - y|^{1-n} dy \to 0 \quad \text{as} \quad \delta \to 0$

2) Put $N(x) = E_0^{1/2}(E_0^{-1} - E(x)^{-1})E_0^{1/2}$. Then

$$\sup_x |N(x)| < 1 . \tag{1.4}$$

Also $N = V(x)W(x)$, where V and W are bounded and satisfy uniform Hölder conditions with exponents $> 1/2$. $D_j V$ is locally square integrable for each j.

3) Put

$$Z_0(x) = \int_{|x-y|<1} (|V(y)|^2 + |W(y)|^2 + |B(y)|) dy$$

and

$$Z(x) = Z_0(x) + \int_{|x-y|<1} \sum |D_j V(y)|^2 dy .$$

Then

$$Z_0(x) \to 0 \quad \text{as} \quad |x| \to \infty \tag{1.5}$$

and there are numbers α, p satisfying

$$\alpha \geqq 0, 1 \leqq p \leqq \infty, \alpha > 1 - [2n/(n + 1)p] \tag{1.6}$$

such that $\rho(x)^\alpha Z(x) \in L^p$, where $\rho(x) = 1 + |x|$.

4) $A_0(D)^2$ is diagonal.

Let J be the identification operator from $\mathscr{H}_0$ to $\mathscr{H}$: $Ju = u$. When they exist, the wave operators for self adjoint operators H on $\mathscr{H}$ and H_0 on $\mathscr{H}_0$ are defined by the strong limits

$$W_{\pm}u = \lim_{t\to\pm\infty} e^{itH}Je^{-itH_0}P_0u \,, \tag{1.7}$$

where P_0 is the projection onto the absolutely continuous subspace of H_0 (for definitions, cf. [2]). We say that the wave operators are complete if their ranges coincide with the absolutely continuous subspace of H. We have

THEOREM 1.1. *Under hypotheses* 1)—4) $H(x, D)$ *has a self adjoint extension* H *such that the wave operators* (1.7) *exist and are complete. Moreover, if* $\varphi(s)$ *is a function satisfying*

$$\int_0^\infty \left| \int_\Gamma e^{-i\tau s - it\varphi(s)} ds \right|^2 d\tau \longrightarrow 0 \quad as \quad t \longrightarrow \infty \tag{1.8}$$

and

$$\int_\Gamma e^{-it\varphi(s)} ds \longrightarrow 0 \quad as \quad t \longrightarrow \infty \tag{1.9}$$

for each bounded Borel set Γ, *then*

$$W_{\pm}u = \lim_{t\to\pm\infty} e^{it\varphi(H)}Je^{-it\varphi(H_0)}u \,. \tag{1.10}$$

The relations $\psi(H)W_{\pm} = W_{\pm}\psi(H_0)$ *hold for each Borel function* ψ. *The spectrum of* H_0 *is absolutely continuous and the singular spectrum of* H *is of measure* 0.

The proof of the theorem will be given in Section 4 after we give the abstract theory in the next section and prove some technical lemmas in Section 3. Here we make the following observations.

1. Hypothesis 1) can be weakened to include functions having Coulomb type singularities at finite points. The method is similar to that of [3].

2. The theorem holds if we replace hypothesis 3) by (1.5) and

$$\sup_x \int Z(y)\rho(x - y)^{-\beta} dy < \infty \tag{1.11}$$

for some $\beta < (1/2)(n - 1)$. This follows from our proof given in Section 4.

3. The Dirac operator is a special case of our system. For then $E = E_0 = I$, $A_0(\xi)^2 = |\xi|^2 + \mu^2$ and

$$Z(x) = Z_0(x) = \int_{|x-y|<1} |B(y)| \, dy \,. \tag{1.12}$$

Thus we have

COROLLARY 1.2. *For the Dirac operator the conclusions of Theorem* 1.1 *hold if* $B(x)$ *satisfies hypothesis* 1) *and* $Z(x)$ *given by* (1.12) *satisfies*

$$(1.13) \qquad Z(x) \to 0 \quad as \quad |x| \to \infty$$

and $\rho^\alpha Z \in L^p$ *for some* α, p *satisfying* (1.6).

4. In [4] we considered systems similar to (1.1). However there are several basic differences which reflect in the results obtained. There we impose more restrictions on $B(x)$ to obtain essential self adjointness for $H(x, D)$. There too $B_0 = 0$ resulting in the homogeneity of $A_0(D)$. This allowed us to drop hypothesis 4) but brought about a corresponding weakness in the results. Dirac operators could not be covered there.

We now show that hypothesis 4) can be dropped when $B_0 = 0$ in (1.2). For instance we have

THEOREM 1.3. *Suppose* $B_0 = 0$ *in* (1.2) *and hypotheses* 1) *and* 2) *hold. Assume* (1.5) *and that* $\rho^\alpha Z \in L^1$ *for some* $\alpha > 0$. *Then the conclusions of Theorem* 1.1 *hold.*

The proof of this theorem will be given in Section 4. October 10, 1974.

