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ON SOME TYPES OF ISOPARAMETRIC HYPERSURFACES IN SPHERES I

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1. Introduction. We shall exhibit two series of non-homogeneous isoparametric hypersurfaces in spheres in this paper, and then give a classification of some types of isoparametric hypersurfaces in a forthcoming paper.

We begin with a few definitions and notations to explain our results more precisely. Let $\bar{M}$ be a Riemannian manifold with metric $(,)$. The induced inner product on cotangent vectors is also denoted by $(,)$. A differentiable function f defined on an open set U in $\bar{M}$ is called *isoparametric* if $df \wedge d(df, df) = 0$ and $df \wedge d(\Delta f) = 0$, where Δ denotes the Laplacian on $\bar{M}$. A hypersurface M (a submanifold of codim 1) in $\bar{M}$ is called *isoparametric* if, for each point p of M, there exist an open neighborhood U of p in $\bar{M}$ and an isoparametric function f defined on U such that

$$U \cap M = \{q \in U \mid f(q) = f(p)\} .$$

Let $\mathscr{F} = \{M_t \mid t \in I\}$ be a family of hypersurfaces in $\bar{M}$ parametrized by an open interval I. $\mathscr{F}$ is called a *family of isoparametric hypersurfaces* if there exist an open set U in $\bar{M}$ and an isoparametric function f on U such that $M_t = f^{-1}(t)$ for each $t \in I$. Two families $\mathscr{F} = \{M_t \mid t \in I\}$ and $\mathscr{F}' = \{M'_{t'} \mid t' \in I'\}$ of isoparametric hypersurfaces in $\bar{M}$ are identified if there exists a diffeomorphism φ of I onto I' such that $M_t = M'_{\varphi(t)}$ for each $t \in I$. Also, if we have an imbedding φ of I into I' such that $M_t \subset M'_{\varphi(t)}$ for each $t \in I$, then we write $\mathscr{F} \subset \mathscr{F}'$.

Now, let $\bar{M} = S^{N-1}$ be the unit sphere in an N-dimensional Euclidean space $\boldsymbol{R}^N$ centered at the origin, and M a locally closed hypersurface in $\bar{M}$. M is said to be *homogeneous* if a suitable subgroup of $O(N)$ acts transitively on M where $O(N)$ denotes the real orthogonal group of $\boldsymbol{R}^N$. It is known that M is isoparametric if and only if M has locally constant principal curvatures (Cartan [2]). Thus, every homogeneous hypersurface in S^{N-1} is isoparametric. Two hypersurfaces M and M' in S^{N-1} are said to be *equivalent* if a suitable orthogonal transformation of $\boldsymbol{R}^N$ transforms M onto M'. Similarly, two families of isoparametric hypersurfaces in

S^{N-1} are *equivalent* if a suitable orthogonal transformation of $\boldsymbol{R}^N$ transforms one to the other.

The following results are due to Münzner [5]. For every connected isoparametric hypersurface M in S^{N-1}, there exists a unique maximal (relative to the above order $\subset$) family $\mathscr{F}_M = \{M_t \mid t \in I\}$ of isoparametric hypersurfaces in S^{N-1} such that each M_t is closed in S^{N-1} and for some t M is an open submanifold of M_t. If M and M' are equivalent, then $\mathscr{F}_M$ and $\mathscr{F}_{M'}$ are equivalent in our sense. Further the classification problem of such maximal families is reduced to an algebraic one in the following way. Let F be a homogeneous polynomial function of degree g on $\boldsymbol{R}^N$. For $g>2$, let m_1 and m_2 be positive such that $m_1+m_2+m_1+m_2+\cdots = N-2$, and let $m_1 = N-2>0$ for $g=1$. Assume F satisfies

$$\text{(M)} \qquad \begin{cases}(dF, dF) = g^2 r^{2g-2} \\ \Delta F = c r^{g-2}\end{cases}$$

where $c=(1/2)(m_2-m_1)g^2$ for $g \geqq 2$ and $c=0$ for $g=1$ and where r is the radius function and Δ is the Laplacian on $\boldsymbol{R}^N$. Then the restriction f of F to S^{N-1} is isoparametric on S^{N-1}, and $\mathscr{F}_F = \{M_t = f^{-1}(t) \mid t \in (-1, 1)\}$ is a maximal family of isoparametric hypersurfaces in S^{N-1} such that each M_t is connected and closed. Conversely, any maximal family of isoparametric hypersurfaces in S^{N-1} is given in the above way. Such two families $\mathscr{F}_F$ and $\mathscr{F}_{F'}$ are equivalent if and only if there exists an element σ in $O(N)$ such that

$$F(\sigma^{-1}x) = \pm F'(X) \qquad x \in \boldsymbol{R}^N .$$

In this case, F and F' are said to be *equivalent*. Münzner also has shown that the above (M) has a solution only if $g=1, 2, 3, 4$ or 6 and that $m_1 = m_2$ if g is 3.

Geometrically, the above integers g, m_1 and m_2 are related to each isoparametric hypersurface M_t as follows. Consider the unit normal vector field $X_t = \operatorname{grad}(f)/(df, df)^{1/2}$ for each M_t. Let

$$k_1(t) > \cdots > k_{g(t)}(t)$$

be the distinct principal curvatures of M_t relative to X_t, and $m_j(t)$ the multiplicity of $k_j(t)$ for each j. Then $g(t)$ and $m_j(t)$ are constant, and we have

$$\begin{aligned} g &= g(t) , \\ m_1 &= m_1(t) = m_3(t) = \cdots , \\ m_2 &= m_2(t) = m_4(t) = \cdots , \\ k_j(t) &= \cot\left(\frac{1}{g}\{(j-1)\pi + \cos^{-1}(t)\}\right) \end{aligned}$$

for $j = 1, 2, \cdots, g$.

We come to the problem of classifying equivalent classes of polynomials F satisfying the above condition (M). In the case where $g = 1$ or $g = 2$ it is easy. Cartan solved it in the case $g = 3$ ([3]) and proposed a problem: Is every closed isoparametric hypersurface in S^{N-1} homogeneous? Recently, Takagi [6] classified the case where $g = 4$ and m_1 or $m_2 = 1$, and his result still shows that the obtained ones are homogeneous.

In the present paper I, we shall investigate a homogeneous polynomial function F satisfying the differential equations (M) of Münzner in the case $g = 4$. To such an F, we associate $m_1 + 1$ quadratic forms $\{p_\alpha\}$ and $m_1 + 1$ cubic forms $\{q_\alpha\}$ in $m_1 + 2m_2$ variables, and give a complete characterization of F in terms of $\{p_\alpha\}$ and $\{q_\alpha\}$ in Theorem 1. Using this, two series of non-homogeneous isoparametric hypersurfaces in spheres will be constructed in Theorem 2.

The polynomial functions F defining them are given explicitly as follows. We denote by $\boldsymbol{F}$ the real quaternion algebra $\boldsymbol{H}$ or the real Cayley algebra $\boldsymbol{K}$, and by $u \to \bar{u}$ the canonical involution of $\boldsymbol{F}$. For the n-column vector space $\boldsymbol{F}^n$ over $\boldsymbol{F}$, the canonical inner product is denoted by $(\ ,\ )$. For each positive integer r, the space $\boldsymbol{F}^{2(r+1)}$ can be identified with $\boldsymbol{R}^N$ where $N = 8(r + 1)$ or $16(r + 1)$. For a point $x = u \times v \in \boldsymbol{F}^{r+1} \times \boldsymbol{F}^{r+1} = \boldsymbol{F}^{2(r+1)}$, we set

$$u = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \qquad v = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}$$

where $u_0, v_0 \in \boldsymbol{F}$, $u_1, v_1 \in \boldsymbol{F}^r$. Then we put

$$F_0(u \times v) = 4\{\| {}^t u\bar{v} \|^2 - (u, v)^2\} + \{\| u_1 \|^2 - \| v_1 \|^2 + 2(u_0, v_0)\}^2$$

where $\|\ \ \|$ denotes the length of a vector, and

$$F = r^4 - 2F_0 \,.$$

Then $M_t = \{x \in S^{N-1} \mid F(x) = t\}$ for each t in $(-1, 1)$ is isoparametric and its multiplicities m_1 and m_2 are given by

$$m_1 = 3 \quad \text{and} \quad m_2 = 4r$$

or

$$m_1 = 7 \quad \text{and} \quad m_2 = 8r$$

respectively according to $\boldsymbol{F} = \boldsymbol{H}$ or $\boldsymbol{K}$.

The homogeneous isoparametric hypersurfaces in spheres have been classified by Hsiang-Lawson [4]. In Part II, we shall give an explicit form of F for each of them, and classify the polynomials F satisfying the condition (M) in the case where $g=4$ and m_1 or $m_2=2$. It will be shown that every closed isoparametric hypersurface in this case is homogeneous.

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2. Preliminaries. First we introduce a few notations for operations on polynomial functions and give some of their elementary properties. These notations and properties will be used consistently throughout our papers I and II.

Let $\boldsymbol{R}^n$ be an n-dimensional Euclidean space with inner product $(\ ,\ )$ and r the radius function of $\boldsymbol{R}^n$. The induced inner product on the dual space is also denoted by $(\ ,\ )$. For any polynomial functions f and g on $\boldsymbol{R}^n$, we denote by $\langle f, g\rangle$ the polynomial function on $\boldsymbol{R}^n$ defined by

$$\langle f, g\rangle(x) = ((df)_x, (dg)_x) \qquad x\in \boldsymbol{R}^n . \tag{2.1}$$

The mapping $(f, g)\rightarrow\langle f, g\rangle$ is bilinear and symmetric, and also satisfies

$$\langle f, g_1g_2\rangle = \langle f, g_1\rangle g_2 + \langle f, g_2\rangle g_1 . \tag{2.2}$$

Let $\{x_1, \cdots, x_n\}$ be an orthonormal coordinate system for $\boldsymbol{R}^n$. Then $\langle f, g\rangle$ is equivalently defined by

$$\langle f, g\rangle = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i} . \tag{2.3}$$

Especially, for a homogeneous polynomial f of degree k on $\boldsymbol{R}^n$, and for any positive integer l we have

$$\langle r^{2l}, f\rangle = 2klfr^{2(l-1)} . \tag{2.4}$$

We denote by $\varDelta$ the Laplacian on $\boldsymbol{R}^n$, that is,

$$\varDelta = \sum_{i=1}^{n} \frac{\partial^2}{(\partial x_i)^2} . \tag{2.5}$$

Then, for any positive integer k, we have

$$\varDelta r^{2k} = 2k(n+2k-2)r^{2(k-1)} . \tag{2.6}$$

Let V be a linear subspace of $\boldsymbol{R}^n$. We introduce the restriction forms of $\langle\ ,\ \rangle$ and $\varDelta$ as follows. Let W be the orthogonal complement of V so that we have $\boldsymbol{R}^n = V\oplus W$ (orthogonal decomposition). Choose orthonormal coordinate systems $\{v_i\}$ and $\{w_j\}$ for V and W respectively. Then any polynomial functions f and g on $\boldsymbol{R}^n$ can be expressed as polynomials in variables $\{v_i\}$ and $\{w_j\}$. We put

$$\langle f, g\rangle_V = \sum_i \frac{\partial f}{\partial v_i}\frac{\partial g}{\partial v_i} \tag{2.7}$$

and

$$\Delta_V f = \sum_i \frac{\partial^2 f}{(\partial v_i)^2} . \tag{2.8}$$

They are determined independently on the choices of coordinate systems, and sometimes they will be also denoted by $\langle f, g\rangle_{\{v_i\}}$ and $\Delta_{\{v_i\}}f$. From the definitions it follows that, for an arbitrary orthogonal decomposition $\boldsymbol{R}^n = V \oplus W$, we have

$$\langle f, g\rangle = \langle f, g\rangle_V + \langle f, g\rangle_W \tag{2.9}$$

and

$$\Delta f = \Delta_V f + \Delta_W f . \tag{2.10}$$

Let f be a polynomial function on $\boldsymbol{R}^n$, and V a linear subspace of $\boldsymbol{R}^n$. f is said to be *homogeneous of degree k on V* if f is homogeneous of degree k with respect to the variables $\{v_i\}$ in the expression of f as a polynomial in $\{v_i\}$ and $\{w_j\}$.

Let V be a linear subspace of $\boldsymbol{R}^n$. Every polynomial function f on V can be considered also as a polynomial function on $\boldsymbol{R}^n$ canonically through the orthogonal decomposition $\boldsymbol{R}^n = V \oplus W$. By this identification, it follows that for polynomial functions f and g on V we have

$$\langle f, g\rangle_V = \langle f, g\rangle \tag{2.11}$$

and

$$\Delta_V f = \Delta f . \tag{2.12}$$

Finally, for a quadratic form f on $\boldsymbol{R}^n$, we define a symmetric linear mapping $\eta(f)$ of $\boldsymbol{R}^n$ by

$$(\eta(f)(x), x') = f(x, x') \qquad x, x' \in \boldsymbol{R}^n \tag{2.13}$$

where f is considered in the usual way as a symmetric bilinear form on $\boldsymbol{R}^n$. The correspondence $f \longrightarrow \eta(f)$ is one to one from the set of quadratic forms on $\boldsymbol{R}^n$ onto the set of symmetric linear mappings of $\boldsymbol{R}^n$.

For quadratic forms f and g on $\boldsymbol{R}^n$, we have

$$\eta(\langle f, g\rangle) = 2(\eta(f)\eta(g) + \eta(g)\eta(f)) , \tag{2.14}$$

and especially

$$\eta(\langle f, f\rangle) = 4(\eta(f))^2 . \tag{2.15}$$

Furthermore, we have

(2.16) $$\Delta f = 2 \operatorname{Tr}(\eta(f)) .$$

They can be verified easily.

Now, let S^{N-1} be the unit sphere in $\boldsymbol{R}^N$ centered at the origin. We need the following preliminary lemmas.

LEMMA 1. *Let F be a homogeneous polynomial function of degree g on $\boldsymbol{R}^N$ satisfying*

$$\langle F, F\rangle = g^2 r^{2g-2} .$$

Then the restriction f of F to S^{N-1} is singular at a point x of S^{N-1} if and only if

$$(dF)_x = \pm (dr^g)_x .$$

PROOF. By definition, f is singular at x if and only if $(df)_x = 0$. Note that a tangent vector X in $T_x(\boldsymbol{R}^n)$ is contained in $T_x(S^{N-1})$ if and only if

$$(dr^g)_x(X) = 0 .$$

Thus, $(df)_x = 0$ if and only if

$$(dF)_x = c(dr^g)_x$$

for some constant c. Since $(dF, dF) = \langle F, F\rangle = (dr^g, dr^g)$ from our assumption, we see that $(df)_x = 0$ if and only if

$$(dF)_x = \pm (dr^g)_x . \qquad \text{q.e.d.}$$

LEMMA 2. *Let F be as in Lemma 1. Then the restriction f of F to S^{N-1} ranges from -1 to 1 unless it is constant, and f is singular at a point x of S^{N-1} if and only if $F(x) = \pm 1$.*

PROOF. Let x be a point of S^{N-1} and choose an orthonormal coordinate system $\{u_1, \cdots, u_{N-1}, z\}$ such that $z(x) = 1$ and $u_i(x) = 0$ for $i = 1, 2, \cdots, N-1$. We expand F as a polynomial in z as

$$F = a_0 z^g + a_1 z^{g-1} + \cdots + a_g$$

where a_h is a homogeneous polynomial of degree h in $u_1, \cdots, u_{N-1}$. We have

$$\begin{aligned}(dF)_x &= \left(\frac{\partial F}{\partial z}\right)(x)(dz)_x + \sum_{i=1}^{N-1}\left(\frac{\partial F}{\partial u_i}\right)(x)(du_i)_x \\ &= g a_0 (dz)_x + \sum_{i=1}^{N-1}\left(\frac{\partial F}{\partial u_i}\right)(x)(du_i)_x\end{aligned}$$

and

$$(dr^g)_x = g(r^{g-2}\, rdr)_x = g(dz)_x \,.$$

First suppose that f is singular at x. Then, by Lemma 1 we have $(dF)_x = \pm(dr^g)_x$, and hence $a_0 = \pm 1$. This shows $F(x) = a_0 = \pm 1$.

Conversely, suppose $F(x) = \pm 1$, i.e., $a_0 = \pm 1$. We have

$$\begin{aligned}\langle F, F\rangle(x) = ((dF)_x, (dF)_x) &= g^2 a_0^2 + \sum_{i=1}^{N-1}\left(\left(\frac{\partial F}{\partial u_i}\right)(x)\right)^2 \\ &= g^2 + \sum_{i=1}^{N-1}\left(\left(\frac{\partial F}{\partial u_i}\right)(x)\right)^2 .\end{aligned}$$

Since $\langle F, F\rangle = g^2 r^{2g-2}$, $\langle F, F\rangle(x) = g^2$, and hence we have $(\partial F/\partial u_i)(x) = 0$ for $i = 1, 2, \cdots, N-1$. Thus, we have $(dF)_x = \pm(dr^g)_x$, and hence f is singular at x by Lemma 1.

We have proved the latter assertion in Lemma 2. The former assertion follows from the latter since S^{N-1} is compact. q.e.d.

LEMMA 3. *Let F be as in Lemma 1, and put*

$$F = \sum a_{i_1\cdots i_N} x_1^{i_1} \cdots x_N^{i_N}$$

where $\{x_1, \cdots, x_N\}$ is an orthonormal coordinate system for $\boldsymbol{R}^N$. Assume that the degree g is even and F satisfies

$$F\,|_{\,x_{k+1}=\cdots=x_N=0} = \left(\sum_{i=1}^{k} x_i^2\right)^{g/2} .$$

Then we have

$$a_{i_1\cdots i_N} = 0$$

whenever $i_1 + \cdots + i_k = g - 1$.

