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GLOBAL ANALYTIC-HYPOELLIPTICITY OF THE $\bar{\partial}$-NEUMANN PROBLEM

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Introduction. The (real-)analytic behavior (near the boundary) of solutions of the so-called $\bar{\partial}$-Neumann problem seems to have been unknown. In this paper we show that the global analytic-hypoellipticity (up to the boundary) holds on certain domains in $\boldsymbol{C}^n$ with analytic boundaries.

A systematic study of the $\bar{\partial}$-Neumann problem was made by Kohn [3], and the most difficult part of his work was the proof of the C^∞ hypoellipticity (up to the boundary). Soon after, Kohn and Nirenberg [5] gave an elegant proof of the C^∞ hypoellipticity by establishing the so-called subelliptic estimate. Their method is today used for various problems as the standard technique. However, it seems difficult, even if possible, to deduce the analytic-hypoellipticity of the $\bar{\partial}$-Neumann problem from the subelliptic estimate.

Under these circumstances we introduce in Lemma 2 a certain special vector field tangential along the boundary, which can be constructed in the case the Levi form is non-degenerate. It possesses the properties nice enough to carry out the commutator estimates (Lemmas 4 and 5), and these estimates together with the a priori estimate (Lemma 1) lead us in the usual way (see, e.g., Morrey and Nirenberg [6]) to our result. Our a priori estimate is suggested by a paper of Kohn [4].

It should be mentioned that the local problem still remains unsolved, and our method may not be applicable.

1. Statement of the theorem. Let $M \subset \boldsymbol{C}^n$ be a bounded domain whose boundary bM is regularly embedded in $\boldsymbol{C}^n$ with real codimension one. In all that follows we shall assume that the standard hermitian metric is given in $\boldsymbol{C}^n$ and that bM is analytic.

Let r denote the geodesic distance to bM measured as positive outside M and negative inside M, and normalized so that $|dr|^2 = 2$ near bM, where $|\cdot|$ is the length defined by the metric in $\boldsymbol{C}^n$. With a sufficiently small constant $\rho > 0$, we denote by Ω'_ρ the tubular neighborhood $bM \times (-\rho, \rho)$, i.e., $\{\mathrm{P} \in \boldsymbol{C}^n; -\rho < r(\mathrm{P}) < \rho\}$, and we set $\Omega_\rho = \bar{M} \cap \Omega'_\rho$, where $\bar{M}$

is the closure $M \cup bM$ of M. By T_t we denote the subbundle of the complexified tangent bundle $\boldsymbol{C}T$ over Ω'_ρ consisting of all vectors X such that $\langle dr, X\rangle = 0$, where $\langle\cdot, \cdot\rangle$ is the duality between covectors and vectors. Letting $T^{1,0} \subset \boldsymbol{C}T$ be the space of vectors of type (1, 0), we set $T_t^{1,0} = T^{1,0} \cap T_t$. Then the *Levi form* at $\mathrm{P} \in \Omega'_\rho$ is defined as the hermitian form given by

$$(T_t^{1,0})_{\mathrm{P}} \times (T_t^{1,0})_{\mathrm{P}} \ni (X_1, X_2) \mapsto \langle \partial\bar{\partial} r, X_1 \wedge \bar{X}_2\rangle ,$$

where $(T_t^{1,0})_{\mathrm{P}}$ denotes the fibre of the vector bundle $T_t^{1,0}$ over P, and $\bar{X}_2$ the complex conjugate of the vector X_2.

Let $\mathscr{A}^{p,q}$ denote the space of forms of type (p, q) on $\bar{M}$ having C^∞ extensions to $\boldsymbol{C}^n$ across the boundary bM. For $\varphi, \psi \in \mathscr{A}^{p,q}$ the L^2-inner product and norm are defined by

$$(\varphi, \psi) = \int_M \langle \varphi, \psi\rangle dV \quad \text{and} \quad \|\varphi\|^2 = (\varphi, \varphi) ,$$

respectively, where $\langle\cdot, \cdot\rangle$ is the pointwise inner product, and dV the volume form on M. The completion of $\mathscr{A}^{p,q}$ under the norm $\|\cdot\|$ is denoted by $\tilde{\mathscr{A}}^{p,q}$. For the Cauchy-Riemann operator $\bar{\partial}: \mathscr{A}^{p,q-1} \to \mathscr{A}^{p,q}$, its formal adjoint $\vartheta: \mathscr{A}^{p,q} \to \mathscr{A}^{p,q-1}$ is defined by the requirement that $(\vartheta\varphi, \psi) = (\varphi, \bar{\partial}\psi)$ for all $\psi \in \mathscr{A}^{p,q-1}$ with compact supports in M. Now for a differential operator D, we denote by $\sigma(D, dr)$ its principal symbol at dr. Then integration by parts gives us

$$(\vartheta\varphi, \psi) = (\varphi, \bar{\partial}\psi) + \int_{bM} \langle \sigma(\vartheta, dr)\varphi, \psi\rangle dS ,$$

for all $\varphi \in \mathscr{A}^{p,q}$ and $\psi \in \mathscr{A}^{p,q-1}$, where dS denotes the volume form on bM defined by the induced metric and normalized so as to avoid the annoying constant. We set

$$\mathscr{D}^{p,q} = \{\varphi \in \mathscr{A}^{p,q};\ \sigma(\vartheta, dr)\varphi = 0 \quad \text{on} \quad bM\} ,$$

and define the quadratic form $Q(\cdot, \cdot)$ on $\mathscr{D}^{p,q}$ by

$$Q(\varphi, \psi) = (\bar{\partial}\varphi, \bar{\partial}\psi) + (\vartheta\varphi, \vartheta\psi) + (\varphi, \psi) , \qquad \varphi, \psi \in \mathscr{D}^{p,q} .$$

By $\tilde{\mathscr{D}}^{p,q}$ we denote the completion of $\mathscr{D}^{p,q}$ under the norm $Q(\varphi, \varphi)^{1/2}$.