Questions in scattering theory for Dirac operators have been considered by several authors, including Birman [5, 6] Prosser [7], Mochizuki [8], Kato [2], Thomson [9], Yamada [10] and Guillot-Schmidt [11]. Because of the added technical difficulties for the systems we have considered, our approach is different from the various methods of these authors. We have borrowed ideas from Guillot-Schmidt [11].

2. **The abstract theory.** In proving our results we shall make use of two abstract theorems in Hilbert space. The first (Theorem 2.1) extends a theorem due to Kato [1], and the second (Theorem 2.8) was proved in [12].

THEOREM 2.1. *Let* $\mathscr{H}$, $\mathscr{H}_1$ *and* $\mathscr{K}$ *be Hilbert spaces, and let* H *be a self adjoint operator on* $\mathscr{H}$. *Suppose there are closed linear operators* A, B *from* $\mathscr{H}$ *to* $\mathscr{K}$ *such that* $D(H) \subset D(A) \cap D(B)$ *and a linear bijective operator* J *from* $\mathscr{H}$ *to* $\mathscr{H}_1$ *such that* $J_0 = J^*J$ *maps* $D(H)$ *into* $D(B)$. *Assume that there is a* $z_0 \in \rho(H)$ *such that* $Q(z) = [AR(z)B^*]$ *is bounded and* $G(z) = I + Q(z)$ *has a bounded inverse for* $z = z_0$ *and* $z = \bar{z}_0$, *where* $R(z) = (z - H)^{-1}$, *and that*

$$(2.1) \quad (Hu, J_0 v) - (Au, BJ_0 v)_{\mathscr{K}} = (J_0 u, Hv) - (BJ_0 u, Av)_{\mathscr{K}} \; u \;, \; v \in D(H) \,.$$

Then there is a unique self adjoint operator H_1 *on* $\mathscr{H}_1$ *such that* $H_1 J \supset J(H - B^*A)$, $D(H_1) \subset D(AJ^{-1}) \cap D(BJ^*)$ *and*

$$(2.2) \quad R(z) - J^{-1}R_1(z)J = [R(z)B^*]AJ^{-1}R_1(z)J = [J^{-1}R_1(z)JB^*]AR(z) \,,$$

where

$$R_1(z) = (z - H_1)^{-1} \,.$$

In proving the theorem we shall make use of a few lemmas.

LEMMA 2.2. *If* $S(z) = G(z)[BJ_0R(z)B^*]$, *then* $S(z)^* = S(\bar{z})$.

PROOF. First note that (2.1) implies

$$J_0R(z) - R(z)J_0 = [R(z)J_0B^*]AR(z) - [R(z)A^*]BJ_0R(z) . \tag{2.3}$$

Now

$$\begin{aligned} S(\bar{z})^* &= [BR(z)J_0B^*]G(\bar{z})^* = [B(R(z)J_0 + [R(z)J_0B^*]AR(z))B^*] \\ &= [B(J_0R(z) + [R(z)A^*]BJ_0R(z))B^*] = S(z) \text{ by (2.3).} \end{aligned}$$ □

LEMMA 2.3. $G(z)BJ_0R(z) = BR(z)J_0 + [BR(z)J_0B^*]AR(z)$.

PROOF. The left hand side equals $(I + [BR(z)A^*])BJ_0R(z)$. Apply (2.3). □

LEMMA 2.4. *If* $F(z) = [R(z)A^*](BJ_0R(z) - G(z)^{-1}BR(z)J_0)$, *then* $F(z)^* = F(\bar{z})$.

PROOF. By Lemma 2.3, $F(z) = [R(z)A^*]G(z)^{-1}[BR(z)J_0B^*]AR(z)$. This equals $F(\bar{z})^*$ by Lemma 2.2. □

LEMMA 2.5. *Put*

$$T(z) = R(z) - [R(z)A^*]G(z)^{-1}BR(z) . \tag{2.4}$$

Then $T(z)$ *is injective and satisfies the first resolvent equation*

$$T(z) - T(z') = (z' - z)T(z)T(z') . \tag{2.5}$$

PROOF. Equation (2.5) is the consequence of a simple computation and that fact that $R(z)$ satisfies it. If $T(z)u = 0$, then $G(z)^{-1}BR(z)u = BT(z)u = 0$. Hence $R(z)u = 0$ by (2.4). Since $R(z)$ is the resolvent of an operator, we must have $u = 0$. □

LEMMA 2.6. $J_0T(z)^* = T(\bar{z})J_0$.

