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ON SACKER-SELL'S THEOREM FOR A LINEAR SKEW PRODUCT FLOW

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1. **Introduction.** Following Hale [1; p. 145] we shall call a linear system of differential equations to be non-critical if it has no non-trivial solution which is bounded on $(-\infty, \infty)$.

It is clear that a periodic linear system (including autonomous case) is non-critical if and only if none of its characteristic exponents has zero real part. By this fact, it is also true that a periodic linear system has an exponential dichotomy if and only if it is non-critical.

The same assertion was not verified for almost periodic systems until recently Sacker and Sell [2] have proved it affirmatively. However, their proof is based on the facts from the algebraic topology and considerably complicated.

In this article, we shall present a simple proof for this fact (Theorem 2 below). Moreover, our first theorem (Theorem 1 below) says more and we can expect many useful applications.

2. **Skew product flow.** As was mentioned in the above, our main object is to give a simple proof of the following theorem due to Sacker and Sell [2].

THEOREM. *A non-critical linear skew product flow $\pi = (\phi, \sigma)$ on $X \times Y$ has an exponential dichotomy on T if X is a Banach space of finite dimension and if Y is compact and minimal.*

Let π be a flow on $X \times Y$ with phase group T, where X and Y are topological spaces. The flow π, or $\pi = (\phi, \sigma)$, is called to be a skew product flow on $X \times Y$ if there is a decomposition

$$\pi(x, y, t) = (\phi(x, y, t), \sigma(y, t)) , \quad x \in X , \quad y \in Y , \quad t \in T ,$$

in which σ is a flow on Y with the phase group T. A skew product flow $\pi = (\phi, \sigma)$ is called to be linear if X is a linear normed space and $\phi(x, y, t)$ is linear in x. In this case, the operator $\Phi(y, t)$ defined by

$$\Phi(y, t)x = \phi(x, y, t)$$

is a bounded, invertible linear operator of X into itself and continuous in (y, t), and we have

(1) $$\Phi(\sigma(y, t), s)\Phi(y, t) = \Phi(y, t + s) .$$

We shall call the linear skew product flow $\pi = (\phi, \sigma)$ to be non-critical if for any $y \in Y$, $\phi(x, y, t)$ is bounded on T only when $x = 0$. In the theorem above the minimality of Y means that

$$\overline{\{\sigma(y, t); t \in T\}} = Y \quad \text{for each} \quad y \in Y .$$

As was mentioned in [2], a linear system

$$\dot{x} = A(t)x$$

on R^n yields a linear skew product flow $\pi = (\phi, \sigma)$. Here, $X = R^n$, $T = R$ and Y is the hull $H(A)$ of $A(t)$, that is, $\overline{\{A_t; t \in R\}}$, where $A_t(s) = A(t + s)$ and the closure is in the sense of the compact-open-topology. Moreover,

$$\sigma(B, t) = B_t , \quad B \in H(A) ,$$

and $\phi(x, B, t)$ is the solution of the system

(2) $$\dot{x} = B(t)x$$

passing through x at $t = 0$.

Clearly the hull $H(A)$ is compact if and only if $A(t)$ is bounded and uniformly continuous on R, and it is minimal with respect to σ given in the above if $A(t)$ is almost periodic. Moreover, by the fact that π is non-critical, we shall mean that for every $B \in H(A)$ the system (2) (or B shortly) is non-critical in the sense of Hale.

REMARK 1. For a periodic case, it is obvious that every $B \in H(A)$ is non-critical if and only if A is non-critical. However, this assertion is not valid for almost periodic systems.

In the following, let X be a Banach space of finite dimension, let Y be a compact space and let $\pi = (\phi, \sigma)$ be a linear skew product flow on $X \times Y$ with phase group T which is R or αZ (for a real α).

For a $y \in Y$, $S(y)$ denotes a subset of X defined by

$$S(y) = \{x \in X; \|\phi(x, y, t)\| \quad \text{is bounded on} \quad T^+\} ,$$

where $T^+ = \{t \in T; t \geqq 0\}$. Clearly $S(y)$ is a linear subspace of X because of the linearity of ϕ.

The linear skew product flow $\pi = (\phi, \sigma)$ is said to have an exponential dichotomy on T^+ (or on T) at $y \in Y$, if we can select a subspace $U(y)$ of X which is complement to $S(y)$, that is,

$$X = S(y) \oplus U(y)$$

is a direct sum and if there exist two positive constants $K = K(y)$ and $\alpha = \alpha(y)$ such that

$$(3) \qquad \| \phi(x, y, t) \| \leqq K e^{-\alpha(t-s)} \| \phi(x, y, s) \| \quad (t \geqq s,\ x \in S(y))$$

and

$$(4) \qquad \| \phi(x, y, t) \| \leqq K e^{\alpha(t-s)} \| \phi(x, y, s) \| \quad (t \leqq s,\ x \in U(y)),$$

where t and s vary on T^+ (or on T). If π has an exponential dichotomy on T at y for every $y \in Y$ and the constants $K(y)$ and $\alpha(y)$ in the relations (3) and (4) can be chosen independently of $y \in Y$, then π is said to have an exponential dichotomy on T.