Consider the following variational problem: Given $\lambda \in \boldsymbol{C}$ and $\alpha \in \tilde{\mathscr{A}}^{p,q}$ with $q > 0$, find $\varphi \in \tilde{\mathscr{D}}^{p,q}$ such that

$$Q(\varphi, \psi) + (\lambda\varphi, \psi) = (\alpha, \psi) \qquad \text{for all} \quad \psi \in \mathscr{D}^{p,q} . \tag{1}$$

Now the purpose of this paper is to prove the following theorem.

THEOREM. *If the Levi form is non-degenerate and does not have* *exactly q negative eigenvalues in Ω'_ρ, then every solution φ of the equation* (1) *is analytic in Ω_ρ whenever α is analytic there.*

In all that follows we shall assume that all forms and functions we consider are of class C^∞ in Ω_ρ, for it has been shown (see, e.g., [1]) that solutions φ of the equation (1) are of class C^∞ in Ω_ρ under the hypothesis of the above theorem.

2. **Preliminaries.** Let $\mathscr{A}_\rho^{p,q}$ denote the subspace of $\mathscr{A}^{p,q}$ whose elements have compact supports in Ω_ρ, and let $\mathscr{D}_\rho^{p,q} = \mathscr{A}_\rho^{p,q} \cap \mathscr{D}^{p,q}$. Then we see that $\varphi \in \mathscr{D}_\rho^{p,q}$ if and only if $\varphi \in \mathscr{A}_\rho^{p,q}$ and $\sigma(\vartheta, dr)\varphi = 0$ on bM. Recall that the principal symbols of the operators $\bar\partial$ and ϑ at dr are given by $\sigma(\bar\partial, dr)\varphi = \bar\partial r \wedge \varphi$ and $\sigma(\vartheta, dr)\varphi = -\bar\partial r \vee \varphi$, respectively, where $\vee$ is the contraction operation defined by $\langle \eta \vee \omega, \theta\rangle = \langle \omega, \eta \wedge \theta\rangle$. Then setting $\bar n = \sigma(-\bar\partial\vartheta, dr)$, we have by the formula of composition that

$$\mathscr{D}_\rho^{p,q} = \{\varphi \in \mathscr{A}_\rho^{p,q};\ \bar n\varphi = 0 \quad \text{on} \quad bM\}\ . \tag{2}$$

It is easily seen that the operator $\bar n\colon \mathscr{A}_\rho^{p,q} \to \mathscr{A}_\rho^{p,q}$ is an orthogonal projection with respect to the inner product $\langle\cdot, \cdot\rangle$.

Let $\Gamma(\Omega'_\rho, E)$ denote the space of C^∞ sections of the vector bundle E over Ω'_ρ, and let $\nabla_X\colon \mathscr{A}_\rho^{p,q} \to \mathscr{A}_\rho^{p,q}$ be the (complex) covariant differentiation along $X \in \Gamma(\Omega'_\rho, CT)$. We define a connection $\tilde\nabla$ on $\mathscr{A}_\rho^{p,q}$ by

$$\tilde\nabla_X = \bar n\nabla_X\bar n + (1-\bar n)\nabla_X(1-\bar n)\ , \qquad X \in \Gamma(\Omega'_\rho, CT)\ .$$

From (2) we see that the operator $\tilde\nabla_X$ maps $\mathscr{D}_\rho^{p,q}$ into itself whenever $X \in \Gamma(\Omega'_\rho, T_t)$. The following formula of integration by parts holds:

$$(\tilde\nabla_X\varphi, \psi) = (\varphi, -(\tilde\nabla_{\bar X} + \operatorname{div}\bar X)\psi) + \int_{bM}\langle dr, X\rangle\langle\varphi, \psi\rangle dS\ , \tag{3}$$

for $X \in \Gamma(\Omega'_\rho, CT)$ and $\varphi, \psi \in \mathscr{A}_\rho^{p,q}$, where $\operatorname{div}\bar X$ denotes the divergence of the vector field $\bar X$. Denoting by $[\cdot, \cdot]$ the commutation operation, and by $\tilde R$ the curvature tensor associated to the connection $\tilde\nabla$, one has

$$[\tilde\nabla_{X_1}, \tilde\nabla_{X_2}] = \tilde\nabla_{[X_1, X_2]} + \tilde R(X_1, X_2)\ , \quad X_1, X_2 \in \Gamma(\Omega'_\rho, CT)\ . \tag{4}$$

Recall that for $\theta, \varphi \in \sum_{p,q}\mathscr{A}_\rho^{p,q}$,

$$\tilde\nabla_X(\theta \wedge \varphi) = \theta \wedge \tilde\nabla_X\varphi + \tilde\nabla_X\theta \wedge \varphi\ , \quad \tilde\nabla_X(\theta \vee \varphi) = \theta \vee \tilde\nabla_X\varphi + \tilde\nabla_{\bar X}\theta \vee \varphi\ . \tag{5}$$

We also employ the local expressions. Let R denote the dual vector field of ∂r and let $T^{*1,0}$ be the space of covectors of type (1, 0). For $\mathrm{P} \in bM$ and $\varepsilon > 0$ we denote by $V(\mathrm{P}; \varepsilon)$ the ε-neighborhood of P in bM.

DEFINITION. An open set $U = V(\mathrm{P}; \varepsilon) \times (-\rho, 0] \subset \Omega_\rho$ with $\mathrm{P} \in bM$ and

$\varepsilon > 0$ is called a *boundary chart* (*b-chart* for short) if an analytic orthonormal basis $(L_1, \cdots, L_n)$ of $\Gamma(U', T^{1,0})$ with $L_n = R$ can be chosen on $U' = V(\mathrm{P}; 2\varepsilon) \times (-\rho, \rho)$. A *b-frame* (L_i) on a b-chart U is the restriction to U of this basis on U', and a *b-coframe* $(\omega^1, \cdots, \omega^n)$ on U is the basis of $\Gamma(U, T^{*1,0})$ dual to some b-frame on U.

Since bM is compact and ρ is sufficiently small, Ω_ρ is covered by a finite number of b-charts.

