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BOUNDARY GROUPS OF A FUCHSIAN GROUP OF THE SECOND KIND

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1. Let G be a kleinian group, $\Omega(G)$ the region of discontinuity of G and $\Lambda(G)$ the limit set of G. Let $D(\subset \Omega(G))$ be a simply connected invariant domain of a kleinian group G with more than two boundary points. Here D is not necessarily a component of $\Omega(G)$. Then there is a conformal bijection $h_D: U \to D$, where U denotes the upper half plane. The group $\Gamma_D = h_D^{-1} G h_D$ is clearly a fuchsian group, which we call the fuchsian equivalent of G corresponding to h_D. The isomorphism $\chi_D: \Gamma_D \to G$, which carries $\gamma_0 \in \Gamma_D$ into $\gamma = h_D \circ \gamma_0 \circ h_D^{-1} \in G$, is called the canonical isomorphism corresponding to h_D.

Let Γ be a non-elementary fuchsian group. We denote by $B(L, \Gamma)$ the Banach space consisting of all holomorphic bounded automorphic quadratic forms in the lower half plane L with respect to Γ. We associate with every $\psi \in B(L, \Gamma)$ a solution W_ψ of the equation $\{W, z\} \equiv (W''/W')' - (1/2)(W''/W')^2 = \psi$ in L such that $W_\psi(z) = \eta_1(z)/\eta_2(z)$, where η_1 and η_2 are holomorphic solutions of the equation $2\eta''(z) + \psi(z)\eta(z) = 0$ in L with the initial conditions $\eta_1 = \eta_2' = 1$, $\eta_1' = \eta_2 = 0$ at $z = -i$. Then W_ψ is a meromorphic function defined in L. For $\psi \in B(L, \Gamma)$ there is a homomorphism $\chi_\psi: \Gamma \to SL'$ satisfying $W_\psi \circ \gamma = \chi_\psi(\gamma) \circ W_\psi$ for $\gamma \in \Gamma$, where SL' denotes the group of all the Möbius transformations.

The Teichmüller space $T(\Gamma)$ of a non-elementary fuchsian group Γ is the set of those $\psi \in B(L, \Gamma)$ for which W_ψ is the restriction, to L, of a quasiconformal self-mapping w of the Riemann sphere $\hat{C}$ compatible with Γ, that is, of a quasiconformal self-mapping w of $\hat{C}$ such that $w\Gamma w^{-1} \subset SL'$.

Bers [2] proved that, for $\psi \in \partial T(\Gamma)$, W_ψ is conformal in L and $\chi_\psi(\Gamma) = W_\psi \Gamma W_\psi^{-1}$ is a kleinian group with an invariant component containing $W_\psi(L)$.

For $\psi \in \partial T(\Gamma)$, every group of the form $\chi_\psi(\Gamma)$ or its conjugate in SL' is called a boundary group of Γ. Also a point $\psi \in \partial T(\Gamma)$ is called a cusp if there is a hyperbolic element $\gamma \in \Gamma$ such that $\chi_\psi(\gamma)$ is parabolic.

Bers [2] also showed that most points of $\partial T(\Gamma)$ are not cusps and, furthermore, he investigated boundary groups of a fuchsian group of the first kind in detail and proved many important theorems, which we often refer to as Bers' theorems or as Bers' propositions with their numbering in [2].

The purpose of this paper is to discuss some properties of boundary groups of a fuchsian group of the second kind.

2. In this section, we state some of Bers' results in somewhat modified form as lemmas. These play important roles in the later discussions.

As to Propositions 1, 2 and 3 in Bers' paper [2], we can easily verify that we may assume Δ in those Propositions to be a simply connected invariant domain of the kleinian group G contained in $\Omega(G)$. In other words, Bers' Propositions 1, 2 and 3 hold also for a simply connected invariant domain Δ, which is not necessarily an invariant component of G. Therefore we have the following Lemmas 1, 2 and 3.