PROOF. Note that

$$T(z)J_0 = R(z)J_0 + F(z) - [R(z)A^*]BJ_0R(z) , \tag{2.6}$$

where $F(z)$ is given in Lemma 2.4. By that lemma

$$J_0T(\bar{z})^* = J_0R(z) + F(z) - [R(z)J_0B^*]AR(z) .$$

But this equals (2.6) by (2.3). □

PROOF OF THEOREM 2.1. Put $R_1(z) = J^{*-1}T(z)J^*$. Then $R_1(z)$ is a bounded operator for z in neighborhoods of z_0 and $\bar{z}_0$. By Lemma 2.5 it is injective and satisfies (2.5). Thus it is the resolvent of a closed operator H_1 on

$\mathscr{H}_1$. Since $J^*R_1(z)J = T(z)J_0$, we see that $J^*R_1(z)^*J = J^*R_1(\bar{z})J$ and consequently $R_1(z)^* = R_1(\bar{z})$. This shows that H_1 is self adjoint. From (2.4) we see that (2.2) holds in neighborhoods of z_0 and $\bar{z}_0$, and consequently for all $z \in \rho(H) \cap \rho(H_1)$. Finally, suppose $u \in D(H) \cap D(B^*A)$ and put $v = (z - H)u$. Then by (2.2), $Ju = R_1(z)Jv + R_1(z)JB^*Au$. This shows that $Ju \in D(H_1)$ and that $(z - H_1)Ju = J(z - H + B^*A)u$. Hence H_1J is an extension of the operator $J(H - B^*A)$. To show uniqueness, let H_1 be any self adjoint operator such that $D(H_1) \subset D(AJ^{-1}) \cap D(BJ^*)$ and (2.2) holds. A simple computation gives

$$BR(z) = G(z)BJ^*R_1(z)J^{*-1} . \tag{2.7}$$

Thus

$$T(z) = R(z) - [R(z)A^*]BJ^*R_1(z)J^{*-1} ,$$

where $T(z)$ is given by (2.4). Taking adjoints, we see by (2.2) that

$$J^{-1}R_1(z)J = T(\bar{z})^* . \tag{2.8}$$

Since $T(z)$ is uniquely determined by H, A and B, the same must be true of H_1. □

REMARK. In Kato's theorem $\mathscr{H}$, $\mathscr{H}_1$ and $\mathscr{K}$ are the same space, $J = I$ and the norm of $Q(z)$ is assumed < 1. Konno-Kuroda [13] replace the last restriction with the assumption that $Q(z)$ is compact for each z. Note that we remove both of these stipulations.

We note the following consequence of Theorem 2.1.

COROLLARY 2.7. *In addition to the hypotheses of Theorem* 2.1, *assume that J_0^{-1} maps $D(H)$ into itself, and either A or B is bounded. Then $J(H - B^*A)J^{-1}$ is self adjoint.*

PROOF. By the theorem there is a self adjoint operator H_1 on $\mathscr{H}_1$ such that

$$H_1J \supset J(H - B^*A) . \tag{2.9}$$

Taking adjoints we get $J^*H_1 \subset (H - A^*B)J^*$ (this is where we use the fact that either A or B is bounded) and consequently $J^*H_1J \subset (H - A^*B)J_0$. Thus $D(H) \subset D(H_1J) \subset D(HJ_0) \subset D(H)$, the last inclusion coming from the fact that J_0^{-1} maps $D(H)$ into itself. Thus $D(H) = D(H_1J)$, and consequently we have equality in (2.9). Since H_1 is self adjoint, the result follows.

THEOREM 2.8. *In addition to the hypotheses of Theorem* 2.1, *assume that*

a. *There is a* $z_0 \in \rho(H)$ *such that* $BR(z_0)[R(z)A^*]$ *is compact for all nonreal* z.

b. *The singular spectrum of* H *is of measure* 0 *in* Λ *and there are locally Hölder continuous functions* $M(s)$, $N(s)$ *from* Λ *to* $B(\mathscr{K})$ *and a dense subset* S *of* $D(A^*)$ *such that*

$$d(E(s)A^*u, A^*v)/ds = (M(s)u, v)_{\mathscr{K}} \text{ a.e., } u, v \in S, \tag{2.10}$$

$$d(E(s)A^*u, B^*v)/ds = (N(s)u, v)_{\mathscr{K}} \text{ a.e., } u, v \in S, \tag{2.11}$$

where $\{E(s)\}$ *is the spectral family of* H.

c. A *is injective*

d. *There is a closed set* e *of measure* 0 *such that* $[J_0 - I]E(\Gamma)$ *is compact for each interval* Γ *having compact closure in* $\Lambda - e$.

Let H_1 *be the self adjoint operator satisfying the conclusions of Theorem* 2.1. *Then the strong limits*

$$W_{\pm}f = \lim_{t\to\pm\infty} e^{itH_1}Je^{itH}E^{ac}(\Lambda)f$$

exist and are complete. Moreover,

$$W_{\pm}f = \lim_{t\to\pm\infty} e^{it\varphi(H_1)}Je^{-it\varphi(H)}E^{ac}(\Lambda)f$$

holds whenever φ *is a function satisfying* (1.8) *and* (1.9) *for any bounded Borel set* Γ. *The relations* $\psi(H_0)W_{\pm} = W_{\pm}\psi(H)$ *hold for any Borel function* ψ.

In Section 4 we use Theorems 2.1 and 2.8 to prove Theorems 1.1 and 1.3. Note that in these applications it is not required that $\|Q(z)\| < 1$ for some z.