PROOF. Put $F = \sum F_h$ where F_h is the homogeneous part of degree h in the variables $x_1, \cdots, x_k$:

$$F_h = \sum_{i_1+\cdots+i_k=h} a_{i_1\cdots i_N} x_1^{i_1} \cdots x_N^{i_N} \,.$$

The assumption says $F_g = (\sum_{i=1}^k x_i^2)^{g/2}$. We shall show $F_{g-1} = 0$. Put

$$G = F_{g-2} + \cdots + F_0 \,,$$

so that we have

$$F = F_g + F_{g-1} + G \,.$$

Now, we have

$$\frac{\partial F}{\partial x_i} = g x_i \left(\sum_{i=1}^{k} x_i^2\right)^{(g/2)-1} + \frac{\partial F_{g-1}}{\partial x_i} + \frac{\partial G}{\partial x_i}$$

for $i = 1, \cdots, k$, and

$$\frac{\partial F}{\partial x_j} = \frac{\partial F_{g-1}}{\partial x_j} + \frac{\partial G}{\partial x_j}$$

for $j = k+1, \cdots, N$, and hence

$$\begin{aligned}\langle F, F\rangle &= \sum_{i=1}^{k}\left(\frac{\partial F}{\partial x_i}\right)^2 + \sum_{j=k+1}^{N}\left(\frac{\partial F}{\partial x_j}\right)^2 \\ &= \sum_{i=1}^{k}\left\{g^2 x_i^2\left(\sum_{i=1}^{k} x_i^2\right)^{g-2} + \left(\frac{\partial F_{g-1}}{\partial x_i}\right)^2 + \left(\frac{\partial G}{\partial x_i}\right)^2\right. \\ &\quad \left. + 2g x_i\left(\sum_{i=1}^{k} x_i^2\right)^{(g/2)-1}\left(\frac{\partial F_{g-1}}{\partial x_i} + \frac{\partial G}{\partial x_i}\right) + 2\frac{\partial F_{g-1}}{\partial x_i}\frac{\partial G}{\partial x_i}\right\} \\ &\quad + \sum_{j=k+1}^{N}\left\{\left(\frac{\partial F_{g-1}}{\partial x_j}\right)^2 + \left(\frac{\partial G}{\partial x_j}\right)^2 + 2\frac{\partial F_{g-1}}{\partial x_j}\frac{\partial G}{\partial x_j}\right\}.\end{aligned}$$

On the other hand, we have

$$\langle F, F\rangle = g^2 r^{2g-2} = g^2\left(\sum_{i=1}^{k} x_i^2 + \sum_{j=k+1}^{N} x_j^2\right)^{g-1}.$$

Comparing the homogeneous terms of degree $2g-2$ in the variables $x_1, \cdots, x_k$ in the above two equations, we get

$$\sum_{j=k+1}^{N}\left(\frac{\partial F_{g-1}}{\partial x_j}\right)^2 = 0\,,$$

and hence

$$\frac{\partial F_{g-1}}{\partial x_j} = 0 \qquad \text{for } j = k+1, \cdots, N\,.$$

Since F_{g-1} is linear in $x_{k+1}, \cdots, x_N$, we have $F_{g-1} = 0$. This proves Lemma 3. q.e.d.

3. Reductions. From now on we shall concern with isoparametric hypersurfaces in S^{N-1} with 4 distinct principal curvatures. So we investigate a homogeneous polynomial function F of degree 4 on $\boldsymbol{R}^N$ satisfying $\langle F, F\rangle = 16r^6$ and $\Delta F = 8(m_2 - m_1)r^2$. These two equations will be replaced by equivalent ones step by step, and in the latter part of this section two families $\{p_\alpha\}$ and $\{q_\alpha\}$ of polynomials will be associated to F on a suitable coordinate system. Our first purpose is to give a complete characterization of such an F in terms of $\{p_\alpha\}$ and $\{q_\alpha\}$ (Theorem 1 in § 4).

Let m_1 and m_2 be two positive integers such that $N = 2(m_1 + m_2 + 1)$, and F a homogeneous polynomial function of degree 4 on $\boldsymbol{R}^N$. Consider the following two conditions on F;

$$\langle F, F\rangle = 16r^6 , \tag{3.1}$$

$$\Delta F = 8(m_2 - m_1)r^2 . \tag{3.2}$$

As a first step of reductions, we choose a unit vector e in $\boldsymbol{R}^N$ such that the restriction f of F to S^{N-1} takes its maximum at the point e. Let X be the orthogonal complement of the 1-dimensional subspace $\boldsymbol{R}e$ so that we have

$$\boldsymbol{R}^N = X \oplus \boldsymbol{R}e . \tag{3.3}$$

Let z be the coordinate function on $\boldsymbol{R}e$ defined by $z(e) = 1$ and $\{x_1, \cdots, x_{N-1}\}$ an orthonormal coordinate system for X.

LEMMA 4. *Assume that F satisfies* (3.1) *and* (3.2). *Then, F can be written in the form*

$$F = z^4 + Az^2 + Bz + C \tag{3.4}$$

where A, B and C are homogeneous polynomial functions on X of degree 2, 3 *and* 4 *respectively, and A, B and C satisfy the following equations* (1-1)$\sim$(1-8) *listed below. Conversely, assume that a homogeneous polynomial function F of the above form* (3.4) *is given with A, B and C satisfying* (1-1)$\sim$(1-8). *Then F satisfies* (3.1) *and* (3.2).

$$\langle A, A\rangle + 16A = 48\Big(\sum_{i=1}^{N-1} x_i^2\Big) \tag{1-1}$$

$$\langle A, B\rangle + 4B = 0 \tag{1-2}$$

$$\langle B, B\rangle + 2\langle A, C\rangle + 4A^2 = 48\Big(\sum_{i=1}^{N-1} x_i^2\Big)^2 \tag{1-3}$$

$$\langle B, C\rangle + 2AB = 0 \tag{1-4}$$

$$\langle C, C\rangle + B^2 = 16\Big(\sum_{i=1}^{N-1} x_i^2\Big)^3 \tag{1-5}$$

$$\Delta A + 12 = 8(m_2 - m_1) \tag{1-6}$$

$$\Delta B = 0 \tag{1-7}$$

$$\Delta C + 2A = 8(m_2 - m_1)\Big(\sum_{i=1}^{N-1} x_i^2\Big) . \tag{1-8}$$

PROOF. Assume that F satisfies (3.1) and (3.2). We first remark that the restriction f of F to S^{N-1} is not a constant. In fact, suppose that f is a constant c on S^{N-1}. Then we have $F = cr^4$. Since $\langle F, F\rangle = 16r^6$, we have $c = \pm 1$. On the other hand,

$$\Delta F = c\Delta r^4 = c(8 + 4N)r^2 = 8(m_2 - m_1)r^2 .$$

Hence, $\pm(8 + 4N) = 8(m_2 - m_1)$. It follows that $m_1 = -1$ or $m_2 = -1$. This is a contradiction.

By Lemma 2, we have $F(e) = 1$. By the choice of coordinates, we have

$$F\,|_{x_1=\cdots=x_{N-1}=0} = (z^2)^2 .$$

Applying Lemma 3, we see that F has the form

$$F = z^4 + Az^2 + Bz + C$$

where A, B and C are homogeneous polynomials in $x_1, \cdots, x_{N-1}$ of degree 2, 3 and 4 respectively. We write (3.1) and (3.2) in terms of A, B and C. We have

$$\begin{aligned}\langle F, F\rangle &= \left(\frac{\partial F}{\partial z}\right)^2 + \langle F, F\rangle_X \\ &= 16z^6 + 4A^2z^2 + B^2 + 16Az^4 + 8Bz^3 + 4ABz + \langle F, F\rangle_X \\ &= 16z^6 + (16A + \langle A, A\rangle)z^4 + (8B + 2\langle A, B\rangle)z^3 \\ &\quad + (4A^2 + \langle B, B\rangle + 2\langle A, C\rangle)z^2 + (4AB + 2\langle B, C\rangle)z \\ &\quad + B^2 + \langle C, C\rangle ,\end{aligned}$$

and

$$\begin{aligned}16r^6 &= 16(z^2 + \textstyle\sum x_i^2)^3 \\ &= 16z^6 + 48(\textstyle\sum x_i^2)z^4 + 48(\textstyle\sum x_i^2)^2z^2 + 16(\textstyle\sum x_i^2)^3 .\end{aligned}$$

Comparing the coefficients of z^h for each h, we see that (3.1) is equivalent to (1-1)$\sim$(1$\sim$5) as a whole.

Next, we have

$$\begin{aligned}\Delta F &= \Delta_{\{z\}}F + \Delta_X F \\ &= 12z^2 + 2A + (\Delta_X A)z^2 + (\Delta_X B)z + \Delta_X C ,\end{aligned}$$

and

$$8(m_2 - m_1)r^2 = 8(m_2 - m_1)(z^2 + \textstyle\sum x_i^2) .$$

Hence, (3.2) is equivalent to (1-6)$\sim$(1-8). Thus, we have the first assertion of Lemma 4.

The converse follows clearly from the above argument. q.e.d.

LEMMA 5. *Let A be a quadratic form on X satisfying* (1-1) *and* (1-6). *Then, X has a unique orthogonal decomposition*

$$X = Y \oplus W \tag{3.5}$$

with $\dim W = m_1 + 1$ *such that* A *has the form*

$$A = 2\Big(\sum_{j=1}^{n} y_j^2\Big) - 6\Big(\sum_{\alpha=0}^{m_1} w_\alpha^2\Big) \tag{3.6}$$

where $\{y_j\}$ *and* $\{w_\alpha\}$ *are orthonormal coordinate systems for* Y *and* W *respectively, and* $n = m_1 + 2m_2$. *Conversely, if* A *is of the above form with respect to an orthogonal decomposition* $X = Y \oplus W$ *with* $\dim W = m_1 + 1$, *then* A *satisfies* (1-1) *and* (1-6).

PROOF. We denote by $\tilde{A}$ the symmetric mapping $\eta(A)$ of X associated to A. Then (1-1) and (1-6) are equivalent to

$$(\tilde{A})^2 + 4\tilde{A} - 12\,1_X = 0 \tag{1-1$'$}$$

and

$$\operatorname{Tr}(\tilde{A}) = 4(m_2 - m_1) - 6 \tag{1-6$'$}$$

respectively, where 1_X denotes the identity mapping of X. Assume (1-1) and (1-6). (1-1)$'$ shows that an eigenvalue of $\tilde{A}$ is 2 or -6. Decompose X into the eigenspaces:

$$X = Y \oplus W$$

where Y and W are the eigenspaces for the eigenvalues 2 and -6 respectively. This is an orthogonal decomposition since $\tilde{A}$ is symmetric. From (1-6)$'$ it follows that $\dim Y = m_1 + 2m_2$ and $\dim W = m_1 + 1$. This shows our first assertion. The converse is easily seen. q.e.d.

LEMMA 6. *Assume* (1-1) *and* (1-6) *for* A. *Then,* B *satisfies* (1-2) *if and only if* B *is homogeneous of degree* 2 *on* Y *and of degree* 1 *on* W.

PROOF. Write

$$B = \sum_{h=0}^{3} B_h$$

where B_h is the homogeneous part of degree h on W and hence of degree $3 - h$ on Y. Consider (1-2). Since $A = 2(\sum y_j^2) - 6(\sum w_\alpha^2)$ by Lemma 5, we have

$$\begin{aligned}
&\langle A, B\rangle + 4B \\
&\quad = \langle A, B\rangle_Y + \langle A, B\rangle_W + 4B \\
&\quad = 2\langle \textstyle\sum y_j^2, B\rangle_Y - 6\langle \textstyle\sum w_\alpha^2, B\rangle_W + 4B \\
&\quad = 2(2B_2 + 4B_1 + 6B_0) - 6(6B_3 + 4B_2 + 2B_1) \\
&\qquad + 4(B_3 + B_2 + B_1 + B_0) \\
&\quad = -32B_3 - 16B_2 + 16B_0\,.
\end{aligned}$$

Thus (1-2) is equivalent to $B_3 = 0$, $B_2 = 0$ and $B_0 = 0$. This shows Lemma 6. q.e.d.

Hereafter we assume (1-1), (1-6) together with (1-2). The orthogonal decomposition $X = Y \oplus W$ in Lemma 5 gives us the second reduction. Let $\{y_j\}$ and $\{w_\alpha\}$ be orthonormal coordinate systems for Y and W respectively where j runs from 1 to $n = m_1 + 2m_2$ and α runs from 0 to m_1. In view of Lemma 6, we can define $m_1 + 1$ quadratic forms $p_0, \cdots, p_{m_1}$ on Y by

$$B = 8 \sum_{\alpha=0}^{m_1} p_\alpha w_\alpha . \tag{3.7}$$

For C, we put

$$C = \sum_{h=0}^{4} C_h \tag{3.8}$$

where C_h is the homogeneous part of degree h on W and hence of degree $4-h$ on Y, and we define $m_1 + 1$ cubic forms $q_0, \cdots, q_{m_1}$ on Y by

$$C_1 = 8 \sum_{\alpha=0}^{m_1} q_\alpha w_\alpha . \tag{3.9}$$

LEMMA 7. *The equation* (1-3) *holds if and only if we have*
(i) $C_4 = (\sum w_\alpha^2)^2$,
(ii) $C_3 = 0$,
(iii) $C_2 = 2 \sum_{\alpha,\beta} \langle p_\alpha, p_\beta \rangle w_\alpha w_\beta - 6(\sum y_j^2)(\sum w_\alpha^2)$,
(iv) $C_0 = (\sum y_j^2)^2 - 2 \sum p_\alpha^2$.

PROOF. Recall (1-3):

$$\langle B, B \rangle + 2\langle A, C \rangle + 4A^2 = 48(\sum x_i^2)^2 .$$

We have

$$\begin{aligned}
4A^2 &= 4\{2(\sum y_j^2) - 6(\sum w_\alpha^2)\}^2 \\
&= 4\cdot 36(\sum w_\alpha^2)^2 - 96(\sum y_j^2)(\sum w_\alpha^2) + 16(\sum y_j^2)^2 , \\
\langle B, B \rangle &= \langle B, B \rangle_Y + \langle B, B \rangle_W \\
&= 64 \sum_{\alpha,\beta} \langle p_\alpha, p_\beta \rangle w_\alpha w_\beta + 64 \sum p_\alpha^2 \\
2\langle A, C \rangle &= 2\langle A, C \rangle_Y + 2\langle A, C \rangle_W \\
&= 4\langle \sum y_j^2, \sum C_h \rangle - 12\langle \sum w_\alpha^2, \sum C_h \rangle \\
&= 8(C_3 + 2C_2 + 3C_1 + 4C_0) \\
&\quad - 24(4C_4 + 3C_3 + 2C_2 + C_1) \\
&= -96C_4 - 64C_3 - 32C_2 + 32C_0
\end{aligned}$$

and

$$48(\sum x_i^2)^2 = 48(\sum w_\alpha^2)^2 + 96(\sum y_j^2)(\sum w_\alpha^2) + 48(\sum y_j^2)^2 .$$

Summarizing their homogeneous terms, (1-3) is equivalent to

$$\begin{aligned}
&4\cdot 36(\sum w_\alpha^2)^2 - 96C_4 = 48(\sum w_\alpha^2)^2 ,\\
&\quad -64C_3 = 0 ,\\
&\quad -96(\sum y_j^2)(\sum w_\alpha^2) + 64\sum \langle p_\alpha, p_\beta\rangle w_\alpha w_\beta - 32C_2 = 96(\sum y_j^2)(\sum w_\alpha^2) ,\\
&\quad 16(\sum y_j^2)^2 + 64\sum p_\alpha^2 + 32C_0 = 48(\sum y_j^2)^2 .
\end{aligned}$$

Now Lemma 7 follows. q.e.d.

REMARK 1. By Lemmas 4, 5, 6 and 7, it follows that the polynomial function F can be constructed uniquely from $\{p_\alpha\}$ and $\{q_\alpha\}$.

Our $\{p_\alpha\}$ and $\{q_\alpha\}$ associated to F depend on the choice of e in S^{N-1} such that $F(e) = 1$ and on the choice of an orthonormal coordinate system $\{w_\alpha\}$ for W. Let F' be another homogeneous polynomial function of degree 4 on $\boldsymbol{R}^N$ satisfying (3.1) and (3.2). Choose e' in S^{N-1} and $\{w'_\alpha\}$ for W' in the same way, so that we have $\{p'_\alpha\}$ and $\{q'_\alpha\}$ on Y' associated to F'. We say that F and F' are *$O(N)$-equivalent* if there exists an element σ in $O(N)$ such that

$$F'(x) = F(\sigma^{-1}x) \qquad \text{for } x \in \boldsymbol{R}^N .$$

Let V and V' be two finite-dimensional vector spaces over $\boldsymbol{R}$. For a linear isomorphism τ of V onto V', and for a polynomial function f on V, we denote by τf the polynomial function on V' obtained by

$$(\tau f)(v') = f(\tau^{-1}v') .$$

With these notations, we state the following two remarks for a later use.

REMARK 2. Suppose that F and F' are $O(N)$-equivalent by an element σ in $O(N)$ such that $\sigma(e) = e'$. Then σ induces orthonormal transformations $\sigma_W\colon W \to W'$ and $\sigma_Y\colon Y \to Y'$. By a suitable choice of $\{w'_\alpha\}$ for W', we have

$$\sigma_Y p_\alpha = p'_\alpha , \qquad \sigma_Y q_\alpha = q'_\alpha$$

for $\alpha = 0, 1, \cdots, m_1$. Conversely, suppose that there exists an orthonormal tansformation τ of Y onto Y' such that

$$\tau p_\alpha = p'_\alpha , \qquad \tau q_\alpha = q'_\alpha$$

for $\alpha = 0, 1, \cdots, m_1$. Then F and F' are $O(N)$-equivalent by an element σ in $O(N)$ such that $\sigma(e) = e'$.