REMARK 2. It is clear that if a linear skew product flow $\pi = (\phi, \sigma)$ has an exponential dichotomy on T, then π is non-critical. In fact, let $\phi(x, y, t)$ be bounded on T. Then $x \in S(y)$. Therefore, by the relation (3) we have

$$\| \phi(x, y, s) \| \geqq \frac{1}{K} e^{\alpha(t-s)} \| \phi(x, y, t) \| \qquad (t \geqq s)$$

which implies that $\| \phi(x, y, s) \| \to \infty$ as $s \to -\infty$ if $x \neq 0$. From this it follows that $\phi(x, y, t)$ is bounded on T only when $x = 0$.

Thus, under the assumptions for X and Y given in theorem, π has an exponential dichotomy on T if and only if it is non-critical.

3. Exponential dichotomy on T^+. Our first theorem is the following.

THEOREM 1. *If the linear skew product flow $\pi = (\phi, \sigma)$ is non-critical, then it has an exponential dichotomy on T^+ at y for every $y \in Y$.*

By choosing any complementary space $U(y)$ to $S(y)$ we shall show the existence of constants K and α for which the relations (3) and (4) hold.

PROOF OF THE RELATION (4). First of all, we shall prove that

$$(5) \qquad \| \phi(x, y, t) \| \leqq K_1 \| \phi(x, y, s) \|$$

for some constant $K_1 > 0$, any s, $t \in T^+$, $s \geqq t$, and any $x \in U(y)$.

Suppose that this is not the case. Then there exist sequences $\{t_k\}$, $\{s_k\}$ and $\{x^k\}$ such that

$$s_k \geqq t_k \geqq 0\,, \quad x^k \in U(y)\,, \quad x^k \neq 0\,, \quad \| \phi(x^k, y, t_k) \| \geqq k \| \phi(x^k, y, s_k) \|\,.$$

By the linearity of $\phi(x, y, t)$, x^k can be assumed to belong to the unit sphere in $U(y)$. Since the unit sphere in a Banach space of the finite dimension is compact, we can assume that $\{x^k\}$ converges to an $x \in U(y)$,

$\|x\| = 1$.

Let σ_k be chosen so that $0 \leqq \sigma_k \leqq s_k$ and

$$(6) \qquad \|\phi(x^k, y, \sigma_k)\| = \max_{0 \leqq t \leqq s_k} \|\phi(x^k, y, t)\| .$$

Then by noting the relation (1), clearly we have

$$\|\phi(\xi^k, \eta^k, s_k - \sigma_k)\| = \frac{\|\phi(x^k, y, s_k)\|}{\|\phi(x^k, y, \sigma_k)\|} \leqq \frac{1}{k} ,$$

where

$$(7) \qquad \xi^k = \frac{\phi(x^k, y, \sigma_k)}{\|\phi(x^k, y, \sigma_k)\|} , \qquad \eta^k = \sigma(y, \sigma_k) ,$$

which implies that

$$\|\phi(\xi, \eta, s)\| = 0$$

if $(\xi^k, \eta^k, s_k - \sigma_k)$ converges to a (ξ, η, s). Since $\|\xi^k\| = 1$ and Y is compact, there arises a contradiction if s is finite, that is, if $\{s_k - \sigma_k\}$ contains a bounded subsequence. Thus $\{s_k - \sigma_k\}$ is divergent to ∞.

Next, suppose that $\{\sigma_k\}$ contains a bounded subsequence. This makes it possible to assume that

$$\|\phi(x^k, y, \sigma_k)\| \leqq K_0$$

for a constant K_0 and for all k, which shows

$$\|\phi(x^k, y, t)\| \leqq K_0 \quad \text{for all} \quad t,\ 0 \leqq t \leqq s_k$$

by (6). Since $\{s_k\}$ is divergent to ∞, we have

$$\|\phi(x, y, t)\| \leqq K_0 \quad \text{for all} \quad t \in T^+$$

by letting $k \to \infty$, and hence, $x \in S(y)$, which contradicts $x \in U(y)$, $\|x\| = 1$. Therefore $\{\sigma_k\}$ is divergent.

Thus, for the (ξ^k, η^k) given by (7), we have

$$\|\phi(\xi^k, \eta^k, t)\| = \frac{\|\phi(x^k, y, t + \sigma_k)\|}{\|\phi(x^k, y, \sigma_k)\|} \leqq 1$$

for all t, $-\sigma_k \leqq t \leqq s_k - \sigma_k$, which implies

$$\|\phi(\xi, \eta, t)\| \leqq 1$$

for all $t \in T$ and for a limit (ξ, η) of $\{\xi^k, \eta^k\}$. This contradicts the fact that π is non-critical. Thus the relation (5) is proved.