3. Some estimates. Before proving the theorems of Section 1, we shall derive some inequalities which are used in the proofs. They concern the operator given by

$$S_z = \bar{F}(z + |\xi|)^{1/2}F. \tag{3.1}$$

First we note

LEMMA 3.1. *If* $L(x)$ *satisfies*

$$\sup_x \int_{|x-y|<\delta} |L(x)|^2 |x-y|^{1-n}\, dy \to 0 \text{ as } \delta \to 0, \tag{3.2}$$

then $\|LS_a^{-1}\| \to 0$ *as* $a \to \infty$.

PROOF. Put $\tilde{S}_z = \bar{F}(z^2 + |\xi|^2)^{1/4}F$. Then

$$LS_a^{-1} = L\tilde{S}_a^{-1}\bar{F}(a^2 + |\xi|^2)^{1/4}(a + |\xi|)^{-1/2}F.$$

Thus $\|LS_a^{-1}\| \leqq C\|L\tilde{S}_a^{-1}\|$, which converges to 0 as $a \to \infty$ (cf. [14, p. 138]). $\square$

LEMMA 3.2. *If $a \geqq 1$, then $\|S_a^2 R(\pm ia)\|$ has a bound independent of a.*

PROOF. By symmetry we have

$$a^2 |E_0^{1/2}(ia - H_0(\xi))^{-1}E_0^{-1/2}|^2 \leqq 1 \tag{3.3}$$

$$|E_0^{1/2}H_0(\xi)(ia - H_0(\xi))^{-1}E_0^{-1/2}| \leqq 1 . \tag{3.4}$$

Since $H_0(D)$ is elliptic there are positive constants c_0 and N such that

$$|H_0(\xi)| \geqq c_0|\xi| , \qquad |\xi| > N . \tag{3.5}$$

Thus

$$\begin{aligned} |\xi||(i - H_0(\xi))^{-1}| &\leqq C_0 , \qquad |\xi| \leqq N \\ &\leqq c_0^{-1}|H_0(\xi)(i - H_0(\xi))^{-1}| \leqq C_1 , \qquad |\xi| > N \end{aligned} \tag{3.6}$$

by (3.4). Thus

$$\begin{aligned} |\xi||(ia - H_0(\xi))^{-1}| = |\xi||&(i - H_0(\xi))^{-1}| \\ &\cdot |(i - H_0(\xi))(ia - H_0(\xi))^{-1}| \leqq C_2 \end{aligned}$$

by (3.3), (3.4) and (3.6). Thus by (3.3)

$$(|a| + |\xi|)|(ia - H_0(\xi))^{-1}| \leqq C_3 . \tag{3.7}$$

The same reasoning applies if we replace a by $-a$. The result now follows from (3.7). $\square$

THEOREM 3.3. *Let $g(x)$ be a bounded function which satisfies a uniform Hölder condition with exponent $\theta > 1/2$. Then*

$$\|S_a(gu)\| \leqq \|g\|_\infty \|S_a u\| + C_\theta \|g\|_{\infty,\theta} \|u\| , \tag{3.8}$$

where C_θ depends only on n and θ and $\|g\|_{\infty,\theta}$ is $\|g\|_\infty$ plus the Hölder constant.

PROOF. Put

$$r(\xi) = \int |e^{i\xi x} - 1|^2 |x|^{-n-1} dx . \tag{3.9}$$

Note that $r(\xi)$ is homogeneous of degree 1. Moreover, it is easily checked that it is invariant under any rotation about the origin. Thus $r(\xi) = c_0|\xi|$ for some constant c_0. Hence

$$\begin{aligned} \|S_a u\|^2 &= \int (a + |\xi|)\,|Fu|^2\,d\xi \\ &= a\|u\|^2 + c_0 \int r(\xi)\,|Fu|^2\,d\xi \\ &= a\|u\|^2 + c_0 \int \|u_x - u\|^2\,|x|^{-n-1}\,dx\,, \end{aligned} \tag{3.10}$$

where $u_x(y) = u(x+y)$. Now

$$\begin{aligned} &\left(\int \|(gu)_x - gu\|^2\,|x|^{-n-1}\,dx\right)^{1/2} \\ &\quad \leqq \left(\int \|g_x(u_x - u)\|^2\,|x|^{-n-1}\,dx\right)^{1/2} \\ &\qquad + \left(\int \|(g_x - g)u\|^2\,|x|^{-n-1}\,dx\right)^{1/2} \\ &\quad \leqq \|g\|\,(\|S_a u\|^2 - a\|u\|^2)^{1/2} c_0^{-1/2} \\ &\qquad + (2C_1\|g\|_\infty + C_2 K_\theta(g))\,\|u\|\,, \end{aligned} \tag{3.11}$$

where

$$C_1^2 = \int_{|x|>1} |x|^{-n-1}\,dx,\quad C_2^2 = \int_{|x|<1} |x|^{2\theta-n-1}\,dx \tag{3.12}$$

and

$$K_\theta(g) = \sup_{|x|<1} \frac{\|g_x - g\|_\infty}{|x|^\theta}\,. \tag{3.13}$$