REMARK 3. Consider the case where the isoparametric hypersurface in S^{N-1} defined by $F = c$ for some constant c is homogeneous by a subgroup of $O(N)$. Then it follows that the singular submanifold

$$M_1 = \{x \in S^{N-1};\, F(x) = 1\}$$

is also homogeneous by the e-component of the same group. Therefore F and F' are $O(N)$-equivalent if and only if there exist an orthogonal matrix $(\tau_{\alpha\beta})$ of degree $m_1 + 1$ and an orthonormal transformation σ of Y onto Y' such that

$$p'_\beta = \sum_\alpha \tau_{\beta\alpha}(\sigma p_\alpha)\,,$$
$$q'_\beta = \sum_\alpha \tau_{\beta\alpha}(\sigma q_\alpha)$$

for $\beta = 0, 1, \cdots, m_1$.

Remarks 2 and 3 are immediate consequences of the preceding lemmas.

4. **A characterization by $\{p_\alpha\}$ and $\{q_\alpha\}$.** We continue the argument of the preceding section under the assumptions (1-1), (1-2), (1-3) and (1-6). The equations (1-4), (1-5), (1-7) and (1-8) will be reformulated first in terms of B, C_0 and C_1, and then in terms of $\{p_\alpha\}$ and $\{q_\alpha\}$, using Lemmas 5, 6 and 7.

First we list the equations:

$$\langle B, C_2\rangle_Y = 8B(\textstyle\sum w_\alpha^2) \tag{2-1}$$
$$\langle B, C_1\rangle_Y = 0 \tag{2-2}$$
$$\langle B, C_2\rangle_W + \langle B, C_0\rangle_Y + 4B(\textstyle\sum y_j^2) = 0 \tag{2-3}$$
$$\langle B, C_1\rangle_W = 0 \tag{2-4}$$
$$\langle C_2, C_2\rangle_Y + 16C_2(\textstyle\sum w_\alpha^2) = 48(\sum y_j^2)(\sum w_\alpha^2)^2 \tag{2-5}$$
$$\langle C_2, C_1\rangle_Y + 4C_1(\textstyle\sum w_\alpha^2) = 0 \tag{2-6}$$
$$\langle C_2, C_2\rangle_W + \langle C_1, C_1\rangle_Y + 2\langle C_2, C_0\rangle_Y + B^2 = 48(\textstyle\sum y_j^2)^2(\sum w_\alpha^2) \tag{2-7}$$
$$\langle C_2, C_1\rangle_W + \langle C_1, C_0\rangle_Y = 0 \tag{2-8}$$
$$\langle C_1, C_1\rangle_W + \langle C_0, C_0\rangle_Y = 16(\textstyle\sum y_j^2)^3 \tag{2-9}$$
$$\Delta_Y B = 0 \tag{2-10}$$
$$\Delta_Y C_2 = (8m_2 - 12m_1)(\textstyle\sum w_\alpha^2) \tag{2-11}$$
$$\Delta_Y C_1 = 0 \tag{2-12}$$
$$\Delta_W C_2 + \Delta_Y C_0 = (8m_2 - 8m_1 - 4)(\textstyle\sum y_j^2)\,. \tag{2-13}$$

LEMMA 8. *The following implications hold:*
(i) (1-4) $\Leftrightarrow$ (2-1), (2-2), (2-3) *and* (2-4),
(ii) (1-5) $\Leftrightarrow$ (2-5), (2-6), (2-7), (2-8) *and* (2-9),
(iii) (1-7) $\Leftrightarrow$ (2-10),
(iv) (1-8) $\Leftrightarrow$ (2-11), (2-12) *and* (2-13).

PROOF. In each of (1-4), (1-5), (1-7) and (1-8), we replace A by $2(\sum y_j^2) - 6(\sum w_\alpha^2)$, C by $C_4 + C_2 + C_1 + C_0$, and then C_4 by $(\sum w_\alpha^2)^2$. Decomposing the results into the homogeneous part with respect to the variables w_α's, we can conclude Lemma 8. We give here the proof of (i). The rest can be shown in a similar way.

Recall (1-4): $\qquad \langle B, C\rangle + 2AB = 0\,.$

We have

$$\begin{aligned}\langle B, C\rangle &= \langle B, C\rangle_Y + \langle B, C\rangle_W \\ &= \langle B, C_4\rangle_Y + \langle B, C_2\rangle_Y + \langle B, C_1\rangle_Y + \langle B, C_0\rangle_Y \\ &\quad + \langle B, C_4\rangle_W + \langle B, C_2\rangle_W + \langle B, C_1\rangle_W + \langle B, C_0\rangle_W\,.\end{aligned}$$

Note $\langle B, C_4\rangle_Y = 0$, $\langle B, C_0\rangle_W = 0$, and $\langle B, C_4\rangle_W = \langle B, (\sum w_\alpha^2)^2\rangle_W = 4B(\sum w_\alpha^2)$. Thus, we have

$$\begin{aligned}&\langle B, C\rangle + 2AB \\ &\quad = \langle B, C_2\rangle_Y - 8B(\textstyle\sum w_\alpha^2) \\ &\qquad + \langle B, C_1\rangle_Y \\ &\qquad + \langle B, C_0\rangle_Y + \langle B, C_2\rangle_W + 4B(\textstyle\sum y_j^2) \\ &\qquad + \langle B, C_1\rangle_W\,,\end{aligned}$$

from which we can see easily (1-4) $\Leftrightarrow$ (2-1)$\sim$(2-4). q.e.d.

Now we reformulate the above equations (2-1)$\sim$(2-13) in terms of $\{p_\alpha\}$ and $\{q_\alpha\}$ as follows:

$$\begin{cases}\langle\langle p_\alpha, p_\alpha\rangle, p_\alpha\rangle = 16p_\alpha\,, \quad \Delta p_\alpha = 0\,, \\ \Delta(\langle p_\alpha, p_\alpha\rangle) = 16m_2 \qquad \text{for each } \alpha\,;\end{cases} \tag{3-1}$$

$$2\langle\langle p_\alpha, p_\beta\rangle, p_\beta\rangle + \langle\langle p_\beta, p_\beta\rangle, p_\alpha\rangle = 16p_\alpha \qquad \text{for distinct } \alpha, \beta\,; \tag{3-2}$$

$$\langle\langle p_\alpha, p_\beta\rangle, p_\gamma\rangle + \langle\langle p_\beta, p_\gamma\rangle, p_\alpha\rangle + \langle\langle p_\gamma, p_\alpha\rangle, p_\beta\rangle = 0 \qquad \text{for mutually distinct } \alpha, \beta, \gamma\,; \tag{3-3}$$

$$\langle p_\alpha, q_\alpha\rangle = 0 \qquad \text{for each } \alpha\,; \tag{3-4}$$

$$\langle p_\alpha, q_\beta\rangle + \langle p_\beta, q_\alpha\rangle = 0 \qquad \text{for distinct } \alpha, \beta\,; \tag{3-5}$$

(3-6) $$\langle\langle p_\alpha, p_\beta\rangle, q_\gamma\rangle + \langle\langle p_\beta, p_\gamma\rangle, q_\alpha\rangle + \langle\langle p_\gamma, p_\alpha\rangle, q_\alpha\rangle = 0$$ for mutually distinct α, β, γ ;

(3-7) $$\sum_{\alpha=0}^{m_1} p_\alpha q_\alpha = 0 ;$$

(3-8) $$16\left(\sum_{\alpha=0}^{m_1} q_\alpha^2\right) = 16G(\sum y_j^2) - \langle G, G\rangle ;$$

(3-9) $$8\langle q_\alpha, q_\alpha\rangle = 8(\langle p_\alpha, p_\alpha\rangle(\sum y_j^2) - p_\alpha^2) + \langle\langle p_\alpha, p_\alpha\rangle, G\rangle - 24G - 2\sum_{\gamma=0}^{m_1}\langle p_\alpha, p_\gamma\rangle^2$$ for each α ;

(3-10) $$8\langle q_\alpha, q_\beta\rangle = 8(\langle p_\alpha, p_\beta\rangle(\sum y_j^2) - p_\alpha p_\beta) + \langle\langle p_\alpha, p_\beta\rangle, G\rangle - 2\sum_{\gamma=0}^{m_1}\langle p_\alpha, p_\gamma\rangle\langle p_\beta, p_\gamma\rangle$$ for distinct α, β ;

where $G = \sum_{\alpha=0}^{m_1} p_\alpha^2$ and the indices α, β, γ run from 0 to m_1.

LEMMA 9. *The following implications hold:*
(i) (2-1), (2-10), (2-11) $\Rightarrow$ (3-1), (3-2), (3-3)
(3-1), (3-2), (3-3) $\Rightarrow$ (2-1), (2-10) ,
(ii) (2-2) $\Leftrightarrow$ (3-4), (3-5),
(iii) (2-6) $\Rightarrow$ (3-6),
(iv) (2-4) $\Leftrightarrow$ (3-7),
(v) (2-9) $\Leftrightarrow$ (3-8),
(vi) (2-7) $\Leftrightarrow$ (3-9), (3-10).

We give here the proofs of (i) and (iii). The rest can be proved similarly.

PROOF OF (i). Recall (2-10): $\Delta_Y B = 0$. This is equivalent to $\Delta p_\alpha = 0$. Consider (2-11):
$$\Delta_Y C_2 = (8m_2 - 12m_1)(\sum w_\alpha^2) .$$
Using $C_2 = 2\sum\langle p_\alpha, p_\beta\rangle w_\alpha w_\beta - 6(\sum y_j^2)(\sum w_\alpha^2)$, we get
$$\Delta_Y C_2 = 2\sum \Delta_Y(\langle p_\alpha, p_\beta\rangle)w_\alpha w_\beta - 12(m_1 + 2m_2)(\sum w_\alpha^2) .$$
Thus, (2-11) can be written as
$$2\sum \Delta_Y(\langle p_\alpha, p_\beta\rangle)w_\alpha w_\beta = \{12(m_1 + 2m_2) + 8m_2 - 12m_1\}(\sum w_\alpha^2) = 32m_2(\sum w_\alpha^2) .$$
And hence we see that (2-11) is equivalent to

(2-11-1) $$\Delta(\langle p_\alpha, p_\alpha\rangle) = 16m_2$$ for each α ,

and

(2-11-2) $\qquad \Delta(\langle p_\alpha, p_\beta\rangle) = 0 \qquad$ for distinct α, β.

Now consider (2-1): $\langle B, C_2\rangle_Y = 8B(\sum w_\alpha^2)$.
We have

$$\begin{aligned}
&\langle B, C_2\rangle_Y - 8B(\sum w_\alpha^2) \\
&\quad = 2\langle B, \sum \langle p_\alpha, p_\beta\rangle w_\alpha w_\beta\rangle_Y - 6\langle B, (\sum y_j^2)(\sum w_\alpha^2)\rangle_Y - 8B(\sum w_\alpha^2) \\
&\quad = 16\langle \sum p_\alpha w_\alpha, \sum \langle p_\alpha, p_\beta\rangle w_\alpha w_\beta\rangle_Y - 32B(\sum w_\alpha^2) \\
&\quad = 16\{\sum \langle\langle p_\alpha, p_\beta\rangle, p_\gamma\rangle w_\alpha w_\beta w_\gamma - 16\sum p_\alpha w_\alpha w_\beta^2\} .
\end{aligned}$$

Now we have the implication (2-1), (2-10), (2-11) $\Rightarrow$ (3-1), (3-2), (3-3). From the above argument, we also have the implication (3-1), (3-2), (3-3) $\Rightarrow$ (2-1), (2-10).

PROOF OF (iii). Recall (2-6): $\langle C_2, C_1\rangle_Y + 4C_1(\sum w_\alpha^2) = 0$. By Lemma 7, $C_2 = 2\sum \langle p_\alpha, p_\beta\rangle w_\alpha w_\beta - 6(\sum y_j^2)(\sum w_\alpha^2)$. We have

$$\begin{aligned}
&\langle C_2, C_1\rangle_Y + 4C_1(\sum w_\alpha^2) \\
&\quad = 16\langle \sum \langle p_\alpha, p_\beta\rangle w_\alpha w_\beta, \sum q_\gamma w_\gamma\rangle_Y \\
&\qquad - 6(\sum w_\alpha^2)\langle(\sum y_j^2), C_1\rangle_Y + 4C_1(\sum w_\alpha^2) \\
&\quad = 16\sum \langle\langle p_\alpha, p_\beta\rangle, q_\gamma\rangle w_\alpha w_\beta w_\gamma - 32C_1(\sum w_\alpha^2) \\
&\quad = 16\{\sum \langle\langle p_\alpha, p_\beta\rangle, q_\gamma\rangle w_\alpha w_\beta w_\gamma - 16\sum q_\alpha w_\alpha w_\beta^2\} .
\end{aligned}$$

Thus, we see that (2-6) is equivalent to the following three conditions as a whole:

(2-6-1) $\qquad \langle\langle p_\alpha, p_\alpha\rangle, q_\alpha\rangle = 16 q_\alpha \qquad$ for each α;

(2-6-2) $\qquad 2\langle\langle p_\alpha, p_\beta\rangle, q_\alpha\rangle + \langle\langle p_\alpha, p_\alpha\rangle, q_\beta\rangle = 16 q_\beta$
for distinct α, β;

(2-6-3) $\qquad \langle\langle p_\alpha, p_\beta\rangle, q_\gamma\rangle + \langle\langle p_\beta, p_\gamma\rangle, q_\alpha\rangle + \langle\langle p_\gamma, p_\alpha\rangle, q_\beta\rangle = 0$
for distinct α, β, γ.

Thus we have (2-6) $\Rightarrow$ (3-6) = (2-6-3). q.e.d.

Lemma 9 shows the first assertion of the following Theorem 1.

THEOREM 1. *Let m_1 and m_2 be positive integers such that $N = 2(m_1 + m_2 + 1)$, and put $n = m_1 + 2m_2$.*

Assume that a homogeneous polynomial function F of degree 4 on $\boldsymbol{R}^N$ satisfies $\langle F, F\rangle = 16r^6$ and $\Delta F = 8(m_2 - m_1)r^2$. Then two families $\{p_\alpha\}$ and $\{q_\alpha\}$ of polynomials associated to F in § 3 satisfy the equations (3-1)$\sim$(3-10).

Conversely, assume that there are given $m_1 + 1$ quadratic forms $p_0, \cdots, p_{m_1}$ and $m_1 + 1$ cubic forms $q_0, \cdots, q_{m_1}$ both on $\boldsymbol{R}^n$ such that they *satisfy the equations* (3-1)$\sim$(3-10). *Then the polynomial function* F *on* $\boldsymbol{R}^N$ *constructed from* $\{p_\alpha\}$ *and* $\{q_\alpha\}$ *as in* § 3 *satisfies* $\langle F, F\rangle = 16r^6$ *and* $\Delta F = 8(m_2 - m_1)r^2$.

To prove "the converse" in Theorem 1, it suffices, in view of Lemma 9, to show that (2-3), (2-5), (2-6), (2-8), (2-11), (2-12) and (2-13) follow from (3-1)$\sim$(3-10). We first show (2-3), (2-8) and (2-13) below, and then reformulate the rest in terms of $\{p_\alpha\}$ and $\{q_\alpha\}$. They will be proved in § 5.

LEMMA 10. (2-3), (2-8) *and* (2-13) *follow from* (3-1)$\sim$(3-10).

PROOF. Recall (2-3): $\langle B, C_2\rangle_W + \langle B, C_0\rangle_Y + 4B(\sum y_j^2) = 0$. We have

$$\begin{aligned}\langle B, C_2\rangle_W &= \langle B, 2\sum \langle p_\alpha, p_\beta\rangle w_\alpha w_\beta\rangle_W - \langle B, 6(\sum y_j^2)(\sum w_\alpha^2)\rangle_W \\ &= 32\sum p_\alpha\langle p_\alpha, p_\beta\rangle w_\beta - 96(\sum p_\alpha w_\alpha)(\sum y_j^2)\end{aligned}$$

and

$$\begin{aligned}\langle B, C_0\rangle_Y &= \langle B, (\sum y_j^2)^2\rangle_Y - \langle B, 2G\rangle_Y \\ &= 8B(\sum y_j^2) - 16\sum \langle p_\alpha, G\rangle_Y w_\alpha .\end{aligned}$$

Thus, we have

$$\begin{aligned}&\langle B, C_2\rangle_W + \langle B, C_0\rangle_Y + 4B(\sum y_j^2) \\ &\quad = 32\sum \langle p_\alpha, p_\beta\rangle p_\beta w_\alpha - 16\sum \langle p_\alpha, G\rangle w_\alpha \\ &\quad = 16\{\sum_\alpha w_\alpha(2\sum_\beta \langle p_\alpha, p_\beta\rangle p_\beta - \langle p_\alpha, G\rangle)\} .\end{aligned}$$

Since $G = \sum p_\beta^2$, we have $\langle p_\alpha, G\rangle = 2\sum_\beta \langle p_\alpha, p_\beta\rangle p_\beta$, and hence we have (2-3).