Letting (L_i) be a b-frame on a b-chart U and (ω^i) be the dual b-coframe of (L_i), one has on U the following local expressions

$$(6) \qquad \bar\partial\varphi = \sum_{i=1}^{n} \bar\omega^i \wedge (\tilde\nabla_{\bar L_i} + \tilde S_{\bar i})\varphi\ , \quad \vartheta\varphi = -\sum_{i=1}^{n} \bar\omega^i \vee (\tilde\nabla_{L_i} + \tilde S_i)\varphi\ ,$$

for $\varphi \in \mathscr{A}_\rho^{p,q}$, where $\tilde S_{\bar i}$ and $\tilde S_i$ are operators of order zero with analytic coefficients defined on the open set U' given in the above definition. Now if we set for a b-frame (L_i) that

$$(7) \qquad \lambda_{i\bar j} = \langle \partial\bar\partial r, L_i \wedge \bar L_j \rangle\ , \qquad 1 \leqq i, j \leqq n\ ,$$

then from the fact $\langle \partial r, L_i \rangle = \delta_i^n$ one can easily verify that

$$(8) \qquad \langle \partial r, [L_i, \bar L_j] \rangle = \lambda_{i\bar j}\ , \quad \langle \partial r, [L_i, L_j] \rangle = 0\ .$$

In view of the fact that $\lambda_{i\bar j}$ with $1 \leqq i, j \leqq n-1$ represent the matrix coefficients of the Levi form, we define the *trace* of the Levi form by $\mathrm{tr}\,(\mathrm{L}) = \sum_{i=1}^{n-1} \lambda_{i\bar i}$, which has an analytic extension to Ω'_ρ.

Letting (L_i) be a b-frame, we set for $\varphi, \psi \in \mathscr{A}_\rho^{p,q}$,

$$(\varphi, \psi)_z = \int_M \sum_{i=1}^{n} \langle \tilde\nabla_{L_i}\varphi, \tilde\nabla_{L_i}\psi \rangle dV\ , \quad (\varphi, \psi)_{z,t} = \int_M \sum_{i=1}^{n-1} \langle \tilde\nabla_{L_i}\varphi, \tilde\nabla_{L_i}\psi \rangle dV\ ,$$

which are well-defined since the integrands are independent of the choice of the b-frame. Replacing L_i by $\bar L_i$ we define $(\varphi, \psi)_{\bar z}$ and $(\varphi, \psi)_{\bar z,t}$ similarly. Finally we define $\|\varphi\|_z$, $\|\varphi\|_{\bar z}$, $\|\varphi\|_{z,t}$ and $\|\varphi\|_{\bar z,t}$ by $\|\varphi\|_z^2 = (\varphi, \varphi)_z$, and so on. Then in view of (4) and (8), we can verify by (3) that there exists a constant $C > 0$ such that for all $\varphi \in \mathscr{A}_\rho^{p,q}$,

$$(9) \qquad \left| \|\varphi\|_{z,t}^2 - \|\varphi\|_{\bar z,t}^2 - \int_{bM} \mathrm{tr}\,(\mathrm{L})|\varphi|^2 dS \right| \leqq C(\|\varphi\|_{\bar z} + \|\varphi\|)\|\varphi\|\ .$$

Similar calculation gives us for $\varphi \in \mathscr{A}_\rho^{p,q}$ vanishing on bM,

$$(10) \qquad | \|\tilde\nabla_R\varphi\|^2 - \|\tilde\nabla_{\bar R}\varphi\|^2 | \leqq C(\|\varphi\|_{\bar z} + \|\varphi\|)\|\varphi\|\ .$$

Now we define a norm $N(\cdot)$ on $\mathscr{A}_\rho^{p,q}$ as follows:

$$N(\varphi)^2 = \|\varphi\|_{\bar z}^2 + \|\varphi\|_{z,t}^2 + \|\varphi\|^2\ , \qquad \varphi \in \mathscr{A}_\rho^{p,q}\ .$$

Since the Levi form is non-degenerate on Ω'_ρ, one can verify by (8) that for each $X \in \Gamma(\Omega'_\rho, CT)$ there exists a constant $C_X > 0$ such that

$$|(\tilde{\nabla}_X \varphi, \psi)| \leqq C_X N(\varphi) N(\psi) \qquad \text{for all } \varphi, \psi \in \mathscr{A}_\rho^{p,q} . \tag{11}$$

3. A priori estimate and a special vector field. We say that the *basic estimate* holds in $\mathscr{D}^{p,q}$ if for some constant $C > 0$,

$$\int_{bM} |\varphi|^2 dS \leqq CQ(\varphi, \varphi) \qquad \text{for all } \varphi \in \mathscr{D}^{p,q} .$$

Recall (see [2]) that *the basic estimate holds in $\mathscr{D}^{p,q}$ if and only if the Levi form has either at least $n-q$ positive or at least $q+1$ negative eigenvalues at every point of bM.* Then it follows from the assumption that the basic estimate holds in $\mathscr{D}^{p,q}$ in the present case.

Now one has the following a priori estimate.

LEMMA 1. *If the basic estimate holds in $\mathscr{D}^{p,q}$, then there exists a constant $C > 0$ such that*

$$C^{-1} N(\varphi)^2 \leqq Q(\varphi, \varphi) \leqq CN(\varphi)^2 \qquad \textit{for all } \varphi \in \mathscr{D}_\rho^{p,q} .$$

PROOF. Since $-\bar{\partial} r \vee \varphi = \sigma(\vartheta, dr)\varphi = 0$ on bM, it follows from (5) and (10) that $\|\bar{\partial} r \vee \tilde{\nabla}_R \varphi\| \leqq CN(\varphi)$, which implies in view of (6) that $Q(\varphi, \varphi) \leqq CN(\varphi)^2$. Now it is well-known (see, e.g., [1]) that if the basic estimate holds in $\mathscr{D}^{p,q}$ then for some $C > 0$,

$$\|\varphi\|_{\frac{1}{2}}^2 + \|\varphi\|^2 + \int_{bM} |\varphi|^2 dS \leqq CQ(\varphi, \varphi) \qquad \text{for all } \varphi \in \mathscr{D}_\rho^{p,q} .$$

Therefore, the estimate $N(\varphi)^2 \leqq CQ(\varphi, \varphi)$ follows from (9) and the above inequality. q.e.d.

Our a priori estimate is weaker than the so-called Gårding's inequality. To cover it up we construct in the following lemma a certain special vector field Y, which will play an essential role in our commutator estimates in the next section.