Clearly,

$$R_\theta(g) \equiv 2C_1\|g\|_\infty + C_2 K_\theta(g) \leqq C_\theta \|g\|_{\infty,\theta}\,, \tag{3.14}$$

where C_θ depends only on n and θ. Thus

$$\begin{aligned} \|S_a(gu)\|^2 &\leqq a\|gu\|^2 \\ &\qquad + c_0(\|g\|_\infty (\|S_a u\|^2 - a\|u\|^2)^{1/2} c_0^{-1/2} \\ &\qquad + R_\theta(g)\|u\|)^2 \\ &\leqq a\|gu\|^2 + \|g\|_\infty^2 (\|S_a u\|^2 - a\|u\|^2) \\ &\qquad + 2c_0^{1/2}\|g\|_\infty R_\theta(g)\|S_a u\|\,\|u\| + R_\theta(g)^2\|u\|^2 \\ &\leqq (\|g\|_\infty \|S_a u\| + R_\theta(g)\|u\|)^2 \end{aligned}$$

This gives (3.8) □

REMARK. A more precise form of (3.8) is

$$\|S_a(gu)\| \leqq \|g\|_\infty \|S_a u\| + R_\theta(g)\|u\|\,. \tag{3.15}$$

COROLLARY 3.4. *Under the same hypotheses, for each $c > 0$ there is* *a constant N so large that*

$$\| S_a g S_a^{-1} \| \leqq \| g \|_\infty + \varepsilon \, , \qquad a > N \, . \tag{3.16}$$

PROOF. By (3.8)

$$\| S_a g S_a^{-1} u \| \leqq \| g \|_\infty \| u \| + C_\theta \| g \|_{\infty,\theta} \| S_a^{-1} u \| \, . \tag{3.17}$$

Since $\| S_a^{-1} u \| \leqq a^{-1/2} \| u \|$ by (3.10), we can take a so large that the right hand side of (3.17) is $\leqq (\| g \|_\infty + \varepsilon) \| u \|$. □

LEMMA 3.5. *Let $g(\xi)$ be a smooth function and put*

$$K_r(x) = \int_{|\xi|=r} e^{-i\xi x} g(\xi) dS \, .$$

Then for each $\alpha > 0$ there is a constant $C_{r,\alpha}$ depending only on g, r and α such that

$$| K_r(x) | + \sup_t | t - r |^{-\alpha} | K_r(x) - K_t(x) | \leqq C_{r,\alpha} \rho(x)^{\nu+\alpha} \, , \tag{3.18}$$

where $\nu = (1/2)(1 - n)$.

PROOF. The estimate for $K_r(x)$ itself follows from a result of Littman [15] (and it is true with $\alpha = 0$). To estimate the Hölder constant, note that $K_r = r^{n-1} f(r)$, where

$$f(r) = \int_{|\omega|=1} e^{ir\omega x} g(r\omega) d\omega$$

(in the notation we suppress the dependence of x). It clearly suffices to estimate the Hölder constant for $f(r)$. Now

$$f'(r) = \int_{|\omega|=1} e^{ir\omega x} [(i\omega x) g(r\omega) + g_r(r\omega)] \, d\omega \, .$$

Applying Littman's estimate we get

$$| f(r) | \leqq C_r \rho(x)^\nu, \; | f'(r) | \leqq C_r \rho(x)^{\nu+1} \, .$$

Let k be an integer such that $k\alpha \geqq 1$. Then

$$\begin{aligned} | f(r)^k - f(t)^k | &= k \, | f(\tau)^{k-1} f'(\tau)(r - t) | \\ &\leqq C \, | r - t | \, \rho(x)^{k\nu+1} \end{aligned}$$

for t close to r, where τ is some value between t and r. Thus we have

$$| f(r) - f'(t) | \leqq C \, | r - t |^\alpha \rho(x)^{\nu+\alpha} \, .$$ □

COROLLARY 3.6. *Let $g(\xi)$ be a smooth function and let $P(\xi)$ be a second degree polynomial with real coefficients. Put*

$$K_r(x) = \int_{P(\xi)=r^2} e^{i\xi x} g(\xi) dS . \tag{3.19}$$

Then (3.18) *holds.*

PROOF. There is an affine transformation $\xi = A(\eta + \beta)$ such that $P(\xi) = |\eta|^2 - c$, where c is a constant. Thus (3.19) becomes

$$K_r(x) = e^{i(A\beta)x} \int_{|\eta|^2 = c+r^2} e^{i(A\eta)x} g(A(\eta + \beta)) A dS .$$