Next recall (2-8): $\langle C_2, C_1\rangle_W + \langle C_1, C_0\rangle_Y = 0$. We have

$$\begin{aligned}\langle C_2, C_1\rangle_W &= \langle 2\sum \langle p_\alpha, p_\beta\rangle w_\alpha w_\beta, 8\sum q_\alpha w_\alpha\rangle_W \\ &\quad - \langle 6(\sum y_j^2)(\sum w_\alpha^2), 8\sum q_\alpha w_\alpha\rangle_W \\ &= 32\sum \langle p_\alpha, p_\beta\rangle q_\alpha w_\beta - 96(\sum y_j^2)(\sum q_\alpha w_\alpha) ,\end{aligned}$$

and

$$\begin{aligned}\langle C_1, C_0\rangle_Y &= \langle C_1, (\sum y_j^2)^2\rangle_Y - 2\langle C_1, G\rangle_Y \\ &= 12C_1(\sum y_j^2) - 2\langle C_1, G\rangle_Y \\ &= 96(\sum y_j^2)\sum q_\alpha w_\alpha - 16\sum \langle q_\alpha, G\rangle w_\alpha .\end{aligned}$$

Hence we have

$$\begin{aligned}&\langle C_2, C_1\rangle_W + \langle C_1, C_0\rangle_Y \\ &\quad = 16\{2\sum \langle p_\alpha, p_\beta\rangle q_\beta w_\alpha - \sum \langle q_\alpha, G\rangle w_\alpha\} .\end{aligned}$$

Now we see that (2-8) is equivalent to

$$2\sum_{\beta}\langle p_\alpha, p_\beta\rangle q_\beta = \langle q_\alpha, G\rangle \qquad \text{for each } \alpha .$$

By definition, $\langle q_\alpha, G\rangle = \langle q_\alpha, \sum p_\beta^2\rangle = 2\sum_\beta \langle q_\alpha, p_\beta\rangle p_\beta$. Using (3-4) and (3-5), we have

$$\langle q_\alpha, G\rangle = -2\sum_{\beta}\langle p_\alpha, q_\beta\rangle p_\beta .$$

Consider (3-7): $\sum p_\beta q_\beta = 0$. We have

$$0 = \langle p_\alpha, \sum p_\beta q_\beta\rangle = \sum_{\beta}\langle p_\alpha, p_\beta\rangle q_\beta + \sum_{\beta}\langle p_\alpha, q_\beta\rangle p_\beta .$$

This proves the required equation.

Finally recall (2-13): $\Delta_W C_2 + \Delta_Y C_0 = \{8(m_2 - m_1) - 4\}(\sum y_j^2)$. We have

$$\begin{aligned}\Delta_W C_2 &= \Delta_W\{2\sum\langle p_\alpha, p_\beta\rangle w_\alpha w_\beta - 6(\sum y_j^2)(\sum w_\alpha^2)\}\\ &= 4\sum\langle p_\alpha, p_\alpha\rangle - 12(m_1+1)(\sum y_j^2)\end{aligned}$$

and

$$\begin{aligned}\Delta_Y C_0 &= \Delta_Y\{(\sum y_j^2)^2 - 2G\}\\ &= (8+4n)(\sum y_j^2) - 2\sum \Delta_Y p_\alpha^2\\ &= (8+4n)(\sum y_j^2) - 2\sum\{2p_\alpha \Delta p_\alpha + 2\langle p_\alpha, p_\alpha\rangle\} .\end{aligned}$$

Since $\Delta p_\alpha = 0$ by (3-1), we have

$$\Delta_W C_2 + \Delta_Y C_0 = \{(8+4n) - 12(m_1+1)\}(\sum y_j^2) .$$

Now

$$\begin{aligned}8 + 4n - 12(m_1+1) &= 4(2m_2 + m_1) - 12m_1 - 4\\ &= 8(m_2 - m_1) - 4\end{aligned}$$

and hence we have (2-13). q.e.d.

LEMMA 11. *(2-5) and (2-12) can be written as:*

(2-5)′
$$\sum_{\alpha,\beta,\gamma,\delta}\langle\langle p_\alpha, p_\beta\rangle, \langle p_\gamma, p_\delta\rangle\rangle w_\alpha w_\beta w_\gamma w_\delta = 16\sum_{\alpha,\beta,\gamma}\langle p_\alpha, p_\beta\rangle w_\alpha w_\beta w_\gamma^2 ;$$

(2-12)′
$$\Delta q_\alpha = 0 \qquad \textit{for each } \alpha$$

respectively.

PROOF. Recall (2-5): $\langle C_2, C_2\rangle_Y + 16C_2(\sum w_\alpha^2) = 48(\sum w_\alpha^2)^2(\sum y_j^2)$, and $C_2 = 2\sum\langle p_\alpha, p_\beta\rangle w_\alpha w_\beta - 6(\sum y_j^2)(\sum w_\alpha^2)$.
We have

$$\langle C_2, C_2\rangle_Y = 4\sum_{\alpha,\beta,\gamma,\delta}\langle\langle p_\alpha, p_\beta\rangle, \langle p_\gamma, p_\delta\rangle\rangle w_\alpha w_\beta w_\gamma w_\delta$$
$$-96\sum_{\alpha,\beta}\langle p_\alpha, p_\beta\rangle w_\alpha w_\beta(\sum w_\gamma^2) + 4\cdot 36(\sum w_\alpha^2)^2(\sum y_j^2)\ ,$$

and

$$16C_2(\sum w_\alpha^2) = 32\sum_{\alpha,\beta,\gamma}\langle p_\alpha, p_\beta\rangle w_\alpha w_\beta w_\gamma^2 - 96(\sum y_j^2)(\sum w_\alpha^2)^2\ .$$

They show that (2-5) is equivalent to (2-5)′.

Recall (2-12): $\Delta_Y C_1 = 0$. Since $C_1 = \sum 8q_\alpha w_\alpha$, clearly (2-12) is equivalent to (2-12)′. q.e.d.

Note that (2-6) and (2-11) have been reformulated in the proof of Lemma 9.

5. The third decomposition of R^N. In this section, first the family $\{p_\alpha\}$ of quadratic forms on Y will be characterized in matricial forms. Then we shall give a further decomposition of the space Y. The proof of Theorem 1 will be completed.

For each quadratic form p_α on Y, we define the symmetric linear mapping P_α of Y as in § 2 by

$$P_\alpha = \eta(p_\alpha)\ . \tag{5.1}$$

We have

LEMMA 12. *The conditions* (3-1), (3-2) *and* (3-3) *on* $\{p_\alpha\}$ *are equivalent to the following conditions* (i), (ii) *and* (iii) *respectively*:

(i) *For each α, we have*

$$P_\alpha^3 = P_\alpha\ ,\quad \mathrm{Tr}\, P_\alpha = 0\ ,\quad \mathrm{rank}\, P_\alpha = 2m_2\ ; \tag{4-1$_\alpha$}$$

(ii) *For each distinct α, β, we have*

$$P_\alpha = P_\beta^2 P_\alpha + P_\alpha P_\beta^2 + P_\beta P_\alpha P_\beta\ ; \tag{4-2$_{\alpha,\beta}$}$$

(iii) *For each mutually distinct α, β, γ we have*

$$\mathfrak{S}(P_\alpha P_\beta P_\gamma) = 0\ , \tag{4-2$_{\alpha,\beta,\gamma}$}$$

where $\mathfrak{S}$ denotes the sum of terms obtained by interchanging the indices over all permutations.

Note $\dim Y = n = m_1 + 2m_2$. Lemma 12 follows by direct verifications, using (2.14), (2.15) and (2.16).

LEMMA 13. (2-5) *follows from* (3-1), (3-2) *and* (3-3).

PROOF. Recall, by Lemma 11, (2-5) $\Leftrightarrow$ (2-5)′:

$$\sum_{\alpha,\beta,\gamma,\delta} \langle\langle p_\alpha, p_\beta\rangle, \langle p_\gamma, p_\delta\rangle\rangle w_\alpha w_\beta w_\gamma w_\delta = 16 \sum_{\alpha,\beta,\gamma} \langle p_\alpha, p_\beta\rangle w_\alpha w_\beta w_\gamma^2 .$$

The monomials of w_α's appearing in (2-5)$'$ are classified in the following types;

$$w_\alpha^4,\ w_\alpha^3 w_\beta,\ w_\alpha^2 w_\beta^2,\ w_\alpha^2 w_\beta w_\gamma,\ w_\alpha w_\beta w_\gamma w_\delta$$

where α, β, γ and δ are all distinct. Now (2-5)$'$ decomposes into the following five equations;

$$\langle\langle p_\alpha, p_\alpha\rangle, \langle p_\alpha, p_\alpha\rangle\rangle = 16\langle p_\alpha, p_\alpha\rangle , \tag{2-5-1}$$

$$\langle\langle p_\alpha, p_\alpha\rangle, \langle p_\alpha, p_\beta\rangle\rangle = 8\langle p_\alpha, p_\beta\rangle , \tag{2-5-2}$$

$$\begin{aligned}\langle\langle p_\alpha, p_\alpha\rangle, \langle p_\beta, p_\beta\rangle\rangle + 2\langle\langle p_\alpha, p_\beta\rangle, \langle p_\alpha, p_\beta\rangle\rangle \\ = 8(\langle p_\alpha, p_\alpha\rangle + \langle p_\beta, p_\beta\rangle) ,\end{aligned} \tag{2-5-3}$$

$$\langle\langle p_\alpha, p_\alpha\rangle, \langle p_\beta, p_\gamma\rangle\rangle + 2\langle\langle p_\alpha, p_\beta\rangle, \langle p_\alpha, p_\gamma\rangle\rangle = 8\langle p_\beta, p_\gamma\rangle , \tag{2-5-4}$$

$$\begin{aligned}\langle\langle p_\alpha, p_\beta\rangle, \langle p_\gamma, p_\delta\rangle\rangle + \langle\langle p_\alpha, p_\gamma\rangle, \langle p_\beta, p_\delta\rangle\rangle \\ + \langle\langle p_\alpha, p_\delta\rangle, \langle p_\beta, p_\gamma\rangle\rangle = 0 ,\end{aligned} \tag{2-5-5}$$

where $\alpha, \beta, \gamma, \delta$ are all distinct.

We give here a proof of (2-5-4). In the following verification, $P_\alpha, P_\beta, \cdots$ are denoted simply by $\alpha, \beta, \cdots$, and the notation $\langle\ ,\ \rangle$ is also used for mappings, i.e., $\langle\alpha, \beta\rangle = 2(\alpha\beta + \beta\alpha)$.

To prove (2-5-4), it suffices to show

$$\langle\langle\alpha, \alpha\rangle, \langle\beta, \gamma\rangle\rangle + 2\langle\langle\alpha, \beta\rangle, \langle\alpha, \gamma\rangle\rangle = 8\langle\beta, \gamma\rangle .$$

The left hand side

$$\begin{aligned}&= 8\{\langle\alpha^2, (\beta\gamma + \gamma\beta)\rangle + \langle(\alpha\beta + \beta\alpha), (\alpha\gamma + \gamma\alpha)\rangle\} \\ &= 16\{\alpha^2\beta\gamma + \alpha^2\gamma\beta + \beta\gamma\alpha^2 + \gamma\beta\alpha^2 \\ &\quad + \alpha\beta\alpha\gamma + \alpha\beta\gamma\alpha + \beta\alpha\alpha\gamma + \beta\alpha\gamma\alpha \\ &\quad + \alpha\gamma\alpha\beta + \alpha\gamma\beta\alpha + \gamma\alpha\alpha\beta + \gamma\alpha\beta\alpha\} .\end{aligned}$$

The right hand side

$$= 16(\beta\gamma + \gamma\beta) .$$

From $(4\text{-}2)_{\gamma,\alpha}$: $\gamma = \alpha^2\gamma + \gamma\alpha^2 + \alpha\gamma\alpha$, we have

$$\gamma\beta = \alpha^2\gamma\beta + \gamma\alpha^2\beta + \alpha\gamma\alpha\beta .$$

From $(4\text{-}2)_{\beta,\alpha}$: $\beta = \alpha^2\beta + \beta\alpha^2 + \alpha\beta\alpha$, we have

$$\beta\gamma = \alpha^2\beta\gamma + \beta\alpha^2\gamma + \alpha\beta\alpha\gamma .$$

Substituting them, we see that it suffices to show

$$\beta\gamma\alpha^2+\gamma\beta\alpha^2+\alpha\beta\gamma\alpha+\beta\alpha\gamma\alpha+\alpha\gamma\beta\alpha+\gamma\alpha\beta\alpha=0\,.$$

Now the left hand side of this equation coincides with $\mathfrak{S}(\alpha\beta\gamma)\alpha$, which is 0 by $(4\text{-}3)_{\alpha,\beta,\gamma}$.

The rest of equations can be proved in a similar way. q.e.d.

From now on in this section we assume (3-1) and (3-2). We choose *an arbitrary index* α, *say* $\alpha=0$.

By virtue of $(4\text{-}1)_\alpha$, each P_α has the eigenvalues 1, -1 and 0. We decompose the space Y into the eigenspaces of P_0;

$$Y=U\oplus V\oplus Z \tag{5.2}$$

where U, V and Z are the eigenspaces of P_0 for the eigenvalues 1, -1 and 0 respectively. Note that the decomposition (5.2) is orthogonal since P_0 is symmetric and that, by $(4\text{-}1)_0$, we have

$$\begin{cases}\dim U=\dim V=m_2\,,\\ \dim Z=m_1\,.\end{cases} \tag{5.3}$$

Now, with respect to orthonormal bases of U, V and W, P_0 is represented by the matrix;

$$P_0\sim\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}$$

where 1 denotes the identity matrix of degree m_2. Similarly, we have

LEMMA 14. *For each* $\alpha>0$, P_α *is represented by the following matrix;*

$$P_\alpha\sim\begin{pmatrix}0&a_\alpha&b_\alpha\\a'_\alpha&0&c_\alpha\\b'_\alpha&c'_\alpha&0\end{pmatrix}$$

where a_α *is* $m_2\times m_2$, b_α *and* c_α *are* $m_2\times m_1$ *and* $'$ *indicates the transpose. Further they satisfy*

$$\begin{cases}a_\alpha a'_\alpha+2b_\alpha b'_\alpha=1\,, & a'_\alpha a_\alpha+2c_\alpha c'_\alpha=1\,,\\ b'_\alpha b_\alpha=c'_\alpha c_\alpha\,;\end{cases} \tag{5.4}$$

$$\begin{cases}b_\alpha c'_\alpha a'_\alpha+a_\alpha c_\alpha b'_\alpha=0\,, & c_\alpha b'_\alpha a_\alpha+a'_\alpha b_\alpha c_\alpha=0\,,\\ c'_\alpha a'_\alpha b_\alpha+b'_\alpha a_\alpha c_\alpha=0\,.\end{cases} \tag{5.5}$$

Conversely, assume that a matrix of the above form is given and satisfies (5.4), (5.5). *Then it satisfies* $(4\text{-}1)_\alpha$, $(4\text{-}2)_{\alpha,0}$ *and* $(4\text{-}2)_{0,\alpha}$.

PROOF. Consider $(4\text{-}2)_{\alpha,0}$:

$$P_\alpha = P_0^2 P_\alpha + P_\alpha P_0^2 + P_0 P_\alpha P_0 \ .$$

This gives the required form for P_α. Similarly, $(4\text{-}2)_{0,\alpha}$:

$$P_0 = P_\alpha^2 P_0 + P_\alpha^2 P_0 + P_\alpha P_0 P_\alpha$$

gives (5.4). If we assume $(4\text{-}2)_{\alpha,0}$, $(4\text{-}2)_{0,\alpha}$, then $(4\text{-}1)_\alpha$ is equivalent to (5.5). Note that the condition: $\operatorname{rank} P_\alpha = 2m_2$ follows from (5.4) and (5.5). q.e.d.

COROLLARY 1. (2-11-2) *holds, i.e., we have*

$$\Delta\langle p_\alpha, p_\beta\rangle = 0$$

for each distinct α, β.

PROOF. Without loss of generality, we may assume $\beta = 0$. We have

$$\Delta\langle p_0, p_\alpha\rangle = 4\operatorname{Tr}(P_0 P_\alpha + P_\alpha P_0) \ .$$

It can be easily verified that $\operatorname{Tr}(P_0 P_\alpha) = 0$ and $\operatorname{Tr}(P_\alpha P_0) = 0$ for $\alpha > 0$ using Lemma 14. q.e.d.

Let $\{u_i\}$, $\{v_i\}$ and $\{z_k\}$ be orthonormal coordinate systems for U, V and Z respectively. We consider the homogeneous degree with respect to the variables $z_1, \cdots, z_{m_1}$ for polynomial functions on Y. Let

$$p_\alpha = \sum_h p_{\alpha,h} \ , \qquad q_\alpha = \sum_h q_{\alpha,h} \tag{5.6}$$

be the decompositions into homogeneous parts with respect to $z_1, \cdots, z_{m_1}$, where h indicates the total degree on $\{z_k\}$.

COROLLARY 2. *For each* $\alpha > 0$, *we have*

(i) $p_{\alpha,2} = 0$,

(ii) $\langle p_0, p_{\alpha,0}\rangle = 0$.

One can verify them using matricial forms given in Lemma 14.

LEMMA 15. *We have, from* (3-8) *and* (3-4),

(i) $q_{\alpha,3} = 0$ *for each* α,

(ii) q_0 *is homogeneous of degree* 1 *on* U, V *and* W.

PROOF. (i) Recall (3-8):

$$16\Big(\sum_\alpha q_\alpha^2\Big) = 16(\sum y_j^2)G - \langle G, G\rangle$$

where $G = \sum_\alpha p_\alpha^2$ and $\sum y_j^2 = \sum u_i^2 + \sum v_i^2 + \sum z_k^2$. In the equation (3-8), consider the homogeneous parts of degree 6 with respect to $z_1, \cdots, z_{m_1}$. Since $p_{\alpha,2} = 0$, the total degree of G with respect to z_k's is less than 4. Similarly, the total degree of $\langle G, G\rangle$ with respect to z_k's is less than 6, since $\langle G, G\rangle = 4\sum \langle p_\alpha, p_\beta\rangle p_\alpha p_\beta$. Thus, we have $\sum q_{\alpha,3}^2 = 0$, and hence $q_{\alpha,3} = 0$ for each α.

(ii) For $\alpha = 0$, (3-4) gives

$$\langle p_0, q_0\rangle = 0 .$$

Now we have $p_0 = \sum u_i^2 - \sum v_i^2$, and hence

$$\langle p_0, q_0\rangle = 2\sum u_i \frac{\partial q_0}{\partial u_i} - 2\sum v_i \frac{\partial q_0}{\partial v_i} .$$

If S is homogeneous of degree k and l with respect to $\{u_i\}$ and $\{v_i\}$ respectively, then we have

$$\langle p_0, S\rangle = 2(k-l)S .$$

Thus, $\langle p_0, q_0\rangle = 0$ implies that each non zero term of q_0 consists of monomials with the same degree on $\{u_i\}$ and $\{v_i\}$. Since q_0 is cubic and $q_{0,3} = 0$ by (i), we have (ii). q.e.d.