LEMMA 2. *Suppose that the Levi form is non-degenerate in Ω'_ρ. If ρ is sufficiently small, then there exists an analytic vector field $Y \in \Gamma(\Omega'_\rho, T_t)$ with $\bar{Y} = -Y$ such that*

$$\langle \partial r, [X, Y] \rangle = 0 \quad \textit{in } \Omega'_\rho \qquad \textit{for all } X \in \Gamma(\Omega'_\rho, T_t^{1,0} \oplus T_t^{0,1}) , \tag{12}$$

$$\langle \partial r, [\bar{R}, Y] \rangle = 0 \textit{ on } bM , \qquad \langle \partial r, Y \rangle = 1 \textit{ on } bM , \tag{13}$$

where $T_t^{0,1}$ denotes the subbundle of T_t consisting of vectors of type (0, 1).

PROOF. We first note that the condition (12) can be rewritten in terms of b-frame as follows: For every b-frame (L_i) on each b-chart U,

$$(14)\qquad \langle \partial r, [L_i, Y]\rangle = \langle \partial r, [\bar{L}_i, Y]\rangle = 0 \qquad \text{in } U \text{ for } i \leqq n-1\,.$$

Suppose that $Y \in \Gamma(\Omega'_\rho, T_t)$ is expressed on U as

$$(15)\qquad Y = u(R - \bar{R}) + \sum_{j=1}^{n-1} v^j L_j - \sum_{j=1}^{n-1} \bar{w}^j \bar{L}_j\,,$$

with unknown functions u, v^j and w^j. Then by (8) we see that the condition (14) is satisfied if and only if

$$(16)\qquad v^j = \sum_{i=1}^{n-1} \lambda^{j\bar{i}}(\bar{L}_i - \lambda_{n\bar{i}})u\,, \quad \bar{w}^j = \sum_{i=1}^{n-1} \lambda^{i\bar{j}}(L_i - \lambda_{i\bar{n}})u\,,$$

where $\lambda_{i\bar{j}}$ are given in (7), and $\lambda^{i\bar{j}}$ with $i, j \leqq n-1$ are defined by $\sum_{j=1}^{n-1} \lambda_{k\bar{j}}\lambda^{i\bar{j}} = \delta^i_k$. Now if v^j and w^j are defined by (16), then the condition (13) is fulfilled if and only if u satisfies

$$(17)\qquad Pu = 0 \text{ on } bM \qquad \text{and} \qquad u = 1 \text{ on } bM\,,$$

where P is a differential operator defined globally on Ω'_ρ by

$$P = \bar{R} - \lambda_{n\bar{n}} - \sum_{i,j=1}^{n-1} \lambda_{j\bar{n}}\lambda^{j\bar{i}}(\bar{L}_i - \lambda_{n\bar{i}})\,.$$

If u is real-valued, then from (15) and (16) it follows that $\bar{Y} = -Y$. Thus it suffices to construct a real-valued analytic function u on Ω'_ρ satisfying (17). Now denoting by $\bar{P}$ the complex conjugate of the differential operator P, we consider the following initial value problem:

$$(18)\qquad (P + \bar{P})u = 0 \quad \text{in} \quad \Omega_\rho\,, \qquad u = 1 \quad \text{on} \quad bM\,.$$

Since $\sigma(P + \bar{P}, dr) = \langle dr, R + \bar{R}\rangle = 2$, the initial surface bM is nowhere characteristic with respect to the operator $P + \bar{P}$. It then follows by virtue of the Cauchy-Kowalewski theorem that there exists a real-valued solution u of the problem (18) having an analytic extension to Ω'_ρ provided ρ is small enough. Meanwhile, from the definition of the operator P we see that the operator $P - \bar{P}$ consists of only first order terms and furthermore satisfies $\sigma(P - \bar{P}, dr) = \langle dr, \bar{R} - R\rangle = 0$. In view of the fact that $u = 1$ on bM, we obtain $(P - \bar{P})u = 0$ on bM, which implies together with (18) that this solution u satisfies (17). q.e.d.

4. Commutator estimates. We begin with some algebraic formulas.

LEMMA 3 (Leibniz' formula). *If $D_1, \cdots, D_m$ and B are linear differential operators, then*

$$(19)\qquad [D_m \cdots D_1, B] = \sum_{k=0}^{m-1} \sum_{\sigma \in (m,k)} (\mathrm{ad}\, D_{\sigma(m)} \cdots \mathrm{ad}\, D_{\sigma(k+1)}(B)) D_{\sigma(k)} \cdots D_{\sigma(1)}\,,$$

(20) $$[B, D_1 \cdots D_m] = \sum_{k=0}^{m-1} (-1)^{m-k} \sum_{\sigma \in (m,k)} D_{\sigma(1)} \cdots D_{\sigma(k)}(\operatorname{ad} D_{\sigma(k+1)} \cdots \operatorname{ad} D_{\sigma(m)}(B)) ,$$

where $\operatorname{ad} D$ *is defined by* $\operatorname{ad} D(B) = [D, B]$, *and* (m, k) *denotes the set of all* $\binom{m}{k}$ *permutations* σ *of* $1, \cdots, m$ *such that* $\sigma(1) < \cdots < \sigma(k)$ *and* $\sigma(k+1) < \cdots < \sigma(m)$.

PROOF. The proof of (19) is contained in [7, pp. 575–576], and (20) can be proved similarly. q.e.d.