We apply Lemma 3.5. □

LEMMA 3.7. *Let $f(\xi)$ be a smooth function such that the surface S given by $f(\xi) = 0$ is bounded and* grad $f \neq 0$ *on S. Let $g(\xi)$ be a function which is bounded on S. Then*

$$\int_S g(\xi) Fu \overline{Fv} dS = (K*u, v) , \qquad u, v \in C_0^\infty , \tag{3.20}$$

where

$$K(x) = \int_S e^{i\xi x} g(\xi) dS . \tag{3.21}$$

PROOF. Put $\delta_a(x) = a/\pi(x^2 + a^2)$, and let $\varphi(\xi)$ be any test function which is 1 on S. Then the left hand side of (3.20) is the limit as $a \to 0$ of

$$\int \delta_a(f(\xi)) \varphi(\xi) g(\xi) Fu \overline{Fv} d\xi = (FK_a Fu, Fv) = (K_a * u, v) ,$$

where

$$K_a(x) = \int e^{i\xi x} \delta_a(f(\xi)) \varphi(\xi) g(\xi) d\xi .$$

But this converges to (3.21). □

Let $P_j(\xi)$ be the j-th eigenvalue of the diagonal matrix $A_0(\xi)^2$. Clearly it is a nonnegative polynomial of degree 2. If I is any Borel subset of the real line, let $I^2 = \{s^2 \mid s \in I\}$. Let $\{E(\lambda)\}$ denote the spectral family of H_0. We have

LEMMA 3.8. *If I is an interval which is a positive distance from the origin, then*

$$(E(I)u, v)_0 = \frac{\operatorname{sgn} I}{2\pi} \sum_{j,k} \int_{P_j(\xi) \in I^2} [\delta_{jk} + P_j(\xi)^{-1/2} A_{jk}(\xi)] Fu_k \overline{Fv_j} d\xi , \tag{3.22}$$

where sgn I *is the sign of the points in I and the $A_{jk}(\xi)$ are the elements of the matrix $A_0(\xi)$.*

PROOF. First we note that H_0 has no eigenvalues. Thus by Stone's theorem

$$(E(I)u, v)_0 = \frac{i}{2\pi} \lim_{a\to 0} \int_I ([R(z) - R(\bar{z})]u, v)_0 ds \ , \tag{3.23}$$

where $z = s + ia$. Suppose $[R(z) - R(\bar{z})]E_0^{-1}u = f$. Then

$$(4a^2A_0^2 + (s^2 + a^2 - A_0^2)^2)f = -2ia(s^2 + a^2 + 2sA_0 + A_0^2)u \ .$$

Taking Fourier transforms and making use of the fact that A_0^2 is diagonal, we get

$$\int_I Ff_j ds \to i \operatorname{sgn} I(Fu_j + P_j(\xi)^{-1/2} \sum A_{jk}(\xi)Fu_k)\chi_j$$

where χ_j is the characteristic function of the set $P_j(\xi) \in I^2$. Substituting into (3.23) we get (3.22). □

COROLLARY 3.9. *There is a closed set e of measure* 0 *such that*

$$d(E(s)u, v)_0/ds = \frac{1}{2\pi} \sum_{j,k} \int_{P_j(\xi)=s^2} [|s| \delta_{jk} + A_{jk}(\xi)]Fu_k\overline{Fv_j}dS, \ s \notin e \ . \tag{3.24}$$

PROOF. Let e be the set of those s for which there is a $\xi \in E^n$ such that $s^2 = P_j(\xi)$, grad $P_j(\xi) = 0$ for some j. This set is clearly closed, and it has measure 0 by Sard's theorem. If I does not intersect e, then $P_j(\xi)$ can be introduced as one of the variables in the integral over the set $P_j(\xi) \in I^2$ in (3.22). Taking the derivative with respect to s, we obtain (3.24). □

COROLLARY 3.10. *The spectrum of H_0 is absolutely continuous and contained in the set $(-\infty, -\mu] \cup [\mu, \infty)$, where*

$$\mu^2 = \min_j \min_\xi P_j(\xi) \ . \tag{3.25}$$

PROOF. This follows immediately from Lemma 3.8 in view of Lemma 3.2, p. 448, of [16]. □

COROLLARY 3.11. *For a.e. s*

$$2\pi d(E(s)u, v)/ds = (K_s * u, v) \tag{3.26}$$

where

$$K_{sjk}(x) = \int_{P_j(\xi)=s^2} e^{ix\xi} [\operatorname{sgn} s\delta_{jk} + s^{-1}A_{jk}(\xi)]dS \ . \tag{3.27}$$

PROOF. If the interval I does not intersect e, then by Lemma 3.8

$$2\pi(E(I)u, v) = (K_I * u, v) \ ,$$

where

$$K_{Ijk}(x) = \int_{P_j(\xi)\in I^2} e^{ix\xi}\,\mathrm{sgn}\, I[\delta_{jk} + P_j(\xi)^{-1/2}A_{jk}(\xi)]d\xi \,.$$

Taking the derivative with respect to the upper endpoint of I, we obtain (3.26) and (3.27).