LEMMA 1. *Let D $(\subset \Omega(G))$ be a simply connected invariant domain of a kleinian group G with more than two boundary points, $h_D\colon U \to D$ a conformal bijection, Γ_D a fuchsian equivalent corresponding to h_D and $\chi_D\colon \Gamma_D \to G$ the canonical isomorphism. Then $h_D\colon U \to D$ can be extended to a mapping $\tilde{h}_D$ of the union of U with the set of non-elliptic fixed points of Γ_D onto the union of D with the set of non-elliptic fixed points of G in such a way that for every $\gamma_0 \in \Gamma_D$ and for every terminal arc $C_0 \subset U$ of γ_0, the restriction of $\tilde{h}_D$ to the closure $\hat{C}_0$ of C_0 is a homeomorphism of $\hat{C}_0$ onto the closure $\hat{C}$ of the terminal arc $C = h_D(C_0)$ of $\gamma = \chi_D(\gamma_0) \in G$.*

LEMMA 2. *Under the hypothesis of Lemma 1, if $\gamma_0 \in \Gamma_0$ is parabolic, so is $\gamma = \chi_D(\gamma_0)$.*

LEMMA 3. *Under the hypothesis of Lemma 1, assume that ζ_1 and ζ_2 are two distinct fixed points of elements of Γ_D with $\tilde{h}_D(\zeta_1) = \tilde{h}_D(\zeta_2)$. Then ζ_1 and ζ_2 are the fixed points of the same element $\gamma_0 \in \Gamma_D$ so that γ_0 is hyperbolic and $\chi_D(\gamma_0)$ is parabolic.*

It is easy to verify that the above lemmas imply the following.

LEMMA 4. *Under the hypothesis of Lemma 1, G is non-elementary if and only if Γ_D is non-elementary.*

The following lemma is well-known.

LEMMA 5. *Under the hypothesis of Lemma 1, assume that G is finitely generated and that Γ_D is of the first kind. Then D is a component of G.*

Before stating Lemma 6, we recall definitions of type and of signature of a fuchsian group.

Let Γ be a non-elementary finitely generated fuchsian group. Then $U/\Gamma = S$ is a Riemann surface and the covering map $U \to S$ is holomorphic. It is well known that S is obtained from a compact Riemann surface of genus g by removing n' points and m holes, where g, n' and m are finite. The mapping $U \to S$ is unramified except over finitely many points $P_1, P_2, \cdots, P_{n''}$, the mapping near a pre-image of P_j being ν_j-to-one. We arrange $\{P_j\}_{j=1}^{n''}$ so that $2 \leqq \nu_1 \leqq \nu_2 \leqq \cdots \leqq \nu_{n''}$ and we set $n = n' + n''$ and $\nu_j = \infty$, $j = n''+1, n''+2, \cdots, n$. Under these circumstances, we say that, if $m > 0$, Γ or U/Γ is of type $(g; n, m)$ and of signature $(g; \nu_1, \cdots, \nu_n; m)$ over U and that, if $m = 0$, Γ or U/Γ is of finite type (g, n) and of signature $(g; \nu_1, \cdots, \nu_n)$ over U. For a non-elementary finitely generated fuchsian group Γ, Γ is of finite type if and only if Γ is of the first kind. The following is due to Bers [2].

LEMMA 6. *Let Γ be a non-elementary finitely generated fuchsian group. Assume that Γ is of the first kind. Then Γ is of finite type (g, n) and of signature $(g; \nu_1, \cdots, \nu_n)$ over U if and only if Γ is generated by $2g + n$ "standard generators" $\alpha_1, \beta_1, \alpha_2, \cdots, \beta_g, \gamma_1, \cdots, \gamma_n$ with the defining relations $[\alpha_1, \beta_1]\circ[\alpha_2, \beta_2]\circ\cdots\circ[\alpha_g, \beta_g]\circ\gamma_1\circ\cdots\circ\gamma_n = id$ and $\gamma_j^{\nu_j} = id$ $(1 \leqq j \leqq n$ and $\nu_j < \infty)$, where $[\alpha, \beta] = \alpha\circ\beta\circ\alpha^{-1}\circ\beta^{-1}$ and $\alpha_1, \cdots, \beta_g$ are hyperbolic, γ_j is elliptic and of order ν_j if $\nu_j < \infty$ and γ_j is parabolic if $\nu_j = \infty$ and these γ_j $(1 \leqq j \leqq n)$ form a complete set of non-conjugate elliptic and parabolic elements of Γ. If Γ is of the second kind, then Γ is of type $(g; n, m)$ and of signature $(g; \nu_1, \cdots, \nu_n; m)$ over U if and only if Γ is generated by $2g + m + n$ "standard generators" $\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, \gamma_1, \cdots \gamma_n, \delta_1, \cdots, \delta_m$ with the defining relations $[\alpha_1, \beta_1]\circ[\alpha_2, \beta_2]\circ\cdots\circ[\alpha_g, \beta_g]\circ\gamma_1\circ\cdots\circ\gamma_n\circ\delta_1\circ\cdots\circ\delta_m = id$, $\gamma_j^{\nu_j} = id$ $(1 \leqq j \leqq n$ and $\nu_j < \infty)$. Here α_j, β_j are hyperbolic, γ_j with $\nu_j < \infty$ are elliptic and of order ν_j, γ_j with $\nu_j = \infty$ are parabolic and these γ_j $(1 \leqq j \leqq n)$ form a complete set of non-conjugate elliptic and parabolic elements of Γ, and δ_j $(1 \leqq j \leqq m)$ form a complete set of non-conjugate primitive hyperbolic elements of Γ whose axis do not intersect the axis of any other hyperbolic element of Γ except for the multiples of δ_j's.*