Now let $X_1, \cdots, X_m$ be arbitrary complex vector fields on Ω'_ρ, θ be a 1-form on Ω'_ρ and $\tilde{B}: \mathscr{A}_\rho^{p,q} \to \mathscr{A}_\rho^{p,q}$ be a linear differential operator. Then in view of (5) we get by induction the following two formulas:

(21) $$(\operatorname{ad} \tilde{\nabla}_m \cdots \operatorname{ad} \tilde{\nabla}_1(\theta \wedge \tilde{B}))\varphi = \sum_{k=0}^{m} \sum_{\sigma \in (m,k)} (\tilde{\nabla}_{\sigma(k)} \cdots \tilde{\nabla}_{\sigma(1)}\theta) \wedge (\operatorname{ad} \tilde{\nabla}_{\sigma(m)} \cdots \operatorname{ad} \tilde{\nabla}_{\sigma(k+1)}(\tilde{B}))\varphi ,$$

(22) $$(\operatorname{ad} \tilde{\nabla}_m \cdots \operatorname{ad} \tilde{\nabla}_1(\theta \vee \tilde{B}))\varphi = \sum_{k=0}^{m} \sum_{\sigma \in (m,k)} (\tilde{\nabla}_{\overline{\sigma(k)}} \cdots \tilde{\nabla}_{\overline{\sigma(1)}}\theta) \vee (\operatorname{ad} \tilde{\nabla}_{\sigma(m)} \cdots \operatorname{ad} \tilde{\nabla}_{\sigma(k+1)}(\tilde{B}))\varphi ,$$

for all $\varphi \in \mathscr{A}_\rho^{p,q}$, where we use the abbreviated notations $\tilde{\nabla}_k = \tilde{\nabla}_{X_k}$ and $\tilde{\nabla}_{\bar{k}} = \tilde{\nabla}_{\bar{X}_k}$.

We shall need two commutator estimates, the first of which is the following.

LEMMA 4. *There exist constants* $C_0, C_1 > 0$ *such that for all* $\varphi \in \mathscr{D}_\rho^{p,q}$ *and all integers* $m \geqq 1$,

$$|Q(\tilde{\nabla}_Y^m\varphi, \tilde{\nabla}_Y^m\varphi) - Q(\varphi, \tilde{\nabla}_Y^{*m}\tilde{\nabla}_Y^m\varphi)| \leqq N(\tilde{\nabla}_Y^m\varphi)\sum_{k=0}^{m-1} C_0C_1^{m-k}\frac{m!}{k!}N(\tilde{\nabla}_Y^k\varphi) ,$$

where $\tilde{\nabla}_Y^*$ *denotes the formal adjoint* $(-\tilde{\nabla}_{\bar{Y}} - \operatorname{div} \bar{Y})$ *of* $\tilde{\nabla}_Y$.

PROOF. Since $\langle dr, Y \rangle = 0$, the formula (3) gives us

$$(\bar{\partial}\tilde{\nabla}_Y^m\varphi, \bar{\partial}\tilde{\nabla}_Y^m\varphi) - (\bar{\partial}\varphi, \bar{\partial}\tilde{\nabla}_Y^{*m}\tilde{\nabla}_Y^m\varphi) = ([\bar{\partial}, \tilde{\nabla}_Y^m]\varphi, \bar{\partial}\tilde{\nabla}_Y^m\varphi) + (\bar{\partial}\varphi, [\tilde{\nabla}_Y^{*m}, \bar{\partial}]\tilde{\nabla}_Y^m\varphi) .$$

From Lemma 1 we first get

$$|([\bar{\partial}, \tilde{\nabla}_Y^m]\varphi, \bar{\partial}\tilde{\nabla}_Y^m\varphi)| \leqq C_0N(\tilde{\nabla}_Y^m\varphi)\|[\bar{\partial}, \tilde{\nabla}_Y^m]\varphi\| .$$

Now if (L_i) is a b-frame on a b-chart U and (ω^i) is its dual b-coframe, then in view of the expression in (6) we have from (19) in Lemma 3 and (21) that on U,

$$[\bar{\partial}, \tilde{\nabla}_Y^m]\varphi = -\sum_{i=1}^{n} \sum_{\substack{j+k+l=m \\ l\neq m}} \frac{m!}{j!k!l!}(\tilde{\nabla}_Y^j \bar{\omega}^i) \wedge ((\mathrm{ad}\,\tilde{\nabla}_Y)^k(\tilde{\nabla}_{\bar{L}_i} + \tilde{S}_{\bar{i}}))\tilde{\nabla}_Y^l \varphi .$$

From (4) we see that the first order term of $(\mathrm{ad}\,\tilde{\nabla}_Y)^k(\tilde{\nabla}_{\bar{L}_i} + \tilde{S}_{\bar{i}})$ is $\tilde{\nabla}_X$ with $X = (\mathrm{ad}\,Y)^k(\bar{L}_i)$, thus by Lemma 2 we have $\langle \partial r, X\rangle = 0$ on bM. Since all quantities are analytic, we obtain in view of (10),

$$\|[\bar{\partial}, \tilde{\nabla}_Y^m]\varphi\| \leqq \sum_{l=0}^{m-1} C_0 C_1^{m-l} \frac{m!}{l!} N(\tilde{\nabla}_Y^l \varphi) .$$

Similarly, the formula (20) in Lemma 3 gives us

$$(\bar{\partial}\varphi, [\tilde{\nabla}_Y^{*m}, \bar{\partial}]\tilde{\nabla}_Y^m \varphi) = -\sum_{j=0}^{m-1} (-1)^{m-j} \frac{m!}{j!(m-j)!} (\tilde{\nabla}_Y^j \bar{\partial}\varphi, ((\mathrm{ad}\,\tilde{\nabla}_Y^*)^{m-j}\bar{\partial})\tilde{\nabla}_Y^m \varphi) .$$

Since $\tilde{\nabla}_Y^* = -\tilde{\nabla}_{\bar{Y}} - \mathrm{div}\,\bar{Y} = \tilde{\nabla}_Y + \mathrm{div}\,Y$, we have

$$\|((\mathrm{ad}\,\tilde{\nabla}_Y^*)^{m-j}\bar{\partial})\tilde{\nabla}_Y^m \varphi\| \leqq C_0 C_1^{m-j}(m-j)!\, N(\tilde{\nabla}_Y^m \varphi) ,$$

while from the fact that $\tilde{\nabla}_Y^j \bar{\partial} = \bar{\partial}\tilde{\nabla}_Y^j + [\tilde{\nabla}_Y^j, \bar{\partial}]$ we get

$$\|\tilde{\nabla}_Y^j \bar{\partial}\varphi\| \leqq \sum_{k=0}^{j} C_0 C_1^{j-k} \frac{j!}{k!} N(\tilde{\nabla}_Y^k \varphi) .$$