4. The proofs. Now we give the proof of Theorem 1.1. We verify that the hypotheses of Theorems 2.1 and 2.8 are satisfied. Note first that there exist hermitian matrices $L(x)$, $M(x)$ such that $LM = B$, M is invertible for a.e. x and

$$\text{(4.1)} \qquad \sup_x \int_{|x-y|<\delta} (|L(y)|^2 + |M(y)|^2)\,|x-y|^{1-n}\,dy \to 0 \text{ as } \delta \to 0$$

(hypothesis 1)). Take $\mathscr{K} = L^2 \oplus L^2$ and define

$$\begin{aligned} Au &= \{Mu,\ S_a^{-1}WE_0^{1/2}H_0u\} \\ Bu &= \{LE^{-1}E_0u,\ S_aVE_0^{1/2}u\} \end{aligned}$$

as operators from $\mathscr{H}$ to $\mathscr{K}$, where S_a is the operator (3.1) and a is a positive number to be chosen later. Thus

$$\begin{aligned} A^*\{v, w\} &= E_0^{-1}(Mv + H_0E_0^{-1/2}WS_a^{-1}w) \\ B^*\{v, w\} &= E^{-1}Lv + E_0^{-1/2}VS_aw \,. \end{aligned}$$

For s real, let $H^{s,2}$ be the completion of C_0^∞ with respect to the norm given by

$$\|v\|_{s,2}^2 = \int \rho(\xi)^{2s}\,|Fv|^2\,d\xi \,.$$

In view of Theorem 7.3, Chapter 7, of [14], A, B, A^* and B^* map $H^{1/2,2}$ boundedly into $\mathscr{K}$ by (4.1), hypothesis 2) and Corollary 3.4. Thus these operators are closable, and the domains of their closures contain $H^{1/2,2}$. Next we note that $S_a(I-N)^{-1}S_a^{-1}$ is a bounded operator on $\mathscr{H}$ for a sufficiently large. This follows from the fact that $S_a(I-N)S_a^{-1} = I - S_aNS_a^{-1}$. The latter has a bounded inverse by hypothesis 2) and Corollary 3.4. Since $J_0 = J^* = E_0^{-1}E = E_0^{-1/2}(I-N)^{-1}E_0^{1/2}$, we see that J_0 maps the space $H^{1/2,2}$ into itself and a fortiori $D(H_0) = H^{1,2}$ into $D(B)$. Put $R(z) = (z - H_0)^{-1}$. Then

$$\begin{aligned} \text{(4.2)} \qquad BR(z)A^*\{u, v\} = \{&LJ_0^{-1}R(z)E_0^{-1}Mv \\ &+ LJ^{-1}R(z)H_0E_0^{-1/2}WS_a^{-1}w,\ S_aVE_0^{1/2}R(z)E_0^{-1}Mv \\ &+ S_aVE_0^{1/2}R(z)H_0E_0^{-1/2}WS_a^{-1}w\} \,. \end{aligned}$$

Take $z = \pm ia$. Then

$$\| LJ_0^{-1}R(z)E_0^{-1}M \| + \| LJ_0^{-1}R(z)H_0E_0^{-1/2}WS_a^{-1} \| + \| S_aVE_0^{1/2}R(z)E_0^{-1}M \| \to 0 \text{ as } a \to \infty \tag{4.3}$$

by Lemmas 3.1 and 3.2 and Corollary 3.4. Put

$$P(z) = S_aVE_0^{1/2}R(z)H_0E_0^{-1/2}WS_a^{-1} .$$

If we can show that $I + P(z)$ has a bounded inverse for a sufficiently large, then the same will be true of $G(z) = I + [BR(z)A^*]$ by (4.2) and (4.3). By (3.4)

$$U(z) = E_0^{1/2}R(z)H_0E_0^{-1/2}$$

is a contraction for each z. Thus if we put $C_a = S_aNS_a^{-1}$, we have

$$\| P(z)^k \| \leqq \| S_aVS_a^{-1} \| \, \| C_a \|^{k-1} \| S_aWS_a^{-1} \| \tag{4.4}$$

for each k. By hypothesis 2) and Corollary 3.4, $\| C_a \|$ can be made < 1 by taking a sufficiently large. Thus for such a we can make $\| P(z)^k \| < 1$ by taking k large enough in view of (4.4). This shows that $I + P(z)^k$ is invertible in L^2 for k large. Consequently the same must be true of $I + P(z)$. Thus $G(z)$ has a bounded inverse on $\mathscr{K}$ for a sufficiently large. Finally, we note that in the present case (2.1) takes on the form

$$(H_0u, J_0v)_0 - (Au, BJ_0v)_{\mathscr{K}} = (J_0u, H_0v)_0 - (BJ_0u, Av)_{\mathscr{K}}$$