COROLLARY. (2-12) *and* (2-6-1) *follow from* (3-1)$\sim$(3-10).

PROOF. Recall (2-12) $\Leftrightarrow$ (2-12)$'$: $\Delta q_\alpha = 0$ for each α. Without loss of generality, we may assume $\alpha = 0$. Then $\Delta q_0 = 0$ follows from (ii) of Lemma 15.

Next, recall (2-6-1): $\langle\langle p_\alpha, p_\alpha\rangle, q_\alpha\rangle = 16 q_\alpha$ for each α. Again we may assume $\alpha = 0$ without loss of generality. Since $p_0 = \sum u_i^2 - \sum v_i^2$, we have

$$\langle p_0, p_0\rangle = 4(\sum u_i^2 + \sum v_i^2) .$$

By (ii) of Lemma 15, $q_0 = q_{0,1}$. Now we have

$$\langle\langle p_0, p_0\rangle, q_0\rangle = \langle\langle p_0, p_0\rangle, q_{0,1}\rangle = 16 q_{0,1} = 16 q_0 .$$

This proves our corollary. q.e.d.

LEMMA 16. (2-6-2) *follows from* (3-1)$\sim$(3-10).

PROOF. Recall (2-6-2): $2\langle\langle p_\alpha, p_\beta\rangle, q_\alpha\rangle + \langle\langle p_\alpha, p_\alpha\rangle, q_\beta\rangle = 16 q_\beta$ for each distinct α, β. Interchanging the indices, it suffices to show

$$2\langle\langle p_0, p_\alpha\rangle, q_0\rangle + \langle\langle p_0, p_0\rangle, q_\alpha\rangle = 16 q_\alpha$$

for $\alpha > 0$. From $\langle p_0, p_0\rangle = 4(\sum u_i^2 + \sum v_i^2)$, we have

$$\langle\langle p_0, p_0\rangle, q_{\alpha,h}\rangle = 8(3-h) q_{\alpha,h}$$

for any h. Since $q_{\alpha,3}=0$ by (i) of Lemma 15, it suffices now to show

$$\langle\langle p_0, p_\alpha\rangle, q_0\rangle = 4q_{\alpha,2} - 4q_{\alpha,0} .$$

By Corollary 2 of Lemma 14, it suffices to show

$$(*) \qquad \langle\langle p_0, p_{\alpha,1}\rangle, q_0\rangle = 4q_{\alpha,2} - 4q_{\alpha,0} .$$

Now we consider the total degree on the variables $u_1, \cdots, u_{m_2}$. Let

$$\begin{aligned} p_{\alpha,1} &= s_1 + s_0 , \\ q_{\alpha,0} &= f_3 + f_2 + f_1 + f_0 , \\ q_{\alpha,1} &= g_2 + g_1 + g_0 , \\ q_{\alpha,2} &= h_1 + h_0 \end{aligned}$$

be the decompositions into homogeneous parts, where each suffix indicates the total degree on $u_1, \cdots, u_{m_2}$. Recall (3-5). We have

$$\langle p_0, q_\alpha\rangle + \langle p_\alpha, q_0\rangle = 0 ,$$

and hence

$$\begin{aligned} &\langle p_0, q_{\alpha,0}\rangle + \langle p_0, q_{\alpha,1}\rangle + \langle p_0, q_{\alpha,2}\rangle \\ &\quad + \langle p_{\alpha,0}, q_{0,1}\rangle + \langle p_{\alpha,1}, q_{0,1}\rangle = 0 . \end{aligned}$$

Equivalently, we have

$$\begin{aligned} &\{\langle p_0, q_{\alpha,2}\rangle + \langle p_{\alpha,1}, q_0\rangle_{\{u_i,v_i\}}\} \\ &\quad + \{\langle p_0, q_{\alpha,1}\rangle + \langle p_{\alpha,0}, q_{0,1}\rangle\} \\ &\quad + \{\langle p_0, q_{\alpha,0}\rangle + \langle p_{\alpha,1}, q_0\rangle_Z\} = 0 . \end{aligned}$$

Observing the degree with respect to $z_1, \cdots, z_{m_1}$ of each term in the above equation, we obtain:

$$(1) \qquad \langle p_0, q_{\alpha,2}\rangle + \langle p_{\alpha,1}, q_0\rangle_{\{u_i,v_i\}} = 0 ,$$

$$(2) \qquad \langle p_0, q_{\alpha,1}\rangle + \langle p_{\alpha,0}, q_0\rangle = 0 ,$$

$$(3) \qquad \langle p_0, q_{\alpha,0}\rangle + \langle p_{\alpha,1}, q_0\rangle_Z = 0 .$$

From $p_0 = \sum u_i^2 - \sum v_i^2$, we obtain:

$$(4) \qquad \langle p_0, q_{\alpha,2}\rangle = 2h_1 - 2h_0 ,$$

$$(5) \qquad \langle p_0, q_{\alpha,1}\rangle = 4g_2 - 4g_0 ,$$

$$(6) \qquad \langle p_0, q_{\alpha,0}\rangle = 2(3f_3 + f_2 - f_1 - 3f_0) .$$

On the other hand, we have

$$\begin{aligned} \langle p_{\alpha,1}, q_0\rangle_{\{u_i,v_i\}} &= \langle s_0, q_0\rangle_{\{u_i,v_i\}} + \langle s_1, q_0\rangle_{\{u_i,v_i\}} \\ &= \langle s_0, q_0\rangle_V + \langle s_1, q_0\rangle_U . \end{aligned}$$

Substituting this and (4) into (1), we get

$$(7)\qquad \begin{cases} 2h_1 + \langle s_0, q_0\rangle_V = 0\,, \\ 2h_0 - \langle s_1, q_0\rangle_U = 0\,. \end{cases}$$

Similarly, substituting $\langle p_{\alpha,1}, q_0\rangle_Z = \langle s_1, q_0\rangle_Z + \langle s_0, q_0\rangle_Z$ and (6) into (3), we get

$$(8)\qquad \begin{cases} f_3 = f_0 = 0\,, \\ 2f_2 + \langle s_1, q_0\rangle_Z = 0\,, \\ 2f_1 - \langle s_0, q_0\rangle_Z = 0\,. \end{cases}$$

Since $\langle p_0, p_{\alpha,1}\rangle = \langle p_0, s_0\rangle + \langle p_0, s_1\rangle = -2s_0 + 2s_1$, (7) and (8) give the required equation $(*)$. q.e.d.

Note that we have completed the proof of Theorem 1.

6. A further characterization. In this section we give a further characterization of $\{p_\alpha\}$ and $\{q_\alpha\}$ under an additional condition (A) for a later use. Let $\{p_\alpha\}$ be $m_1 + 1$ quadratic forms on Y satisfying (3-1) and (3-2). With the notations in § 5, we state

LEMMA 17. *The following three conditions are mutually equivalent*:
(i) $\langle p_\alpha, p_\beta\rangle = 0$ *for distinct* α, β;
(ii) $\langle p_\alpha, p_\alpha\rangle = \langle p_\beta, p_\beta\rangle$ *for distinct* α, β;
(iii) $p_{\alpha,1} = 0$ *for each* α.

PROOF. As one can see easily, to prove Lemma 17, it suffices to show that, for each $\alpha > 0$, the following three conditions are mutually equivalent:
(i)′ $\langle p_0, p_\alpha\rangle = 0$;
(ii)′ $\langle p_0, p_0\rangle = \langle p_\alpha, p_\alpha\rangle$;
(iii)′ $p_{\alpha,1} = 0$.

Using Lemma 14, we give matricial representations for $\langle p_0, p_\alpha\rangle$, $\langle p_\alpha, p_\alpha\rangle$ and $p_{\alpha,1}$. In the following, the indices for submatrices are omitted. We have

$$\langle p_0, p_\alpha\rangle \sim 2\begin{pmatrix} 0 & 0 & b \\ 0 & 0 & -c \\ b' & -c' & 0 \end{pmatrix},$$

$$\langle p_\alpha, p_\alpha\rangle \sim 4\begin{pmatrix} aa' + bb' & bc' & ac \\ cb' & a'a + cc' & a'b \\ c'a' & b'a & b'b + c'c \end{pmatrix}$$

$$p_{\alpha,1} \sim \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ b' & c' & 0 \end{pmatrix}.$$

Thus, (i)$' \Leftrightarrow$ (iii)$'$ and (iii)$' \Rightarrow$ (ii)$'$ are clear. Suppose (ii)$'$. Then $aa' + bb' = 1$. Since $aa' + 2bb' = 1$ by Lemma 14, we see $bb' = 0$, and hence $b = 0$. Similarly we have $c = 0$. This proves (ii)$' \Rightarrow$ (iii)$'$. q.e.d.

From now on we denote by (A) one of the three conditions in Lemma 17. Now assume that $\{p_\alpha\}$ satisfy the condition (A) together with (3-1) and (3-2). We remark here that the image and the kernel of P_α are independent on α and that the condition (3-3) follows automatically. We put, for each α,

(6.1) $$R_\alpha = P_\alpha|_{U\oplus V}.$$

We see that R_α is a symmetric mapping of $U \oplus V$ into itself and for $\alpha = 0$, $R_0|_U = 1_U$, $R_0|_V = -1_V$. Furthermore it is easily seen that the family $\{R_\alpha\}$ satisfies the following two conditions:

(5-1) $$R_\alpha^2 = 1_{U\oplus V}, \qquad \mathrm{Tr}\, R_\alpha = 0 \qquad \text{for each } \alpha;$$

(5-2) $$R_\alpha R_\beta + R_\beta R_\alpha = 0 \qquad \text{for distinct } \alpha, \beta.$$

Conversely, we have

LEMMA 18. *Let $\{R_\alpha\}$ be $m_1 + 1$ symmetric mappings of $U \oplus V$ into itself satisfying* (5-1) *and* (5-2). *Then we can associate $m_1 + 1$ quadratic forms $\{p_\alpha\}$ on Y satisfying* (3-1), (3-2) *and the condition* (*A*) *with the relation* (6.1) *for each α.*

PROOF. For each R_α, we define P_α by

$$P_\alpha = \begin{cases} R_\alpha & \text{on } U \oplus V \\ 0 & \text{on } Z. \end{cases}$$

Then P_α is a symmetric mapping of $Y = U \oplus V \oplus Z$. Now (5-1) implies $(4\text{-}1)_\alpha$ for each α. From the construction of P_α, it follows that $(4\text{-}2)_{\alpha,\beta}$ is a consequence of (5-2). Let p_α be the quadratic form on Y corresponding to P_α. $\{p_\alpha\}$ satisfy the required conditions. q.e.d.

LEMMA 19. *Assume that $\{p_\alpha\}$ satisfy* (3-1), (3-2) *and the condition* (*A*). *Let $\{q_\alpha\}$ be $m_1 + 1$ cubic forms on Y. Then* (3-3) *and* (3-6) *follow immediately. The conditions* (3-8), (3-9) *and* (3-10) *can be written equivalently as*

(5-8) $$\sum q_\alpha^2 = G(\sum z_k^2),$$

(5-9) $$\langle q_\alpha, q_\alpha\rangle = G - p_\alpha^2 + 4(\sum u_i^2 + \sum v_i^2)(\sum z_k^2) \qquad \textit{for each } \alpha\,,$$

(5-10) $$\langle q_\alpha, q_\beta\rangle = -p_\alpha p_\beta \qquad \textit{for distinct } \alpha, \beta$$

respectively.

PROOF. By Lemma 17, we see that (3-3) and (3-6) follow immediately from (A). For $G = \sum p_\alpha^2$, consider $\langle G, G\rangle$. We have

$$\begin{aligned}\langle G, G\rangle &= \sum_{\alpha,\beta} \langle p_\alpha^2, p_\beta^2\rangle = 4\sum_{\alpha,\beta} p_\alpha p_\beta \langle p_\alpha, p_\beta\rangle \\ &= 4\sum_\alpha p_\alpha^2\langle p_\alpha, p_\alpha\rangle = 4\Bigl(\sum_\alpha p_\alpha^2\Bigr)\langle p_0, p_0\rangle \\ &= 16G(\sum u_i^2 + \sum v_i^2)\,.\end{aligned}$$

This gives (3-8) $\Leftrightarrow$ (5-8). Since each p_β is a quadratic form on $U \oplus V$, we have

$$\begin{aligned}\langle\langle p_\alpha, p_\alpha\rangle, p_\beta\rangle &= \langle\langle p_0, p_0\rangle, p_\beta\rangle \\ &= \langle\langle p_0, p_0\rangle, p_\beta\rangle_{U\oplus V} = 16p_\beta\,.\end{aligned}$$

Thus, we have

$$\begin{aligned}\langle\langle p_\alpha, p_\alpha\rangle, G\rangle &= \sum_\beta \langle\langle p_\alpha, p_\alpha\rangle, p_\beta^2\rangle \\ &= \sum_\beta 2p_\beta\langle\langle p_0, p_0\rangle, p_\beta\rangle = 32G\,.\end{aligned}$$

This and Lemma 17 give (3-9) $\Leftrightarrow$ (5-9). Lemma 17 gives also (3-10) $\Leftrightarrow$ (5-10). q.e.d.

By Lemmas 18 and 19, it follows that for a given $\{R_\alpha\}$ satisfying (5-1) and (5-2), the required conditions for $\{q_\alpha\}$ in Theorem 1 are now (3-4), (3-5), (3-7), (5-8), (5-9) and (5-10).

For a later use, we give the following lemma.

LEMMA 20. *Let $\{p_\alpha\}$ be $m_1 + 1$ quadratic forms on Y satisfying* (3-1), (3-2) *and* (*A*). *Then $p_0, \cdots, p_{m_1}$ are algebraically independent over* $\boldsymbol{R}$.

PROOF. First we prove that $p_0, \cdots, p_{m_1}$ are linearly independent over $\boldsymbol{R}$. Suppose $\sum a_\alpha p_\alpha = 0$, $a_\alpha \in \boldsymbol{R}$. We have for any β,

$$\Bigl\langle p_\beta, \sum_\alpha a_\alpha p_\alpha\Bigr\rangle = a_\beta\langle p_\beta, p_\beta\rangle\,,$$

and hence $a_\beta = 0$. Next suppose

$$\sum a_{i_0\cdots i_{m_1}} p_0^{i_0} \cdots p_{m_1}^{i_{m_1}} = 0\,.$$

Since each p_α is a quadratic form, we have

$$\sum_{i_0+\cdots+i_{m_1}=l} a_{i_0\cdots i_{m_1}} p_0^{i_0} \cdots p_{m_1}^{i_{m_1}} = 0$$

for each l. We shall show $a_{i_0\cdots i_{m_1}} = 0$ for all $i_0, \cdots, i_{m_1}$. This will be shown by the induction on $l = i_0 + \cdots + i_{m_1}$. The case $l = 1$ has been proved. For each β, we have

$$\begin{aligned}&\langle p_\beta, \sum a_{i_0\cdots i_{m_1}} p_0^{i_0} \cdots p_{m_1}^{i_{m_1}} \rangle \\ &\quad = \sum i_\beta a_{i_0\cdots i_{m_1}} p_0^{i_0} \cdots p_\beta^{i_\beta - 1} \cdots p_{m_1}^{i_{m_1}} \langle p_0, p_0 \rangle .\end{aligned}$$

Using this, one can complete easily the proof. q.e.d.

7. **Representations of a Clifford algebra.** In this section we prove certain lemmas concerning representations of a Clifford algebra for a later use.

Let $\boldsymbol{F}$ be an associative division algebra over $\boldsymbol{R}$, i.e., $\boldsymbol{F} = \boldsymbol{R}, \boldsymbol{C}$ or the real quaternion algebra $\boldsymbol{H}$. We denote by $M_m(\boldsymbol{F})$ the algebra of all $m \times m$ matrices with coefficients in $\boldsymbol{F}$, and by 1_m the unit matrix in $M_m(\boldsymbol{F})$. $M_m(\boldsymbol{F})$ is called the total matrix algebra over $\boldsymbol{F}$ of degree m.

For each non-negative integer κ, we denote by C_κ the Clifford algebra over $\boldsymbol{R}$ associated to the negative definite quadratic form $-(\ ,\ )$ on $\boldsymbol{R}^\kappa$, where $(\ ,\ )$ is the usual inner product on $\boldsymbol{R}^\kappa$. Let $\{e_1, \cdots, e_\kappa\}$ be an orthonormal base for $\boldsymbol{R}^\kappa$ with respect to $(\ ,\ )$. Then C_κ is the associative algebra over $\boldsymbol{R}$ with the unit 1 generated by $e_1, \cdots, e_\kappa$ with the relations:

$$\begin{cases} e_k^2 = -1 & \text{for each } k \ , \\ e_k e_l + e_l e_k = 0 & \text{for each distinct } k, l \ , \end{cases}$$

and $\{1, e_{k_1} \cdots e_{k_r}; k_1 < \cdots < k_r, 1 \leqq r \leqq \kappa\}$ forms a basis of the underlying vector space of C_κ, and hence $\dim C_\kappa = 2^\kappa$. We denote by $x \to x^*$ the canonical involution of C_κ, that is, the anti-automorphism of C_κ satisfying $e_k = -e_k$ for each k. A homomorphism

$$\rho\colon C_\kappa \to M_m(\boldsymbol{R}) \quad \text{with} \quad \rho(1) = 1_m$$

is called a *representation* of C_κ of degree m. Two representations $\rho, \tilde{\rho}$ of C_κ of degree m are said to be *equivalent* if there exists $A \in GL(m, \boldsymbol{R})$ such that $\tilde{\rho}(x) = A\rho(x)A^{-1}$ for each $x \in C_\kappa$. The set of equivalence classes of representations of C_κ of degree m will be denoted by $\mathscr{R}_m(C_\kappa)$.