Therefore,

$$|(\bar{\partial}\varphi, [\tilde{\nabla}_Y^{*m}, \bar{\partial}]\tilde{\nabla}_Y^m \varphi)| \leqq N(\tilde{\nabla}_Y^m \varphi) \sum_{k=0}^{m-1} C_0^2 (2C_1)^{m-k} \frac{m!}{k!} N(\tilde{\nabla}_Y^k \varphi) .$$

Next we consider the terms for ϑ. Similarly to the case for $\bar{\partial}$, the term $[\vartheta, \tilde{\nabla}_Y^m]\varphi$ can be expanded by (22) into the sum of terms of the form

$$(\tilde{\nabla}_Y^j \bar{\omega}^i) \vee ((\mathrm{ad}\,\tilde{\nabla}_Y)^k(\tilde{\nabla}_{L_i} + \tilde{S}_i))\tilde{\nabla}_Y^l \varphi .$$

The same argument for $\bar{\partial}$ applies when $i \leqq n-1$. In the case $i = n$, if we notice that $(\tilde{\nabla}_Y^j \bar{\partial} r) \vee \tilde{\nabla}_Y^l \varphi = 0$ on bM, we can again use the inequality (10) to obtain

$$|([\vartheta, \tilde{\nabla}_Y^m]\varphi, \vartheta\tilde{\nabla}_Y^m \varphi)| \leqq N(\tilde{\nabla}_Y^m \varphi) \sum_{k=0}^{m-1} C_0 C_1^{m-k} \frac{m!}{k!} N(\tilde{\nabla}_Y^k \varphi) .$$

The term $(\vartheta\varphi, [\tilde{\nabla}_Y^{*m}, \vartheta]\tilde{\nabla}_Y^m \varphi)$ can be estimated similarly. q.e.d.

Now the Gram-Schmidt orthogonalization process gives us analytic vector fields $Z_1, \cdots, Z_{2n} \in \Gamma(\Omega'_\rho, T_t^{1,0} \oplus T_t^{0,1})$ which span $T_t^{1,0} \oplus T_t^{0,1}$ at every point of Ω'_ρ. Letting $|K| = l$ and $\tilde{\nabla}_Z^K = \tilde{\nabla}_{Z_{\kappa_1}} \cdots \tilde{\nabla}_{Z_{\kappa_l}}$ for an ordered multi-index $K = (\kappa_1, \cdots, \kappa_l)$ with $1 \leqq \kappa_i \leqq 2n$, we set

$$N(\varphi;\, l,\, m)=\frac{1}{(l+m)!}\max_{|K|=l} N(\tilde{\nabla}_Z^K\tilde{\nabla}_Y^m\varphi)\qquad \text{for}\quad \varphi\in\mathscr{A}_\rho^{p,q}\,.$$

Then our second commutator estimate can be stated as follows.

LEMMA 5. *There exist C_0, $C_1>0$ such that for all $\varphi\in\mathscr{D}_\rho^{p,q}$, integers $m\geqq 0$ and ordered multi-indices K with $|K|=l\geqq 1$,*

$$\begin{aligned}
&(l+m)!^{-2}|Q(\tilde{\nabla}_Z^K\tilde{\nabla}_Y^m\varphi,\, \tilde{\nabla}_Z^K\tilde{\nabla}_Y^m\varphi)-Q(\varphi,\, (\tilde{\nabla}_Z^K\tilde{\nabla}_Y^m)^*\tilde{\nabla}_Z^K\tilde{\nabla}_Y^m\varphi)|\\
&\quad\leqq C_0\Big(\sum_{j=0}^{l} C_1^{l-j}N(\varphi;\, j,\, m)+\sum_{j=1}^{l} C_1^{l-j}N(\varphi;\, j-1,\, m+1)+C_1^l\frac{1}{m!}\|\tilde{\nabla}_Y^{m+1}\varphi\|\Big)\\
&\qquad\cdot\Big(\sum_{\substack{j\leqq l,\, k\leqq m\\ j+k\neq l+m}} C_1^{l-j+m-k}N(\varphi;\, j,\, k)+\sum_{\substack{j\leqq l,\, k\leqq m\\ j\neq 0}} C_1^{l-j+m-k}N(\varphi;\, j-1,\, k+1)\\
&\qquad\qquad+\sum_{k=0}^{m} C_1^{l+m-k}\frac{1}{k!}\|\tilde{\nabla}_Y^{k+1}\varphi\|\Big)\,,
\end{aligned}$$

where $(\tilde{\nabla}_Z^K\tilde{\nabla}_Y^m)^*$ denotes the formal adjoint of $\tilde{\nabla}_Z^K\tilde{\nabla}_Y^m$.

PROOF. Similarly to the proof of Lemma 4, we get from (19) in Lemma 3,

$$\begin{aligned}
&(l+m)!^{-2}|([\bar{\partial},\, \tilde{\nabla}_Z^K\tilde{\nabla}_Y^m]\varphi,\, \bar{\partial}\tilde{\nabla}_Z^K\tilde{\nabla}_Y^m\varphi)|\\
&\quad\leqq C_0N(\varphi;\, l,\, m)\Big(\sum_{\substack{j\leqq l,\, k\leqq m\\ j+k\neq l+m}} C_1^{l-j+m-k}N(\varphi;\, j,\, k)+\sum_{k=0}^{m} C_1^{l+m-k}\frac{1}{k!}\|\tilde{\nabla}_Y^{k+1}\varphi\|\\
&\qquad+\sum_{k=0}^{m}\sum_{j=1}^{l-1} C_1^{l-j+m-k}\frac{(l-j)!}{(l+k)!}\sum_{\sigma\in(l,j)}\|\tilde{\nabla}_Y\tilde{\nabla}_{\kappa_{\sigma(j)}}\cdots\tilde{\nabla}_{\kappa_{\sigma(1)}}\tilde{\nabla}_Y^k\varphi\|\Big)\,,
\end{aligned}$$

where we abbreviate $\tilde{\nabla}_{Z_i}$ to $\tilde{\nabla}_i$. Taking the commutator between $\tilde{\nabla}_Y$ and $\tilde{\nabla}_{\sigma(j)}\cdots\tilde{\nabla}_{\sigma(1)}$, we get from (20) in Lemma 3,

$$\begin{aligned}
&\frac{(l-j)!}{(l+k)!}\sum_{\sigma\in(l,j)}\|\tilde{\nabla}_Y\tilde{\nabla}_{\kappa_{\sigma(j)}}\cdots\tilde{\nabla}_{\kappa_{\sigma(1)}}\tilde{\nabla}_Y^k\varphi\|\\
&\quad\leqq C_0\Big(N(\varphi;\, j-1,\, k+1)+\sum_{j'=0}^{j} C_1^{j-j'}N(\varphi;\, j',\, k)+C_1^j\frac{1}{k!}\|\tilde{\nabla}_Y^{k+1}\varphi\|\Big)\,.
\end{aligned}$$