To prove the relation (4), it is sufficient to show the existence of a $\tau = \tau(\varepsilon) \in T^+$ for any given $\varepsilon > 0$ such that for any $t \in T^+$ and any $x \in U(y)$, there is an s, $t \leqq s \leqq t + \tau$, for which

$$(8) \qquad \| \phi(x, y, t) \| \leqq \varepsilon \| \phi(x, y, s) \| .$$

Then, from (5) and (8) we would have

$$\begin{aligned} \| \phi(x, y, t) \| &\leqq \varepsilon K_1 \| \phi(x, y, t + \tau) \| \\ &\leqq (\varepsilon K_1)^k \| \phi(x, y, t + k\tau) \| . \end{aligned}$$

Therefore, if we choose an ε and an integer k so that

$$0 < \varepsilon < \frac{1}{K_1} , \qquad t + (k + 1)\tau > s \geqq t + k\tau$$

for a given $s \geqq t$, then we have (4) by putting

$$K = \frac{1}{\varepsilon} , \qquad \alpha = - \frac{1}{\tau} \log (\varepsilon K_1) .$$

Suppose that for an $\varepsilon > 0$ we can not choose such a $\tau > 0$ as in the above. Then there are sequences $\{t_k\}$ and $\{x^k\}$ such that $t_k \in T^+$, $x^k \in U(y)$ and

$$\| \phi(x^k, y, t_k) \| \geqq \varepsilon \| \phi(x^k, y, s) \| \quad \text{for all} \quad s \in T^+ , \quad t_k \leqq s \leqq t_k + 2k .$$

By setting

$$(9) \qquad \bar{\xi}^k = \frac{\phi(x^k, y, t_k + k)}{\| \phi(x^k, y, t_k) \|} , \quad \bar{\eta}^k = \sigma(y, t_k + k) ,$$

we have

$$\| \phi(\bar{\xi}^k, \bar{\eta}^k, t) \| \leqq \frac{1}{\varepsilon} \quad \text{for all} \quad t \in T^+ , \quad -k \leqq t \leqq k .$$

On the other hand, since

$$\frac{1}{\varepsilon} \geqq \| \bar{\xi}^k \| \geqq \frac{1}{K_1} ,$$

we may assume that $(\bar{\xi}^k, \bar{\eta}^k)$ converges to a $(\bar{\xi}, \bar{\eta})$, $\| \bar{\xi} \| \neq 0$. Again $\phi(\bar{\xi}, \bar{\eta}, t)$ becomes to be bounded on T which yields a contradiction and completes the proof.

Proof of the relation (3). The relation (3) has been proved by Sacker and Sell [2], but we shall give a proof for the self-contained, which is slightly different from that in [2] and will be done in a similar manner to the proof for the relation (4). Moreover, the proof in the below also shows that the constants K and α can be chosen independently of each $y \in Y$ as was shown in [2].

First, assuming that the relation (5) does not hold for $t \geqq s$, $y \in Y$,

$x \in S(y)$, we shall select sequences $\{t_k\}$, $\{s_k\}$, $\{y^k\}$ and $\{x^k\}$ such that

$$t_k \geqq s_k \geqq 0 , \quad x^k \in S(y^k) , \quad x^k \neq 0 , \quad y^k \in Y ,$$
$$\| \phi(x^k, y^k, t_k) \| \geqq k \| \phi(x^k, y^k, s_k) \| .$$

Let $\sigma_k \geqq s_k$ be chosen so that

$$\| \phi(x^k, y^k, \sigma_k) \| \geqq \frac{1}{2} \sup_{t \geqq s_k} \| \phi(x^k, y^k, t) \| \tag{10}$$

instead of (6). Hence we have

$$\| \phi(\xi^k, \eta^k, s_k - \sigma_k) \| \leqq \frac{2}{k} ,$$

where replacing y with y^k, ξ^k and η^k are given by (7), which shows that $\{\sigma_k - s_k\}$ must be divergent to ∞. On the other hand, by (10),

$$\| \phi(\xi^k, \eta^k, t) \| \leqq 2 \quad \text{for} \quad t \in T , \quad s_k - \sigma_k \leqq t < \infty ,$$

which yields a contradiction since π is non-critical. Thus we have the relation (5) for $t \geqq s$, $y \in Y$ and $x \in S(y)$.