We consider a representation ρ of C_κ of degree m satisfying

$$\rho(x^*) = \rho(x)' \qquad \text{for each } x \in C_\kappa \ , \tag{7.1}$$

where $'$ indicates the transpose of a matrix. Two representations $\rho, \tilde{\rho}$ of C_κ satisfying (7.1) are said to be *orthogonally equivalent* if there exists

$\sigma \in O(m)$ such that $\tilde{\rho}(x) = \sigma\rho(x)\sigma^{-1}$ for each $x \in C_\kappa$. The set of orthogonal equivalence classes of representations of C_κ of degree m satisfying (7.1) will be denoted by $\mathscr{R}_m(C_\kappa, *)$.

LEMMA 21. *The natural map:*

$$\mathscr{R}_m(C_\kappa, *) \to \mathscr{R}_m(C_\kappa)$$

is a bijection.

PROOF.* The bracket operation $[x, y] = xy - yx$ on C_κ defines a Lie algebra over $\boldsymbol{R}$, which is denoted by $\mathfrak{g}$. Since C_κ is a semi-simple algebra over $\boldsymbol{R}$, it is the direct sum of a finite number of total matrix algebras. It follows that $\mathfrak{g}$ has a natural structure of reductive algebraic Lie algebra over $\boldsymbol{R}$. Now the canonical involution $x \to x^*$ of C_κ is a positive involution in the sense that the symmetric bilinear form $\mathrm{Tr}\,(L_{xy^*})$ on C_κ is positive definite, L_x being the left regular representation of C_κ: $L_x y = xy$. In fact, for $x_0 = e_{i_1} \cdots e_{i_r}$, $y_0 = e_{j_1} \cdots e_{j_s} (i_1 < \cdots < i_r, j_1 < \cdots < j_s)$, we have

$$x_0 y_0^* = \begin{cases} 1 & r = s,\ \{i_1, \cdots, i_r\} = \{j_1, \cdots, j_s\} \\ \pm e_{k_1} \cdots e_{k_t},\ t > 0 & \text{otherwise}\ , \end{cases}$$

where

$$\{k_1, \cdots, k_t\} = \{i_1, \cdots, i_r\} \cup \{j_1, \cdots, j_s\} - \{i_1, \cdots, i_r\} \cap \{j_1, \cdots, j_s\}\ .$$

Thus we have

$$\mathrm{Tr}\,(L_{x_0 y_0^*}) = \begin{cases} \dim C_\kappa = 2^\kappa > 0 & r = s,\ \{i_1, \cdots, i_r\} = \{j_1, \cdots, j_s\} \\ 0 & \text{otherwise}\ , \end{cases}$$

and hence $\mathrm{Tr}\,(L_{xy^*})$ is positive definite on C_κ. Thus, by a theorem of Weil [8], the map θ of $\mathfrak{g}$ defined by $x \to -x^*$ is a Cartan involution of $\mathfrak{g}$.

We shall show first the surjectivity. Let ρ be a representation of C_κ of degree m. Then the representation

$$\rho\colon \mathfrak{g} \to \mathfrak{gl}(m, \boldsymbol{R})$$

is completely reducible. Hence there exists a Cartan involution θ_0 of $\mathfrak{gl}(m, \boldsymbol{R})$ such that

$$\theta_0(\rho(x)) = \rho(\theta(x)) \qquad \text{for each } x \in \mathfrak{g}\ .$$

θ_0 can be expressed as

$$\theta_0(X) = -P^{-1}X'P \qquad \text{for } X \in \mathfrak{gl}(m, \boldsymbol{R})$$

* The proof of surjectivity is due to I. Satake.

by a positive definite symmetric matrix $P \in M_m(\boldsymbol{R})$. Thus we have

$$\rho(x^*) = P^{-1}\rho(x)'P \qquad \text{for } x \in C_\kappa .$$

Put $A = P^{1/2}$ and

$$\tilde{\rho}(x) = A\rho(x)A^{-1} \qquad \text{for } x \in C_\kappa .$$

Then we have for each $x \in C_\kappa$

$$\begin{aligned}\tilde{\rho}(x^*) &= A\rho(x^*)A^{-1} = AP^{-1}\rho(x)'PA^{-1}\\ &= A'^{-1}\rho(x)'A' = \tilde{\rho}(x)' ,\end{aligned}$$

and hence $\tilde{\rho}$ satisfies (7.1). This proves the surjectivity of the map.

To prove the injectivity, let ρ and $\tilde{\rho}$ be mutually equivalent representations of C_κ satisfying (7.1). Let $A \in GL(m, \boldsymbol{R})$ such that

$$\tilde{\rho}(x) = A\rho(x)A^{-1} \qquad \text{for } x \in C_\kappa . \tag{7.2}$$

Then we have $\tilde{\rho}(x^*) = A\rho(x^*)A^{-1}$ for each $x \in C_\kappa$. From the condition (7.1) we have $\tilde{\rho}(x)' = A\rho(x)'A^{-1}$ and hence

$$\tilde{\rho}(x) = A'^{-1}\rho(x)A' \qquad \text{for } x \in C_\kappa \tag{7.3}$$

(7.2) and (7.3) imply that the symmetric matrix $A'A$ commutes with each $\rho(x)$. Now write A as the product: $A = \sigma P$ of $\sigma \in O(m)$ and a positive definite symmetric matrix P. Then $A'A = P^2$ commutes with each $\rho(e_k)$. From the condition (7.1), $\tau_t = \exp t\rho(e_k)$ is in $O(m)$ for each $t \in \boldsymbol{R}$, and hence $\tau_t P \tau_t^{-1}$ is also a positive definite symmetric matrix. It follows from $\tau_t P^2 \tau_t^{-1} = (\tau_t P \tau_t^{-1})^2 = P^2$ that each τ_t commutes with P and hence each $\rho(e_k)$ commutes with P. Since C_κ is generated by $e_1, \cdots, e_\kappa$, we have

$$\tilde{\rho}(x) = \sigma\rho(x)\sigma^{-1} \qquad \text{for } x \in C_\kappa .$$

Thus ρ and $\tilde{\rho}$ are orthogonally equivalent. q.e.d.

The subspace of C_κ spanned by $e_1, \cdots, e_\kappa$ is identified with $\boldsymbol{R}^\kappa$ in a natural way, and any orthogonal transformation σ of $\boldsymbol{R}^\kappa (\sigma \in O(\kappa))$ is extended uniquely to an automorphism σ of C_κ. For a representation ρ of C_κ of degree m, we define another representation $\sigma\rho$ by

$$(\sigma\rho)(x) = \rho(\sigma^{-1}x) \qquad \text{for } x \in C_\kappa .$$

If ρ satisfies (7.1), then $\sigma\rho$ also satisfies (7.1), since the automorphism σ of C_κ commutes with the canonical involution $x \to x^*$. The correspondence $(\sigma, \rho) \to \sigma\rho$ gives an action of $O(\kappa)$ on $\mathscr{R}_m(C_\kappa)$ and on $\mathscr{R}_m(C_\kappa, *)$. Let $O(\kappa)\backslash\mathscr{R}_m(C_\kappa)$ and $O(\kappa)\backslash\mathscr{R}_m(C_\kappa, *)$ denote the spaces of $O(\kappa)$-orbits respectively. Since the natural map $\mathscr{R}_m(C_\kappa, *) \to \mathscr{R}_m(C_\kappa)$ is $O(\kappa)$-equivariant, Lemma 21 gives us the natural bijection

$$O(\kappa)\backslash\mathscr{R}_m(C_\kappa, *) \to O(\kappa)\backslash\mathscr{R}_m(C_\kappa) .$$

We cite Atiyah-Bott-Shapiro [1]: We have an isomorphism

$$C_{\kappa+8} \cong C_\kappa \otimes M_{16}(\boldsymbol{R}) , \tag{7.4}$$

and the Clifford algebras C_κ's for $\kappa \leqq 8$ are given by the following table;

κ	C_κ	$d(\kappa)$
1	$\boldsymbol{C}$	2
2	$\boldsymbol{H}$	4
3	$\boldsymbol{H} \oplus \boldsymbol{H}$	4
4	$M_2(\boldsymbol{H})$	8
5	$M_4(\boldsymbol{C})$	8
6	$M_8(\boldsymbol{R})$	8
7	$M_8(\boldsymbol{R}) \oplus M_8(\boldsymbol{R})$	8
8	$M_{16}(\boldsymbol{R})$	16

where $d(\kappa)$ denotes the degree of irreducible representations of C_κ. We have

$$d(\kappa + 8) = 16d(\kappa) \tag{7.5}$$

in virtue of the isomorphism (7.4).

LEMMA 22. *For $\kappa \geqq 1$, $O(\kappa)\backslash\mathscr{R}_{\kappa+1}(C_\kappa, *)$ is not empty if and only if $\kappa = 1, 3$ or 7. For $\kappa = 1, 3$ or 7, $O(\kappa)\backslash\mathscr{R}_{\kappa+1}(C_\kappa, *)$ consists of exactly one element, represented by an irreducible representation of C_κ.*

PROOF. By Lemma 21, it suffices to show the above for the set $O(\kappa)\backslash\mathscr{R}_{\kappa+1}(C_\kappa)$. From (7.5) we have

$$\begin{aligned} d(\kappa + 8) - (\kappa + 8) &= 16d(\kappa) - \kappa - 8 \\ &= (15d(\kappa) - 8) + (d(\kappa) - \kappa) > d(\kappa) - \kappa . \end{aligned}$$

It follows that if $\mathscr{R}_{\kappa+1}(C_\kappa)$ is not empty, then $\kappa \leqq 8$ and $\mathscr{R}_{\kappa+1}(C_\kappa)$ consists of equivalent classes of irreducible representations. From the table cited above we get the first assertion of Lemma 22.

In case $\kappa = 1$, $C_1 = \boldsymbol{C}$ and $\mathscr{R}_2(C_1)$ consists of just one class. In case $\kappa = 3$, $C_3 = \boldsymbol{H} \oplus \boldsymbol{H}$ and $\mathscr{R}_4(C_3)$ consists of two classes. Putting $z = e_1e_2e_3$ in C_3, we define $f_+, f_- \in C_3$ by

$$f_+ = \frac{1}{2}(1 + z),\ f_- = \frac{1}{2}(1 - z) .$$

Then they are primitive idempotents of C_3 defining the decomposition $C_3 = \boldsymbol{H} \oplus \boldsymbol{H}$. Since $-1_3 \in O(3)$ transforms f_+ into f_-, $O(3)\backslash\mathscr{R}_4(C_3)$ consists exactly one element. In case $\kappa = 7$, we see similarly that $O(7)\backslash\mathscr{R}_8(C_7)$ consists exactly one element, making use of the element $z = e_1e_2 \cdots e_7 \in C_7$. q.e.d.

For $\kappa = 1, 3, 7$, we have $C_{\kappa-1} \cong \boldsymbol{R}, \boldsymbol{H}, M_8(\boldsymbol{R})$ respectively. Hence we have

LEMMA 23. *For $\kappa = 1, 3, 7$, the set $\mathscr{R}_m(C_{\kappa-1}, *)$ is not empty if and only if m is a multiple of* 1, 4, 8 *respectively. In these cases, $\mathscr{R}_m(C_{\kappa-1}, *)$ consists of exactly one class.*

Now, let κ, m be positive integers. Consider a family $\{a_k\}_{1\leqq k\leqq\kappa}$ of κ matrices in $M_m(\boldsymbol{R})$ satisfying the following condition:

$$(7.6) \qquad \begin{cases} a'_k a_k = 1_m & \text{for each } k \\ a'_k a_l + a'_l a_k = 0 & \text{for distinct } k,\, l\,. \end{cases}$$

Two such families $\{a_k\}$, $\{\tilde{a}_k\}$ are said to be *equivalent* and denoted by $\{a_k\} \sim \{\tilde{a}_k\}$ if there exist $\sigma, \tau \in O(m)$ such that

$$\tilde{a}_k = \sigma a_k \tau^{-1} \qquad \text{for each } k\,.$$

They are classified in terms of representations of Clifford algebras as follows.

LEMMA 24. *The set of equivalence classes of families $\{a_k\}$ of κ matrices in $M_m(\boldsymbol{R})$ satisfying the condition* (7.6) *is in a bijective correspondence with the set $\mathscr{R}_m(C_{\kappa-1}, *)$.*

PROOF. Let ρ be a representation of $C_{\kappa-1}$ of degree m satisfying (7.1). We define κ matrices $a_1, \cdots, a_\kappa$ by

$$\begin{cases} a_k = \rho(e_k) & 1 \leqq k \leqq \kappa - 1\,, \\ a_\kappa = 1_m\,. \end{cases}$$

Since we have

$$\begin{cases} a'_k = -a_k,\ a_k^2 = -1_m & \text{for each } k,\ 1 \leqq k \leqq \kappa - 1 \\ a_k a_l + a_l a_k = 0 & \text{for distinct } k,\, l,\ 1 \leqq k,\, l \leqq \kappa - 1\,, \end{cases}$$

the family $\{a_k\}$ satisfies the condition (7.6). The correspondence $\rho \to \{a_k\}$ induces a map of $\mathscr{R}_m(C_{\kappa-1}, *)$ into the set of equivalence classes of families $\{a_k\}$ satisfying (7.6). One can see easily that it is bijective. q.e.d.

Next, consider a family $\{A_k\}_{1\leqq k\leqq\kappa}$ of κ matrices in $M_m(\boldsymbol{R})$ satisfying the following condition:

$$(7.7)\qquad \begin{cases} A_k' = -A_k,\ A_k^2 = -1_m & \text{for each } k\,, \\ A_kA_l + A_lA_k = 0 & \text{for distinct } k, l\,. \end{cases}$$

Note that the condition (7.7) implies the condition (7.6). Two such families $\{A_k\}$, $\{\widetilde{A}_k\}$ are said to be *equivalent* and denoted by $\{A_k\} \approx \{\widetilde{A}_k\}$ if there exist $\sigma \in O(m)$ and $\tau = (\tau_{kl}) \in O(\kappa)$ such that

$$\widetilde{A}_k = \sum_{l=1}^{\kappa} \tau_{kl}(\sigma A_l \sigma^{-1}) \qquad \text{for each } k\,.$$

They are also classified in terms of representations of Clifford algebras as follows.

LEMMA 25. *The set of equivalence classes of families $\{A_k\}$ of κ matrices in $M_m(\boldsymbol{R})$ satisfying the condition* (7.7) *is in a bijective correspondence with the set $O(\kappa)\backslash\mathscr{R}_m(C_\kappa, *)$.*

PROOF. For each representation ρ of C_κ of degree m satisfying (7.1), we define κ matrices $A_1, \cdots, A_\kappa$ by

$$A_k = \rho(e_k) \qquad \text{for each } k\,.$$

Then the family $\{A_k\}$ satisfies the condition (7.7). The correspondence $\rho \longrightarrow \{A_k\}$ induces a bijection required in our lemma. q.e.d.

From Lemmas $22 \sim 25$, we have

LEMMA 26. *There exists a family $\{A_k\}$ of κ matrices in $M_{\kappa+1}(\boldsymbol{R})$ satisfying the condition* (7.7) *if and only if $\kappa = 1, 3, 7$. For $\kappa = 1, 3, 7$, there exists a family $\{a_k\}$ of κ matrices in $M_m(\boldsymbol{R})$ satisfying the condition* (7.6) *if and only if m is a multiple of* 1, 4, 8 *respectively. In these cases, both of equivalence classes of $\{A_k\}$ and $\{a_k\}$ are unique.*

8. Examples of non-homogeneous isoparametric hypersurfaces. Now we come back to families of quadratic forms $\{p_\alpha\}$ and cubic forms $\{q_\alpha\}$ on $Y = \boldsymbol{R}^n$. In this section we shall classify polynomials $\{p_\alpha\}$, $\{q_\alpha\}$ under certain conditions and construct two series of non-homogeneous isoparametric hypersurfaces.

As in §5, let

$$Y = U \oplus V \oplus Z$$

be the eigenspace decomposition of the symmetric mapping P_0 corresponding to p_0, where U, V and Z are the eigenspaces for the eigenvalues 1, -1 and 0 respectively. Recall $\dim U = \dim V = m_2$ and $\dim Z = m_1$. We choose orthonormal coordinate systems $\{u_i\}$, $\{v_i\}$ and $\{z_k\}$ for U, V and Z respectively. Each symmetric mapping P_k corresponding to p_k for $k \geqq 1$ will be represented by a matrix with respect to these coordinates as in Lemma 14.

LEMMA 27. *Assume that P_0 is represented in the above way. Then the family $\{p_\alpha\}$ satisfies* (3-1), (3-2) *and the condition* (*A*) *if and only if* (1) *each $P_k (1 \leqq k \leqq m_1)$ is represented by a matrix of the form*

$$\begin{pmatrix} 0 & a_k & 0 \\ a'_k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $a_k \in M_{m_2}(\boldsymbol{R})$ and (2) *the family $\{a_k\}$ satisfies the condition* (7.6) *for $\kappa = m_1$ and $m = m_2$.*

PROOF. First suppose $\{p_\alpha\}$ satisfies (3-1), (3-2) and (A). Then the family $\{R_\alpha\}$ of symmetric mappings of $U \oplus V$ associated to $\{p_\alpha\}$ in §6 satisfies (5-1) and (5-2). The condition (5-2) for $\alpha = 0$ and $\beta = k$ implies that R_k is represented by a matrix of the form

$$\begin{pmatrix} 0 & a_k \\ a'_k & 0 \end{pmatrix}$$

with $a_k \in M_{m_2}(\boldsymbol{R})$. Now (5-1) gives

$$\text{(i)} \qquad a_k a'_k = a'_k a_k = 1_{m_2} \qquad \text{for each } k\,,$$

and also (5-2) gives

$$\text{(ii)} \qquad \begin{cases} a_k a'_l + a_l a'_k = 0 \\ a'_k a_l + a'_l a_k = 0 \end{cases} \qquad \text{for distinct } k, l$$

where $1 \leqq k, l \leqq m_1$. (i) and (ii) together are equivalent to the condition (7.6), thereby obtaining (1) and (2) of Lemma 27.