Meanwhile, if we notice that $(\tilde{\nabla}_Z^K\tilde{\nabla}_Y^m)^*=\tilde{\nabla}_Y^{*m}\tilde{\nabla}_{\kappa_l}^*\cdots\tilde{\nabla}_{\kappa_1}^*$, then similar calculation gives us

$$\begin{aligned}
&(l+m)!^{-2}|(\bar{\partial}\varphi,\, [(\tilde{\nabla}_Z^K\tilde{\nabla}_Y^m)^*,\, \bar{\partial}]\tilde{\nabla}_Z^K\tilde{\nabla}_Y^m\varphi)|\\
&\quad\leqq N(\varphi;\, l,\, m)\sum_{k=0}^{m-1} C_0C_1^{m-k}\frac{1}{(l+k)!}\|\tilde{\nabla}_Z^K\tilde{\nabla}_Y^k\bar{\partial}\varphi\|\\
&\qquad+\Big(N(\varphi;\, l,\, m)+\frac{1}{(l+m)!}\|\tilde{\nabla}_Y\tilde{\nabla}_Z^K\tilde{\nabla}_Y^m\varphi\|\Big)
\end{aligned}$$

$$\cdot \sum_{k=0}^{m} \sum_{j=0}^{l-1} C_0 C_1^{l-j+m-k} \frac{(l-j)!}{(l+k)!} \sum_{\sigma \in (l,j)} \|\tilde{\nabla}_{\kappa_{\sigma(j)}} \cdots \tilde{\nabla}_{\kappa_{\sigma(1)}} \tilde{\nabla}_Y^k \bar{\partial}\varphi\|$$

$$\leqq N(\varphi; l, m) \sum_{k=0}^{m-1} C_0 C_1^{m-k} \Big(N(\varphi; l, k) + \frac{1}{(l+k)!} \| [\tilde{\nabla}_Z^K \tilde{\nabla}_Y^k, \bar{\partial}]\varphi \| \Big)$$

$$+ \Big(N(\varphi; l, m) + N(\varphi; l-1, m+1) + \frac{1}{(l+m)!} \| [\tilde{\nabla}_Y, \tilde{\nabla}_Z^K] \tilde{\nabla}_Y^m \varphi \| \Big)$$

$$\cdot \sum_{k=0}^{m} \sum_{j=0}^{l-1} C_0 C_1^{l-j+m-k} \Big(N(\varphi; j, k)$$

$$+ \frac{(l-j)!}{(l+k)!} \sum_{\sigma \in (l,j)} \| [\tilde{\nabla}_{\kappa_{\sigma(j)}} \cdots \tilde{\nabla}_{\kappa_{\sigma(1)}} \tilde{\nabla}_Y^k, \bar{\partial}]\varphi \| \Big) .$$

These commutators have been estimated, and we obtain the estimate for $\bar{\partial}$. Similar argument also applies for ϑ. q.e.d.

5. Proof of Theorem. With the lemmas established in the previous sections, we shall prove our theorem stated in Section 1.

We first refer to the fact (see, e.g., [1]) that the solution φ of the variational equation (1) satisfies, along with the so-called $\bar{\partial}$-*Neumann conditions*

$$\varphi \in \tilde{\mathscr{D}}^{p,q} , \qquad \bar{\partial}\varphi \in \tilde{\mathscr{D}}^{p,q+1} , \tag{23}$$

the second order differential equation

$$\Box\varphi + (1+\lambda)\varphi = \alpha , \tag{24}$$

where $\Box$ denotes the *complex Laplacian* $\bar{\partial}\vartheta + \vartheta\bar{\partial}$. Since the operator $\Box$ is of elliptic type and has analytic coefficients, the analyticity of φ in $\Omega_\rho - bM$ follows from that of α. Recalling that the boundary bM is nowhere characteristic with respect to the operator $\Box$, the analyticity of φ in a neighborhood of bM will be obtained by virtue of the Holmgren's theorem from that of the Cauchy data of φ on bM.

Now let $\zeta = \zeta(r)$ be a real-valued C^∞ function of r satisfying $\zeta(r) = 1$ for $r > -\rho/3$ and $\zeta(r) = 0$ for $r < -2\rho/3$. Recalling that

$$\|\psi\|_z + \|\psi\|_{\bar{z}} + \|\psi\| \leqq C(N(\psi) + \|\tilde{\nabla}_Y \psi\|) \quad \text{for all} \quad \psi \in \mathscr{A}_\rho^{p,q} ,$$

we see by the routine calculation that the analyticity of the Cauchy data of φ follows from the estimates of the rearranged form

$$N(\zeta\varphi; l, m) \leqq C_0 C_1^l C_2^m \qquad \text{for all} \quad l, m \geqq 0 . \tag{25}$$

Now we shall prove (25) by induction. We first show (25) in the case $l = 0$, then for $l > 0$. In the following, the letters B_0 and B_1 will be used to denote known positive constants, depending only on the given data, which may change from instance to instance, and the letters C_0, C_1 and C_2 constants which should be determined in the induction process.