3. After preparing two more lemmas, we shall prove a theorem.

LEMMA 7. *Let G be a non-elementary finitely generated kleinian group with a simply connected invariant component Δ. Assume that there exists a simply connected invariant domain D of G, which is contained in Δ properly, and let $h_D\colon U \to D$ be a conformal bijection, Γ_D the fuchsian equivalent of G corresponding to h_D, and $\chi_D\colon \Gamma_D \to G$ the canonical isomorphism. Then G has a parabolic element γ such that*

$\chi_D^{-1}(\gamma)$ *is hyperbolic.*

PROOF. Let $h_\Delta: U \to \Delta$ be a conformal bijection, Γ_Δ the fuchsian equivalent of G corresponding to h_Δ and χ_Δ the canonical isomorphism corresponding to h_Δ. From Ahlfors' finiteness theorem, Δ/G is of finite type, so is U/Γ_Δ. Hence Γ_Δ is a finitely generated fuchsian group of the first kind. Clearly $h_\Delta^{-1}(D)$ is a simply connected invariant domain of Γ_Δ, which is contained in U properly. The map $h_\Delta^{-1}\circ h_D: U \to h_\Delta^{-1}(D)$ is a conformal bijection and we see that the fuchsian equivalent of Γ_Δ corresponding to $h_\Delta^{-1}\circ h_D$ is Γ_D and that the canonical isomorphism corresponding to $h_\Delta^{-1}\circ h_D$ is $\chi = \chi_\Delta^{-1}\circ\chi_D$. As D is not a component of G, we see from Lemma 5 that Γ_D is of the second kind. Obviously Γ_D is finitely generated, and Lemma 4 implies that Γ_D is non-elementary. Therefore Γ_D is a non-elementary finitely generated fuchsian group of the second kind. Now we may assume that Γ_Δ is of type (g, n) and of signature $(g; \nu_1, \cdots, \nu_n)$ over U. Then we see from Lemma 6 that Γ_Δ is generated by $2g+n$ "standard generators" $\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, \gamma_1, \cdots, \gamma_n$ with the defining relations $[\alpha_1, \beta_1]\circ[\alpha_2, \beta_2]\circ\cdots\circ[\alpha_g, \beta_g]\circ\gamma_1\circ\cdots\circ\gamma_n = id$, $\gamma_j^{\nu_j} = id$ $(1 \leqq j \leqq n, \nu_j < \infty)$, where α_j, β_j are hyperbolic and γ_j with $\nu_j = \infty$ are parabolic. Since $\chi = \chi_\Delta^{-1}\circ\chi_D: \Gamma_D \to \Gamma_\Delta$ is isomorphic, Γ_D is also generated by $2g+n$ generators $\alpha_j' = \chi^{-1}(\alpha_j)$, $\beta_j' = \chi^{-1}(\beta_j)$ $(1 \leqq j \leqq g)$, $\gamma_j' = \chi^{-1}(\gamma_j)$ $(1 \leqq j \leqq n)$ with the defining relations $[\alpha_1', \beta_1']\circ[\alpha_2', \beta_2']\circ\cdots\circ[\alpha_g', \beta_g']\circ\gamma_1'\circ\cdots\circ\gamma_n' = id$, $\gamma_j'^{\nu_j} = id$ $(1 \leqq j \leqq n, \nu_j < \infty)$. We see from Lemma 2 that α_j', β_j' $(1 \leqq j \leqq g)$ are hyperbolic. As χ is isomorphic and γ_j $(1 \leqq j \leqq n, \nu_j < \infty)$ form a complete set of non-conjugate elliptic elements of Γ_Δ, γ_j' $(1 \leqq j \leqq n, \nu_j < \infty)$ also form that of Γ_D. Similarly, if all γ_j' $(1 \leqq j \leqq n, \nu_j = \infty)$ are parabolic, these elements form a complete set of non-conjugate parabolic elements of Γ_D, and hence from Lemma 6, Γ_D is of the first kind. This contradiction shows that there exists a parabolic element $\gamma_j \in \Gamma_\Delta$ such that $\gamma_j' = \chi^{-1}(\gamma_j) \in \Gamma_D$ is hyperbolic. From Lemma 2, $\chi_D(\gamma_j') = \chi_D\circ\chi^{-1}(\gamma_j) = \chi_\Delta(\gamma_j)$ is parabolic. Thus we have our Lemma 7.