The converse follows from the above argument and Lemma 18.

q.e.d.

Now let $\{p_\alpha\}$ be a family of quadratic forms on Y satisfying (3-1), (3-2) and (A), and let $\{q_\alpha\}$ be a family of cubic forms on Y. We assume the following additional condition:

(B) For each α, q_α is expressed as

$$q_\alpha = \sum_\beta \lambda_{\alpha\beta} p_\beta$$

where $\lambda_{\alpha\beta}$'s are linear forms on Z.

First note that the above expression of q_α is unique by virtue of Lemma 20. We put

$$\lambda_{\alpha\beta} = \sum_{k=1}^{m_1} a_{\alpha\beta k} z_k \tag{8.1}$$

for each α, β, and define m_1 matrices $A_1, \cdots, A_{m_1}$ in $M_{m_1+1}(\boldsymbol{R})$ by

$$A_k = (a_{\alpha\beta k})_{0 \leqq \alpha, \beta \leqq m_1} \tag{8.2}$$

for each k, $1 \leqq k \leqq m_1$.

LEMMA 28. *As in the above, suppose that* $\{p_\alpha\}$ *and* $\{q_\alpha\}$ *satisfy* (3-1) *and* (3-2) *together with* (*A*) *and* (*B*). *Then,* $\{p_\alpha\}$ *and* $\{q_\alpha\}$ *satisfy the conditions* (3-4), (3-5), (3-7), (5-8), (5-9) *and* (5-10) *if and only if the family* $\{A_k\}$ *of* m_1 *matrices in* $M_{m_1+1}(\boldsymbol{R})$ *satisfies the condition* (7.7) *and the following condition*:

$$\frac{1}{2} \sum_k (a_{\alpha\gamma k} a_{\beta\delta k} + a_{\alpha\delta k} a_{\beta\gamma k}) = \delta_{\alpha\beta} \delta_{\gamma\delta} \tag{8.3}$$

for each $\alpha, \beta, \gamma, \delta$ *with* $\{\alpha, \beta\} \cap \{\gamma, \delta\} = \varnothing$.

PROOF. Note that the above condition (8.3) is equivalent to the following two conditions:

$$\sum_k a_{\alpha\beta k} a_{\alpha\gamma k} = \delta_{\beta\gamma} \text{ for each } \alpha, \beta, \gamma \text{ with } \beta \neq \alpha, \gamma \neq \alpha \, ; \tag{8.3.1}$$

$$\sum_k (a_{\alpha\gamma k} a_{\beta\delta k} + a_{\alpha\delta k} a_{\beta\gamma k}) = 0 \text{ for mutually distinct } \alpha, \beta, \gamma, \delta \, . \tag{8.3.2}$$

Similarly, the condition (7.7) decomposes into

$$A_k + A'_k = 0 \quad \text{for each} \quad k \, ; \tag{7.7.1}$$

$$\begin{cases} A'_k A_k = 1_{m_1+1} & \text{for each} \quad k \, , \\ A'_k A_l + A'_l A_k = 0 & \text{for distinct} \quad k, l \, . \end{cases} \tag{7.7.2}$$

First we show the following implications: (3-7) $\Leftrightarrow$ (7.7.1); (7.7.1) $\Rightarrow$ (3-4) and (3-5); and then (5-8) $\Leftrightarrow$ (7.7.2).

Recall (3-7): $\sum p_\alpha q_\alpha = 0$. We have

$$\sum_\alpha p_\alpha q_\alpha = \sum_{\alpha,\beta} \lambda_{\alpha\beta} p_\alpha p_\beta = \frac{1}{2} \sum_k \left\{ \sum_{\alpha,\beta} (a_{\alpha\beta k} + a_{\beta\alpha k}) p_\alpha p_\beta \right\} z_k \, .$$

By Lemma 20, we see (3-7) $\Leftrightarrow$ (7.7.1). Since each $\lambda_{\beta\gamma}$ is a linear form on Z, we have $\langle p_\alpha, \lambda_{\beta\gamma} \rangle = 0$. Thus, we have

$$\langle p_\alpha, q_\beta \rangle = \sum_\gamma \lambda_{\beta\gamma} \langle p_\alpha, p_\gamma \rangle = \lambda_{\beta\alpha} \langle p_0, p_0 \rangle \, ,$$

using Lemma 17. Therefore we can write

$$\langle p_\alpha, q_\beta \rangle + \langle p_\beta, q_\alpha \rangle = (\lambda_{\alpha\beta} + \lambda_{\alpha\beta}) \langle p_0, p_0 \rangle \, .$$

This shows (7.7.1) $\Rightarrow$ (3-4) and (3-5). Recall (5-8): $\sum q_\alpha^2 = G(\sum z_k^2)$. We have

$$\sum_\alpha q_\alpha^2 = \sum_\alpha \Big(\sum_\beta \lambda_{\alpha\beta} p_\beta\Big)^2 = \sum_{\alpha,\beta,\gamma} \lambda_{\alpha\beta}\lambda_{\alpha\gamma} p_\beta p_\gamma$$
$$= \frac{1}{2}\sum_{\alpha,\beta,\gamma,k,l} (a_{\alpha\beta k}a_{\alpha\gamma l} + a_{\alpha\beta l}a_{\alpha\gamma k}) p_\beta p_\gamma z_k z_l \, ,$$

and

$$G\Big(\sum_k z_k^2\Big) = \Big(\sum_k z_k^2\Big)\Big(\sum_\alpha p_\alpha^2\Big) .$$

Now (5-8) is equivalent to

$$\begin{cases} \sum_{\alpha,\beta,\gamma} a_{\alpha\beta k}a_{\alpha\gamma k} p_\beta p_\gamma = \sum_\beta p_\beta^2 & \text{for each } k \, , \\ \sum_{\alpha,\beta,\gamma} (a_{\alpha\beta k}a_{\alpha\gamma l} + a_{\alpha\beta l}a_{\alpha\gamma k}) p_\beta p_\gamma = 0 & \text{for distinct } k, l \, , \end{cases}$$

which is, by Lemma 20, equivalent to

$$\begin{cases} \sum_\alpha a_{\alpha\beta k}a_{\alpha\gamma k} = \delta_{\beta\gamma} & \text{for each } \beta, \gamma, k \, , \\ \sum_\alpha (a_{\alpha\beta k}a_{\alpha\gamma l} + a_{\alpha\beta l}a_{\alpha\gamma k}) = 0 & \text{for each } \beta, \gamma \text{ and distinct } k, l \, . \end{cases}$$

This is nothing but (7.7.2), thereby obtaining the implications described first.

Henceforth we assume the condition (7.7). Consider the condition (5-9). We have

$$\langle q_\alpha, q_\alpha\rangle = \Big\langle \sum_\beta \lambda_{\alpha\beta} p_\beta, \sum_\gamma \lambda_{\alpha\gamma} p_\gamma \Big\rangle$$
$$= \sum_{\beta,\gamma} \langle \lambda_{\alpha\beta}, \lambda_{\alpha\gamma}\rangle p_\beta p_\gamma + \sum_{\beta,\gamma} \lambda_{\alpha\beta}\lambda_{\alpha\gamma} \langle p_\beta, p_\gamma\rangle$$
$$= \sum_{\beta,\gamma,k} a_{\alpha\beta k}a_{\alpha\gamma k} p_\beta p_\gamma + 4\Big(\sum u_i^2 + \sum v_i^2\Big) \sum_{\beta,k,l} a_{\alpha\beta k}a_{\alpha\beta l} z_k z_l \, ,$$

and

$$G - p_\alpha^2 + 4\Big(\sum u_i^2 + \sum v_i^2\Big)\Big(\sum z_k^2\Big)$$
$$= \sum_{\alpha\neq\beta} p_\beta^2 + 4\Big(\sum u_i^2 + \sum v_i^2\Big)\Big(\sum z_k^2\Big) .$$

Again by Lemma 20, we see that (5.9) is equivalent to the following three conditions:

(i) $\quad \sum_k a_{\alpha\beta k}a_{\alpha\gamma k} = \delta_{\beta\gamma}$ for each α, β, γ with $\beta \neq \alpha, \gamma \neq \alpha$;

(ii) $\quad \sum_k a_{\alpha\alpha k}a_{\alpha\alpha k} = 0$ for each α ;

(iii) $\quad \sum_\beta a_{\alpha\beta k}a_{\alpha\beta l} = \delta_{kl}$ for each α, k, l .

Since (ii) and (iii) follow from (7.7), we have (5-9) $\Leftrightarrow$ (i) = (8.3.1). By a similar computation, we can see (5-10) $\Rightarrow$ (8.3.2) and (8.3) $\Rightarrow$ (5-10).

q.e.d.

Now we recall some properties of inner products on division algebras over $\boldsymbol{R}$. Let $\boldsymbol{F}$ be a (not necessarily associative) division algebra over $\boldsymbol{R}$, i.e., $\boldsymbol{F}=\boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}$ or the real Cayley algebra $\boldsymbol{K}$. Let $c_0=1, c_1, \cdots, c_{d-1}$ be the standard units of $\boldsymbol{F}$ with $d=\dim \boldsymbol{F}$. $u \rightarrow \bar{u}$ denotes the canonical involution of $\boldsymbol{F}$. We put $\mathfrak{J}\boldsymbol{F}=\{u \in \boldsymbol{F} \mid \bar{u}=-u\}$. Then $\mathfrak{J}\boldsymbol{F}$ is a $(d-1)$-dimensional subspace of $\boldsymbol{F}$ spanned by $c_1, \cdots, c_{d-1}$. The subspace $\boldsymbol{R}1=\{u \in \boldsymbol{F} \mid \bar{u}=u\}$ will be identified with $\boldsymbol{R}$ in a natural way. On $\boldsymbol{F}$,

$$(u, v)=\frac{1}{2}(u\bar{v}+v\bar{u})$$

defines an inner product with the following properties:

$$\begin{aligned}
&(\bar{u}, \bar{v})=(u, v)\,, \\
&(uv, w)=(v, \bar{u}w)=(u, w\bar{v})\,, \\
&\bar{u}(vw)+v(\bar{u}w)=(wu)\bar{v}+(wv)\bar{u}=2(u, v)w\,.
\end{aligned}$$

$\{c_0, c_1, \cdots, c_{d-1}\}$ forms an orthonormal basis of $\boldsymbol{F}$ with respect to the above inner product. The dual base $\{u_0, u_1, \cdots, u_{d-1}\}$ of $\{c_0, c_1, \cdots, c_{d-1}\}$ forms an orthonormal coordinate system for $\boldsymbol{F}$, which we call *standard*. $(\ ,\ )$ is extended to the m-column vector space $\boldsymbol{F}^m$ by

$$(u, v)=\frac{1}{2}(u'\bar{v}+v'\bar{u})$$

for $u, v \in \boldsymbol{F}^m$, where $'$ denotes the transpose. The *standard* orthonormal coordinate system for $\boldsymbol{F}^m$ consists of $\{u_i^{(\lambda)} \mid 0 \leqq i \leqq d-1, 1 \leqq \lambda \leqq m\}$ where $\{u_i^{(\lambda)} \mid 0 \leqq i \leqq d-1\}$ denotes the standard orthonormal coordinates for the λ-th component $u^{(\lambda)}$ of $u \in \boldsymbol{F}^m$. We write also $\|u\|$ for the norm $(u, u)^{1/2}$ of a vector u.

THEOREM 2. *Let m_1 and m_2 be positive integers such that $N=2(m_1+m_2+1)$, and set $n=m_1+2m_2$.*

(i) *There exist m_1+1 quadratic forms $\{p_\alpha\}$ and m_1+1 cubic forms $\{q_\alpha\}$ on $Y=\boldsymbol{R}^n$ satisfying the equations (3-1) $\sim$ (3-10) together with the conditions (A) and (B) if and only if the pair (m_1, m_2) is one of the following three types: $(1, r)$, $(3, 4r)$, $(7, 8r)$ for some positive integer r. In these cases, the polynomial F associated to such $\{p_\alpha, q_\alpha\}$ is unique up to (ON)-equivalence.*

(ii) *The polynomial F on $\boldsymbol{R}^N$ associated to such $\{p_\alpha, q_\alpha\}$ is given explicitly as follows:*

(a) $(m_1, m_2)=(1, r)$; *We define a polynomial F_0 on $\boldsymbol{R}^{2(r+2)}=\boldsymbol{C}^{r+2}$ by*

$$F_0(\xi) = \left\|\sum_{i=1}^{r+2} \xi_i^2\right\|^2 \quad for \quad \xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{r+2} \end{pmatrix} \in \boldsymbol{C}^{r+2} ,$$

and set $F = r^4 - 2F_0$.

(b) $(m_1, m_2) = (3, 4r)$ *or* $(7, 8r)$; F *denotes* $\boldsymbol{H}$ *or* $\boldsymbol{K}$ *according to* $m_1 = 3$ *or* 7. *We define a polynomial* F_0 *on* $\boldsymbol{R}^N = \boldsymbol{F}^{2(r+1)} = \boldsymbol{F}^{r+1} \times \boldsymbol{F}^{r+1}$ *by*

$$F_0(u \times v) = 4\{\|u'\bar{v}\|^2 - (u, v)^2\} + \{\|u_1\|^2 - \|v_1\|^2 + 2(u_0, v_0)\}^2$$

for

$$u = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} , \quad v = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} , \quad u_0, v_0 \in \boldsymbol{F} , \quad u_1, v_1 \in \boldsymbol{F}^r ,$$

and set $F = r^4 - 2F_0$.

In each case, F *satisfies the differential equations* (M) *of Münzner.*

REMARK. Takagi-Takahashi [7] gave the multiplicities of principal curvatures for homogeneous isoparametric hypersurfaces in spheres. Our pairs (m_1, m_2) of multiplicities in the case (b) do not appear in their table except $(m_1, m_2) = (3, 4)$. Hence our isoparametric hypersurfaces given in the above case (b) are not homogeneous, possibly except the case where $(m_1, m_2) = (3, 4)$. However, in Part II it will be shown that our isoparametric hypersurfaces for $(m_1, m_2) = (3, 4)$ are also non-homogeneous.

PROOF OF (i). The "only if" part follows immediately from Lemmas 26, 27, 28. Conversely, assume that (m_1, m_2) is $(1, r)$, $(3, 4r)$ or $(7, 8r)$. Let $\boldsymbol{F} = \boldsymbol{C}, \boldsymbol{H}$ or $\boldsymbol{K}$ respectively, so that $\dim \boldsymbol{F} = m_1 + 1$. In the following, indices $k, l, \cdots$ and $\alpha, \beta, \cdots$ run through $1, 2, \cdots, m_1$ and $0, 1, \cdots, m_1$ respectively. For $u, v \in \boldsymbol{F}$ we have

$$\begin{aligned}
&(c_k u, v) = (u, \bar{c}_k v) = -(c_k v, u) && \text{for each } k \\
&c_k(c_k u) = -\bar{c}_k(c_k u) = -(c_k, c_k)u = -u && \text{for each } k \\
&c_k(c_l u) + c_l(c_k u) = -\bar{c}_k(c_l u) - \bar{c}_l(c_k u) = -2(c_k, c_l)u = 0 && \\
& && \text{for distinct } k, l .
\end{aligned}$$

We define $A_1, \cdots, A_{m_1} \in M_{m_1+1}(\boldsymbol{R})$ by

$$A_k = (a_{\alpha\beta k})_{0 \leq \alpha, \beta \leq m_1} \quad \text{with} \quad a_{\alpha\beta k} = (c_k c_\beta, c_\alpha)$$

for each k. Then $\{A_k\}$ satisfy (7.7) as is easily seen from the above properties. Consider (8.3). For each $\alpha, \beta, \gamma, \delta$ with $\{\alpha, \beta\} \cap \{\gamma, \delta\} = \emptyset$, we have

$$\begin{aligned}
&\sum_k (a_{\alpha\gamma k}a_{\beta\delta k} + a_{\alpha\delta k}a_{\beta\gamma k}) \\
&\quad = \sum_k (c_k c_\gamma, c_\alpha)(c_k c_\delta, c_\beta) + \sum_k (c_k c_\delta, c_\alpha)(c_k c_\gamma, c_\beta) \\
&\quad = \sum_\varepsilon (c_\varepsilon c_\gamma, c_\alpha)(c_\varepsilon c_\delta, c_\beta) + \sum_\varepsilon (c_\varepsilon c_\delta, c_\alpha)(c_\varepsilon c_\gamma, c_\beta) \\
&\quad = \sum_\varepsilon (c_\varepsilon, c_\alpha \bar{c}_\gamma)(c_\varepsilon, c_\beta \bar{c}_\delta) + \sum_\varepsilon (c_\varepsilon, c_\alpha \bar{c}_\delta)(c_\varepsilon, c_\beta \bar{c}_\gamma) \\
&\quad = (c_\alpha \bar{c}_\gamma, c_\beta \bar{c}_\delta) + (c_\alpha \bar{c}_\delta, c_\beta \bar{c}_\gamma) \\
&\quad = (\bar{c}_\beta(c_\alpha \bar{c}_\gamma), \bar{c}_\delta) + (\bar{c}_\delta, \bar{c}_\alpha(c_\beta \bar{c}_\gamma)) \\
&\quad = 2(c_\beta, c_\alpha)(\bar{c}_\gamma, \bar{c}_\delta) = 2(c_\alpha, c_\beta)(c_\gamma, c_\delta) \\
&\quad = 2\delta_{\alpha\beta}\delta_{\gamma\delta} ,
\end{aligned}$$

and hence we have (8.3) for $\{A_k\}$.

Next, we define m_1 matrices $\{a_k\}$ in $M_{m_2}(\boldsymbol{R})$ as follows: for $m_1 = 1$

$$a_k = 1_r ,$$

and for $m_1 = 3$ or 7

$$a_k = \begin{pmatrix} A_k & & 0 \\ & \ddots & \\ 0 & & A_k \end{pmatrix}$$

where A_k appears r-times in the diagonal. One sees easily that $\{a_k\}$ satisfy (7.6).