PROOF OF (25) FOR $l=0$. From Lemma 1 we have

$$\begin{aligned} B_0^{-1}N(\tilde{\nabla}_Y^m\zeta\varphi)^2 &\leqq Q(\tilde{\nabla}_Y^m\zeta\varphi, \tilde{\nabla}_Y^m\zeta\varphi) \\ &= \{Q(\zeta\varphi, \tilde{\nabla}_Y^{*m}\tilde{\nabla}_Y^m\zeta\varphi) + (\lambda\zeta\varphi, \tilde{\nabla}_Y^{*m}\tilde{\nabla}_Y^m\zeta\varphi)\} - (\lambda\tilde{\nabla}_Y^m\zeta\varphi, \tilde{\nabla}_Y^m\zeta\varphi) \\ &\quad + \{Q(\tilde{\nabla}_Y^m\zeta\varphi, \tilde{\nabla}_Y^m\zeta\varphi) - Q(\zeta\varphi, \tilde{\nabla}_Y^{*m}\tilde{\nabla}_Y^m\zeta\varphi)\} . \end{aligned}$$

Recalling the fact $\tilde{\mathscr{D}}^{p,q}\cap\mathscr{A}^{p,q}=\mathscr{D}^{p,q}$ (see, e.g., [1]), we see by (2) that $\zeta\varphi$ satisfies the $\bar\partial$-Neumann conditions (23), or more precisely, satisfies $\zeta\varphi\in\mathscr{D}_\rho^{p,q}$ and $\bar\partial(\zeta\varphi)\in\mathscr{D}_\rho^{p,q+1}$, from which we have

$$\begin{aligned} Q(\zeta\varphi, \tilde{\nabla}_Y^{*m}\tilde{\nabla}_Y^m\zeta\varphi) + (\lambda\zeta\varphi, \tilde{\nabla}_Y^{*m}\tilde{\nabla}_Y^m\zeta\varphi) &= ((\Box+1+\lambda)\zeta\varphi, \tilde{\nabla}_Y^{*m}\tilde{\nabla}_Y^m\zeta\varphi) \\ &= (\tilde{\nabla}_Y^m(\Box+1+\lambda)\zeta\varphi, \tilde{\nabla}_Y^m\zeta\varphi) . \end{aligned}$$

Since φ is analytic in $\Omega_\rho - bM$ and so is α in Ω_ρ, we have from the equation (24) that

$$m!^{-2}|(\tilde{\nabla}_Y^m(\Box+1+\lambda)\zeta\varphi, \tilde{\nabla}_Y^m\zeta\varphi)| \leqq B_0B_1^mN(\zeta\varphi;\, 0,\, m) .$$

Meanwhile, from the inequality (11) we get

$$m!^{-2}|(\lambda\tilde{\nabla}_Y^m\zeta\varphi, \tilde{\nabla}_Y^m\zeta\varphi)| \leqq B_0N(\zeta\varphi;\, 0,\, m)N(\zeta\varphi;\, 0,\, m-1) .$$

Therefore, in view of Lemma 4 we obtain finally

$$N(\zeta\varphi;\, 0,\, m) \leqq B_0B_1^m + \sum_{k=0}^{m-1} B_0B_1^{m-k}N(\zeta\varphi;\, 0,\, k) ,$$

which imply (25) for $l=0$.

PROOF OF (25) FOR $l>0$. We proceed by induction on the pair (l, m). To show (25) for (l, m), we assume (25) for the pairs (j, k) with $j+k<l+m$, and with $j+k=l+m$ and $j<l$. Now letting K be an arbitrary ordered multi-index with $|K|=l$, we have

$$\begin{aligned} &B_0^{-1}N(\tilde{\nabla}_Z^K\tilde{\nabla}_Y^m\zeta\varphi)^2 \\ &\quad\leqq |(\tilde{\nabla}_Z^K\tilde{\nabla}_Y^m(\Box+1+\lambda)\zeta\varphi, \tilde{\nabla}_Z^K\tilde{\nabla}_Y^m\zeta\varphi)| + |(\lambda\tilde{\nabla}_Z^K\tilde{\nabla}_Y^m\zeta\varphi, \tilde{\nabla}_Z^K\tilde{\nabla}_Y^m\zeta\varphi)| \\ &\qquad + |Q(\tilde{\nabla}_Z^K\tilde{\nabla}_Y^m\zeta\varphi, \tilde{\nabla}_Z^K\tilde{\nabla}_Y^m\zeta\varphi) - Q(\zeta\varphi, (\tilde{\nabla}_Z^K\tilde{\nabla}_Y^m)^*\tilde{\nabla}_Z^K\tilde{\nabla}_Y^m\zeta\varphi)| . \end{aligned}$$

The sum of the first and the second terms on the right is dominated by

$$(l+m)!^2B_0(B_1^{l+m} + N(\zeta\varphi;\, l-1,\, m))N(\zeta\varphi;\, l-1,\, m) .$$

Then using Lemma 5, taking the maximum for $|K|=l$ and shifting $N(\zeta\varphi;\, l,\, m)$ to the left, we obtain finally

$$N(\zeta\varphi;\, l,\, m) \leqq B_0 B_1^{l+m} + \sum_{k=0}^{m} B_0 B_1^{l+m-k} \frac{1}{k!} \|\widetilde{\nabla}_Y^{k+1} \zeta\varphi\|$$
$$+ \sum_{\substack{j \leqq l,\, k \leqq m \\ j+k \neq l+m}} B_0 B_1^{l-j+m-k} N(\zeta\varphi;\, j,\, k) + \sum_{\substack{j \leqq l,\, k \leqq m \\ j \neq 0}} B_0 B_1^{l-j+m-k} N(\zeta\varphi;\, j-1,\, k+1)\, .$$

If we notice that

$$k!^{-1} \|\widetilde{\nabla}_Y^{k+1} \zeta\varphi\| \leqq B_0(k+1) N(\zeta\varphi;\, 0,\, k+1) \leqq B_0^2 B_1^k\, ,$$

then the induction hypothesis gives us

$$(C_0 C_1^l C_2^m)^{-1} N(\zeta\varphi;\, l,\, m) \leqq (B_0/C_0)(B_1/C_1)^l (B_1/C_2)^m$$
$$+ \sum_{j+k \neq 0} B_0 (B_1/C_1)^j (B_1/C_2)^k + \sum_{j,k} B_0 (C_2/C_1)(B_1/C_1)^j (B_1/C_2)^k\, ,$$

which indicates that (25) holds for the pair (l, m). This completes the proof.

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