and this is easily checked to be an identity. Thus all of the hypotheses of Theorem 2.1 are satisfied. Now we turn our attention to Theorem 2.8. First we note that $BR(z)$ is compact for any nonreal z. To see this note that we have already observed that LJ_0^{-1} is a bounded operator from $H^{1/2,2}$ to $\mathscr{H}$. By Theorem 4.1, Chapter 6, of [14] it is compact from $H^{1,2}$ to $\mathscr{H}$ in view of (1.5). By Corollary 3.10, H_0 has no singular spectrum. Next we verify (2.10) and (2.11). Assume first that $p = \infty$ in (1.6) and put $t = (1/2)\alpha > 1/2$. Let s be such that (3.24) holds. Since the set S_s of those ξ such that $P_j(\xi) = s^2$ is bounded, there is a $\varphi \in \mathscr{S}$ such that $F\varphi = 1$ on a neighborhood of S_s. Since $t > 1/2$, the restriction of functions in $\mathscr{S}$ to S_s is a bounded operator from $H^{t,2}$ to $L^2(S_s)$ which depends Hölder continuously on s (cf. [17]). Therefore in view of Corollary 3.9 it suffices to show that the operators defined by $F(\varphi^*Mv)$, $F(\varphi^*A_0(D)WS_a^{-1}w)$, $F(\varphi^*Lv)$ and $F(\varphi^*VS_aw)$ are bounded from L^2 to $H^{t,2}$. For the first three of these operators this follows readily from Lemma 3.6 of [12]. To obtain the same result for the last, we must show that the operator $F(\varphi^*Vu)$ is bounded from $H^{-1/2,2}$ to $H^{t,2}$. This will follow if we can show that $F(\varphi^*Vu)$, $F(\varphi^*D_j(Vu))$ and $F(\varphi^*(D_jV)u)_{t,2}$ are all bounded from L^2 to $H^{t,2}$ for each j. Again this follows from Lemma 3.6 of [12]. Thus (2.10) and (2.11) are verified when $p = \infty$ in (1.6). To verify it for the other cases, note that in view of Corollary 3.11, a sufficient condition for (2.10) and (2.11) to hold is that the operators AK_s*A^* and BK_s*A^* are bounded operators on $\mathscr{K}$ and depend in a locally Hölder continuous way on s. This will follow if we can show that the operators $LK*M$, $S_a VK_s*M$, $LK_s*A_0 WS_a^{-1}$, $S_a VK_s*A_0 WS_a^{-1}$, MK_s*M, $S_a^{-1} WH_0K_s*M$, $MK_s*A_0 WS_a^{-1}$ and $S_a^{-1} WH_0K_s*A_0 WS_a^{-1}$ are such operators on L^2. With the possible exception of the second and fourth of these operators, a sufficient condition for them to have the desired properties is that

$$\sup_x \int Z(y)\rho(x-y)^{-\beta}dy < \infty \tag{4.5}$$

holds for some $\beta < (1/2)(n-1)$ (Lemma 7.2 of [12] and Corollary 3.6). To prove it for them it suffices to show that VK_s*M, $(D_j V)K_s*M$, $VK_s*A_0 WS_a^{-1}$ and $(D_j V)K_s*A_0 WS_a^{-1}$ also have the desired properties for each j. This is also implied by (4.5). Now (4.5) holds if $Z \in L^p$ for some $p < 2n/(n+1)$. Thus (2.10) and (2.11) are established if hypothesis 3) holds either in the case $p = \infty$ or in the case $p < 2n/(n+1)$. The intermediate cases are handled by an interpolation theorem as in the proof of Theorem 4.9 of [18]. Thus (2.10) and (2.11) hold under hypothesis 3). That A is injective follows from the way M was chosen. Finally we note that $[J_0 - I]E(\Gamma)u = E_0^{-1/2} V(I-N)^{-1} WE_0^{1/2}(\varphi^* E(\Gamma)u)$ for Γ a bounded set not intersecting the set e of Corollary 3.9. Now the operator $\varphi^* w$ is bounded from L^2 to $H^{1,2}$ and W is a compact operator from the latter space back to the former by (1.5). Since the remaining operators are bounded, we see that $(J_0 - I)E(\Gamma)$ is a compact operator on L^2. Thus all of the hypotheses of Theorem 2.8 are satisfied. The conclusions of that theorem together with Corollary 3.10 give Theorem 1.1.

Now we turn to the proof of Theorem 1.3. The verification of the hypotheses of Theorems 2.1 and 2.8 are the same as given above with the exception of (2.10) and (2.11). Under the present circumstances we cannot use Lemma 3.8. To find a substitute, let $\lambda_j(\xi)$ be the roots of det $(\lambda - H_0(\xi)) = 0$. Then

$$d(E(I)u, v)_0 = \sum \int_{\lambda_j(\xi) \in I} Fv^* S_j(\xi) Fu d\xi ,$$

where the $S_j(\xi)$ are bounded homogeneous matrices of degree 0 (cf., e.g., [19]). This leads to (3.26) with $K_s(x)$ satisfying $K_s(x) = s^{n-1}K_1(sx)$ (see [12]). This implies that for each $\alpha > 0$

$$| K_s(x) - K_t(x) | \leqq C | s - t |^{\alpha} \rho(x)^{\alpha} .$$

The rest of the verification proceeds as above (cf. [4]). That the spectrum of H_0 is absolutely continuous was shown in [20].

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