LEMMA 8. *Under the hypothesis of Lemma* 1, *assume that* G *is non-elementary finitely generated and has a non-invariant component* Δ_1. *Let* $G_{\Delta_1}(\subset G)$ *be the subgroup of* G *that leaves* Δ_1 *invariant. Then the index of* G_{Δ_1} *in* G *is infinite, and* G_{Δ_1} *contains a parabolic element* γ *such that* $\chi_D^{-1}(\gamma)$ *is hyperbolic, where* $\chi_D: \Gamma_D \to G$ *is the canonical isomorphism corresponding to a conformal bijection* $h_D: U \to D$ *and* Γ_D *is the fuchsian equivalent corresponding to* h_D.

PROOF. Clearly G has an invariant component containing the invariant domain D of G. In his paper [2], Bers proved Proposition 7 by using Propositions 2 and 5 (both in [2]). So, in a completely similar manner to that in [2], we can verify our Lemma 8 by using Lemma 2 in stead of Proposition 2 in [2].

Now we can prove the following.

THEOREM. *Let Γ be a non-elementary finitely generated fuchsian group of the second kind and let $\psi \in \partial T(\Gamma)$. Then $\chi_\psi(\Gamma)$ is a non-elementary finitely generated kleinian group with an invariant component Δ which contains $W_\psi(L)$ properly. Moreover, let $\psi \in \partial T(\Gamma)$ be not a cusp. Then the region $\Omega(\chi_\psi(\Gamma))$ of discontinuity of $\chi_\psi(\Gamma)$ is only one invariant component of $\chi_\psi(\Gamma)$ and is not simply connected.*

PROOF. It has been proved by Bers [2] that, for $\psi \in \partial T(\Gamma)$, $\chi_\psi(\Gamma)$ is a kleinian group with an invariant component Δ which contains $W_\psi(L)$. Since $W_\psi: L \rightarrow W_\psi(L)$ is conformal, $W_\psi(L)$ is a simply connected, $\chi_\psi(\Gamma)$-invariant domain and the fuchsian equivalent of $\chi_\psi(\Gamma)$ corresponding to W_ψ is $\Gamma = W_\psi^{-1}\chi_\psi(\Gamma)W_\psi$. As Γ is non-elementary and finitely generated, Lemma 4 implies that $\chi_\psi(\Gamma)$ is also non-elementary and finitely generated. Hence $\Delta/\chi_\psi(\Gamma)$ is of finite type. On the other hand, $W_\psi(L)/\chi_\psi(\Gamma)$ is conformally equivalent to L/Γ which is not of finite type. Therefore $W_\psi(L) \subsetneqq \Delta$. Now assume that Δ is simply connected. Then, $\chi_\psi(\Gamma)$ is a non-elementary finitely generated kleinian group with a simply connected invariant component Δ, and Δ contains a simply connected invariant domain $W_\psi(L)$ of $\chi_\psi(\Gamma)$ properly. Hence we see from Lemma 7 that $\psi \in \partial T(\Gamma)$ is a cusp. Therefore, if $\psi \in \partial T(\Gamma)$ is not a cusp, then Δ is not simply connected. In this case, from Accola's theorem [1], $\chi_\psi(\Gamma)$ has no invariant components of $\chi_\psi(\Gamma)$ other than Δ. From Lemma 8, for $\psi \in \partial T(\Gamma)$ which is not a cusp, $\chi_\psi(\Gamma)$ has no non-invariant components of $\chi_\psi(\Gamma)$. Thus, for $\psi \in \partial T(\Gamma)$ which is not a cusp, we have $\Delta = \Omega(\chi_\psi(\Gamma))$. The proof of Theorem is complete.

4. **Example.** According to a result of Maskit [3], a non-elementary finitely generated kleinian group G is a Schottky group if and only if G is free and every element of G other than the identity is loxodromic. Now let Γ be a non-elementary finitely generated fuchsian Schottky group. Then the above Maskit's result yields that, for $\psi \in \partial T(\Gamma)$ which is not a cusp, $\chi_\psi(\Gamma)$ is a Schottky group. Thus $\Omega(\chi_\psi(\Gamma))$ is only one invariant component of $\chi_\psi(\Gamma)$ and is not simply connected.

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