Now by Lemma 27 we can associate to the matrices $\{a_k\}$ $m_1 + 1$ quadratic forms $\{p_\alpha\}$ on Y, satisfying (3-1), (3-2) and (A). From the matrices $\{A_k\}$, using (8.1) we can define $m_1 + 1$ cubic forms on Y, satisfying (B). Our polynomials $\{p_\alpha\}$, $\{q_\alpha\}$ satisfy, in virtue of Lemma 28, (3-4), (3-5), (3-7), (5-8), (5-9), (5-10), and hence the equations (3-1) $\sim$ (3-10) by Lemma 19, which proves the "if" part of (i).

It remains to prove the uniqueness. Let $\{p_\alpha, q_\alpha\}$ and $\{\tilde{p}_\alpha, \tilde{q}_\alpha\}$ be two families of polynomials on Y satisfying the conditions in (i), and let F and $\tilde{F}$ be the associated polynomials on $\boldsymbol{R}^N$ respectively. Let

$$(1) \qquad Y = U \oplus V \oplus Z ,$$

$$(2) \qquad Y = \tilde{U} \oplus \tilde{V} \oplus \tilde{Z}$$

be the eigenspace decompositions of symmetric mappings P_0, $\tilde{P}_0$ corresponding to p_0, $\tilde{p}_0$ respectively. We take orthonormal coordinate systems $\{u_i\}$, $\{v_i\}$, $\{z_k\}$ for U, V, W respectively. Linear mappings of Y will be represented by matrices with respect to these coordinates.

Choosing $\sigma_1 \in O(n)$ such that $\sigma_1 U = \tilde{U}$, $\sigma_1 V = \tilde{V}$ and $\sigma_1 Z = \tilde{Z}$, we put

$$p_\alpha^{(1)} = \sigma_1^{-1}\tilde{p}_\alpha\,, \quad q_\alpha^{(1)} = \sigma_1^{-1}\tilde{q}_\alpha\,.$$

Then the polynomials $\{p_\alpha^{(1)}, q_\alpha^{(1)}\}$ also satisfy the conditions in (i) and the eigenspace decomposition of $P_0^{(1)}$ corresponding to $p_0^{(1)}$ is the same as (1). The condition (B) for $\{p_\alpha, q_\alpha\}$ and $\{p_\alpha^{(1)}, q_\alpha^{(1)}\}$ gives $\{A_k\}$ and $\{A_k^{(1)}\}$ in $M_{m_1+1}(\boldsymbol{R})$ respectively, which satisfy (7.7) by Lemma 28. It follows from Lemma 26 that $\{A_k\} \approx \{A_k^{(1)}\}$, that is, there exist $\varphi = (\varphi_{kl}) \in O(m_1)$ and $\tau = (\tau_{\alpha\beta}) \in O(m_1 + 1)$ such that

$$A_k^{(1)} = \sum_l \varphi_{kl}(\tau A_l \tau^{-1}) \qquad \text{for each } k\,.$$

We put

$$p_\alpha^{(2)} = \sum_\beta \tau_{\alpha\beta} p_\beta\,.$$

Then the quadratic forms $\{p_\alpha^{(2)}\}$ also satisfy (3-1), (3-2), (A). Let

$$Y = U^{(2)} \oplus V^{(2)} \oplus Z$$

be the eigenspace decomposition of $P_0^{(2)}$ corresponding to $p_0^{(2)}$. Choosing $\sigma_2 \in O(n)$ such that $\sigma_2 U^{(2)} = U$, $\sigma_2 V^{(2)} = V$, $\sigma_2 | Z =$ identity, we put

$$p_\alpha^{(3)} = \sigma_2 p_\alpha^{(2)}.$$

Then $\{p_\alpha^{(3)}\}$ also satisfy (3-1), (3-2), (A), and the eigenspace decomposition of $P_0^{(3)}$ corresponding to $p_0^{(3)}$ is the same as (1). It follows from Lemma 27 that $\{p_\alpha^{(1)}\}$ and $\{p_\alpha^{(3)}\}$ define $\{a_k^{(1)}\}$ and $\{a_k^{(3)}\}$ in $M_{m_2}(\boldsymbol{R})$ respectively, satisfying (7.6). By Lemma 26, we have $\{a_k^{(1)}\} \sim \{a_k^{(3)}\}$, that is, we can find $\sigma_3, \sigma_4 \in O(m_2)$ such that

$$\sigma_3 a_k^{(3)} \sigma_4^{-1} = a_k^{(1)} \quad \text{for each} \quad k\,.$$

Putting together σ_3, σ_4 and φ^{-1}, we get an element $\sigma_3 \times \sigma_4 \times \varphi^{-1} \in O(m_2) \times O(m_2) \times O(m_1) \subset O(n)$. Put $\sigma = \sigma_1(\sigma_3 \times \sigma_4 \times \varphi^{-1})\sigma_2 \in O(n)$. Then we have

$$\tilde{p}_\alpha = \sum_\beta \tau_{\alpha\beta}(\sigma p_\beta),\ \tilde{q}_\alpha = \sum_\beta \tau_{\alpha\beta}(\sigma q_\beta) \quad \text{for each} \quad \alpha\,,$$

which gives the required uniqueness. In fact,

$$\sum_\beta \tau_{\alpha\beta}(\sigma p_\beta) = \sigma p_\alpha^{(2)} = \sigma_1(\sigma_3 \times \sigma_4 \times \varphi^{-1}) p_\alpha^{(3)} = \sigma_1 p_\alpha^{(1)} = \tilde{p}_\alpha\,.$$

Denoting by $a_{\alpha\beta k}$, $a_{\alpha\beta k}^{(1)}$ the (α, β)-elements of A_k, $A_k^{(1)}$ respectively, we have

$$\begin{aligned}
\sigma_1^{-1}\Big(\sum_\beta \tau_{\alpha\beta}(\sigma q_\beta)\Big) &= (\sigma_3 \times \sigma_4 \times \varphi^{-1})\sigma_2\Big(\sum_{\beta,\gamma,l} \tau_{\alpha\beta} a_{\beta\gamma l} z_l p_\gamma\Big) \\
&= \sum_{\beta,\gamma,l} \tau_{\alpha\beta} a_{\beta\gamma l}(\varphi^{-1} z_l)(\sigma_3 \times \sigma_4 \times \varphi^{-1})\sigma_2\Big(\sum_\delta \tau_{\delta\gamma} p_\delta^{(2)}\Big) \\
&= \sum_{\beta,\gamma,\delta,l,k} \tau_{\alpha\beta} a_{\beta\gamma l} \varphi_{kl} \tau_{\delta\gamma} z_k p_\delta^{(1)} = \sum_{\delta,k} a_{\alpha\delta k}^{(1)} z_k p_\delta^{(1)} = q_\alpha^{(1)}\,,
\end{aligned}$$

and hence

$$\sum_{\beta} \tau_{\alpha\beta}(\sigma q_\beta) = \tilde{q}_\alpha .$$

It follows that F and $\tilde{F}$ are $O(N)$-equivalent.

PROOF OF (ii). (b) $m_1 = 3$ or 7. Let $\boldsymbol{F} = \boldsymbol{H}$ or $\boldsymbol{K}$ respectively. Let

$$U = \boldsymbol{F}^r,\ V = \boldsymbol{F}^r,\ \hat{Z} = \boldsymbol{F},\ W = \boldsymbol{F},\ Z = \mathfrak{J}\boldsymbol{F} \subset \hat{Z} ,$$

and let

$$\boldsymbol{R}^N = U \oplus V \oplus \hat{Z} \oplus W ,$$
$$Y = U \oplus V \oplus Z$$

be the orthogonal direct sums. Elements of U, V, Z, W will be denoted by u, v, z, w respectively. The standard orthonormal coordinate systems for U, V, $\hat{Z}$, W are denoted by $\{u_i^{(\lambda)}\}$, $\{v_i^{(\lambda)}\}$, $\{z_\alpha\}$, $\{w_\alpha\}$ respectively, and they as a whole form an orthonormal coordinate system for $\boldsymbol{R}^N$. As a base point e in $\boldsymbol{R}^N$, we take the unit c_0 in $\hat{Z}$ so that we have $z = z_0$ in the notation of §3. We compute polynomials $\{p_\alpha\}$, $\{q_\alpha\}$ on Y corresponding to matrices $\{a_k\}$, $\{A_k\}$ given in the proof of (i), with respect to the above orthonormal coordinate system. We have

$$p_0 = \sum_{\substack{0 \leq i \leq m_1 \\ 1 \leq \lambda \leq r}} \{(u_i^{(\lambda)})^2 - (v_i^{(\lambda)})^2\} = \|u\|^2 - \|v\|^2 ,$$

$$p_k = 2 \sum_{\substack{0 \leq i,j \leq m_1 \\ 1 \leq \lambda \leq r}} (c_k c_j, c_i) u_i^{(\lambda)} v_j^{(\lambda)} = 2 \sum_{1 \leq \lambda \leq r} (c_k v^{(\lambda)}, u^{(\lambda)}) = 2(c_k, u'\bar{v}) ,$$

$$\begin{aligned} q_\alpha &= \sum_{\beta,k} (c_k c_\beta, c_\alpha) z_k p_\beta \\ &= \sum_k \Big\{ (c_k c_0, c_\alpha) p_0 + \sum_l (c_k c_l, c_\alpha) p_l \Big\} z_k \\ &= \sum_k \Big\{ (c_k c_0, c_\alpha)(\|u\|^2 - \|v\|^2) + 2 \sum_l (c_k c_l, c_\alpha)(c_l, u'\bar{v}) \Big\} z_k \\ &= (c_0, \bar{z} c_\alpha)(\|u\|^2 - \|v\|^2) + 2 \sum_l (c_l, \bar{z} c_\alpha)(c_l, u'\bar{v}) , \end{aligned}$$

where we have

$$\begin{aligned} &(c_0, \bar{z} c_\alpha) = (z, c_\alpha) , \\ &\sum_l (c_l, \bar{z} c_\alpha)(c_l, u'\bar{v}) = (\bar{z} c_\alpha, u'\bar{v}) - (c_0, \bar{z} c_\alpha)(c_0, u'\bar{v}) \\ &\qquad = (\bar{z} c_\alpha, u'\bar{v}) - (z, c_\alpha)(u, v) . \end{aligned}$$

Hence we have

$$q_\alpha = (z, c_\alpha)(\|u\|^2 - \|v\|^2 - 2(u, v)) + 2(\bar{z} c_\alpha, u'\bar{v}) .$$

In particular, $q_0 = 2(\bar{z}, u'\bar{v})$. Now we have

$$\begin{aligned}
\sum_\alpha p_\alpha w_\alpha &= (\|u\|^2 - \|v\|^2)w_0 + 2\sum_k (c_k, u'\bar{v})w_k \\
&= (\|u\|^2 - \|v\|^2)w_0 + 2(w, u'\bar{v}) - 2(c_0, u'\bar{v})w_0 \\
&= (\|u\|^2 - \|v\|^2 - 2(u, v))w_0 + 2(w, u'\bar{v}) , \\
\sum_\alpha q_\alpha w_\alpha &= (z, w)(\|u\|^2 - \|v\|^2 - 2(u, v)) + 2(\bar{z}w, u'\bar{v}) , \\
\sum_\alpha p_\alpha^2 &= (\|u\|^2 - \|v\|^2)^2 + 4\sum_k (c_k, u'\bar{v})^2 \\
&= (\|u\|^2 - \|v\|^2)^2 + 4\|u'\bar{v}\|^2 - 4(u, v)^2 .
\end{aligned}$$

Furthermore we have

$$\langle p_\alpha, p_\beta \rangle = 4(\|u\|^2 + \|v\|^2)\delta_{\alpha,\beta} \quad \text{for each} \quad \alpha, \beta .$$

Recall Lemmas 4, 5, 6, 7. The polynomial F on $\boldsymbol{R}^N$ associated to $\{p_\alpha\}$, $\{q_\alpha\}$ is given by

$$\begin{aligned}
F = z_0^4 &+ z_0^2\{2(\|u\|^2 + \|v\|^2 + \|z\|^2) - 6\|w\|^2\} \\
&+ 8z_0\{\|u\|^2 - \|v\|^2 - 2(u, v))w_0 + 2(w, u'\bar{v})\} \\
&+ (\|u\|^2 + \|v\|^2 + \|z\|^2)^2 - 2\{(\|u\|^2 - \|v\|^2)^2 + 4\|u'\bar{v}\|^2 - 4(u, v)^2\} \\
&+ 8\{(z, w)(\|u\|^2 - \|v\|^2 - 2(u, v)) + 2(\bar{z}w, u'\bar{v})\} \\
&+ 8(\|u\|^2 + \|v\|^2)\|w\|^2 - 6(\|u\|^2 + \|v\|^2 + \|z\|^2)\|w\|^2 + \|w\|^4 \\
= z_0^4 &+ 2z_0^2(\|u\|^2 + \|v\|^2 + \|z\|^2) + (\|u\|^2 + \|v\|^2 + \|z\|^2)^2 \\
&- 6z_0^2\|w\|^2 - 6(\|u\|^2 + \|v\|^2 + \|z\|^2)\|w\|^2 \\
&+ 8z_0w_0(\|u\|^2 - \|v\|^2 - 2(u, v)) + 8(z, w)(\|u\|^2 - \|v\|^2 - 2(u, v)) \\
&+ 16z_0(w, u'\bar{v}) + 16(\bar{z}w, u'\bar{v}) \\
&- 2(\|u\|^2 - \|v\|^2)^2 - 8\|u'\bar{v}\|^2 + 8(u, v)^2 \\
&+ 8(\|u\|^2 + \|v\|^2)\|w\|^2 + \|w\|^4 .
\end{aligned}$$

Putting $\zeta = z_0c_0 + z \in \hat{Z}$ $(z \in Z)$, we have

$$\begin{aligned}
F &= (\|u\|^2 + \|v\|^2 + \|\zeta\|^2)^2 - 6(\|u\|^2 + \|v\|^2 + \|\zeta\|^2)\|w\|^2 \\
&\quad + 8(\zeta, w)(\|u\|^2 - \|v\|^2 - 2(u, v)) + 16(\bar{\zeta}w, u'\bar{v}) \\
&\quad - 2(\|u\|^2 - \|v\|^2)^2 - 8\|u'\bar{v}\|^2 + 8(u, v)^2 \\
&\quad + 8(\|u\|^2 + \|v\|^2)\|w\|^2 + \|w\|^4 \\
&= (\|u\|^2 + \|v\|^2 + \|\zeta\|^2 + \|w\|^2)^2 - 8\|\zeta\|^2\|w\|^2 \\
&\quad + 8(\zeta, w)(\|u\|^2 - \|v\|^2 - 2(u, v)) + 16(\bar{\zeta}w, u'\bar{v}) \\
&\quad - 2(\|u\|^2 - \|v\|^2)^2 - 8\|u'\bar{v}\|^2 + 8(u, v)^2 .
\end{aligned}$$

Seieng

$$||u'\bar{v} - \bar{\zeta}w||^2 = ||u'\bar{v}||^2 - 2(\bar{\zeta}w, u'\bar{v}) + ||\zeta||^2||w||^2$$
$$(||u||^2 - ||v||^2 - 2(\zeta, w))^2 = (||u||^2 - ||v||^2)^2 - 4(\zeta, w)(||u||^2 - ||v||^2) + 4(\zeta, w)^2$$
$$((u, v) - (\zeta, w))^2 = (u, v)^2 - 2(\zeta, w)(u, v) + (\zeta, w)^2 ,$$

we get

$$F = r^4 - 2F_0$$

where

$$F_0 = 4\{||u'\bar{v} - \bar{\zeta}w||^2 - ((u, v) - (\zeta, w))^2\} + (||u||^2 - ||v||^2 - 2(\zeta, w))^2 .$$

We put $u_0 = \bar{\zeta}$, $v_0 = -\bar{w}$, and

$$u_1 = \begin{pmatrix} u_0 \\ u \end{pmatrix}, \quad v_1 = \begin{pmatrix} v_0 \\ v \end{pmatrix} \in \boldsymbol{F}^{r+1}.$$

Then we have

$$F_0 = 4\{||u_1'\bar{v}_1||^2 - (u_1, v_1)^2\} + (||u||^2 - ||v||^2 + 2(u_0, v_0))^2 ,$$

which shows the case (b) of (ii).

(a) $m_1 = 1$. Let

$$U = \boldsymbol{R}^r,\ V = \boldsymbol{R}^r,\ \hat{Z} = \boldsymbol{C},\ W = \boldsymbol{C},\ Z = \mathfrak{J}\boldsymbol{C} \subset \hat{Z}$$

and let

$$\boldsymbol{R}^{2(r+2)} = U \oplus V \oplus \hat{Z} \oplus W ,$$
$$Y = U \oplus V \oplus Z$$

be the orthogonal direct sums. In the same way as (b), we get

$$F = r^4 - 2F_0$$

where

$$F_0 = 4((u, v) - z_0w_1 + z_1w_0)^2 + (||u||^2 - ||v||^2 - 2(\zeta, w))^2 .$$

We put

$$\xi_\lambda = u_0^{(\lambda)} + \sqrt{-1}\, v_0^{(\lambda)} \quad \text{for} \quad \lambda = 1, \cdots, r ,$$
$$\xi_{r+1} = \frac{1}{\sqrt{2}}\{(z_1 - w_1) + \sqrt{-1}(z_0 + w_0)\} ,$$
$$\xi_{r+2} = \frac{1}{\sqrt{2}}\{(-z_0 + w_0) + \sqrt{-1}(z_1 + w_1)\} .$$

Then we have

$$\sum_{i=1}^{r+2} \xi_i^2 = (||u||^2 - ||v||^2 - 2(\zeta, w)) + 2\sqrt{-1}((u, v) - z_0w_1 + z_1w_0) .$$

Thus we have

$$F_0 = \left\| \sum_{i=1}^{r+1} \xi_i^2 \right\|^2,$$

which shows (a) of (ii). q.e.d.

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