Now we shall prove the relation (3). As before, it is sufficient to show that we can not choose sequences $\{t_k\}$, $\{y^k\}$ and $\{x^k\}$ such that

$$t_k \in T^+ , \quad y^k \in Y , \quad x^k \in S(y^k) , \quad x^k \neq 0 ,$$

$\| \phi(x^k, y^k, t) \| \geqq \varepsilon \| \phi(x^k, y^k, t_k) \|$ for all $t \in T^+$, $t_k \leqq t \leqq t_k + 2k$. This will be done easily by noting

$$K_1 \geqq \| \phi(\bar{\xi}^k, \bar{\eta}^k, t) \| \geqq \varepsilon \quad \text{for all} \quad t, \ -k \leqq t \leqq k ,$$

where replacing y with y^k, $\bar{\xi}^k$ and $\bar{\eta}^k$ are given by (9).

4. **Exponential dichotomy on** T. We shall denote by $\Lambda^+(y)$ the positive limiting set of the motion $\sigma(y, t)$, namely, the set of the limiting points of $\sigma(y, \tau_k)$ for a divergent sequence $\{\tau_k\}$ of T^+.

Since Y is compact, Y is minimal if and only if

$$\Lambda^+(y) = Y \quad \text{for every} \quad y \in Y .$$

Therefore, the theorem mentioned in the Section 2 follows from a more general theorem:

THEOREM 2. *Suppose that the linear skew product flow on* $X \times Y$ *is non-critical. Then it has an exponential dichotomy on* T *if*

$$\bigcup_{y \in Y} \Lambda^+(y) = Y .$$

Moreover, owing to Theorem 1, Theorem 2 is an immediate consequence of the following two theorems. Theorem 4 is closely related to

Lemma 4 in [3].

THEOREM 3. *Let the linear skew product flow π be non-critical, and let Y_0 be a subset of Y such that π has an exponential dichotomy on T at y for every $y \in Y_0$.*

Then we can choose the constants K and α in the relations (3) *and* (4) *independently of each y in Y_0.*

PROOF. The relation (4) on T says that

$$\| \phi(x, y, t) \| \leqq K \| x \|$$

for all $t \leqq 0$ and all $x \in U(y)$. Conversely, if

$$x = x^1 + x^2 , \quad x^1 \in S(y) , \quad x^2 \in U(y) ,$$

then

$$\begin{aligned} \| \phi(x, y, t) \| &\geqq \| \phi(x^1, y, t) \| - \| \phi(x^2, y, t) \| \\ &\geqq \frac{1}{K} e^{-\alpha t} \| x^1 \| - K \| x^2 \| \end{aligned}$$

for all $t \leqq 0$. Therefore, for every $y \in Y_0$ the set $U(y)$ equals to the set

$$S^-(y) = \{x \in X; \phi(x, y, t) \text{ is bounded on } T^-\} ,$$

where $T^- = \{t \in T; t \leqq 0\}$.

On the other hand, by changing the sign of t the proof for the relation (3) in Theorem 1 verifies that the constants K and α in the relation (4) can be chosen independently of y for every $y \in Y$ if $U(y)$ is replaced by $S^-(y)$. Thus the proof of the theorem follows immediately.

THEOREM 4. *If the linear skew product flow on $X \times Y$ has an exponential dichotomy on T^+ at $y \in Y$, then it has an exponential dichotomy on T at every $z \in \Lambda^+(y)$.*

Before the proof will be given, we shall mention some facts.

Let $y \in Y$ be fixed, and consider a decomposition

$$X = S(y) \oplus U(y) .$$

Let $P_0(y)$ be a projection operator of X onto $S(y)$ along the space $U(y)$, that is, an idempotent operator with the properties

$$P_0(y)X = S(y) , \qquad (I - P_0(y))X = U(y) ,$$

where I denotes the identity operator on X.

LEMMA 1. *Let $P_0(y)$ be as above and put*

$$P(\sigma(y, t)) = \Phi(y, t)P_0(y)\Phi^{-1}(y, t) .$$

Then $P(\sigma(y, t))$ is a projection operator of X onto $S(\sigma(y, t))$.

PROOF. Clearly $P(\sigma(y, t))$ is idempotent. Therefore, it remains to prove that

$$P(\sigma(y, t))X = S(\sigma(y, t)) . \tag{11}$$

From the definition and the relation (1),

$$\begin{aligned} \Phi(\sigma(y, t), s)P(\sigma(y, t))X &= \Phi(y, t+s)P_0(y)[\Phi^{-1}(y, t)X] \\ &= \Phi(y, t+s)P_0(y)X = \Phi(y, t+s)S(y) . \end{aligned}$$

Hence the relation (11) follows immediately.

LEMMA 2. *The linear skew product flow $\pi = (\phi, \sigma)$ has an exponential dichotomy on T^+(or T) at $y \in Y$ if and only if every projection operator $P(y)$ of X onto $S(y)$ along $U(y)$, appeared in the relation* (4), *satisfies*

$$\begin{aligned} &\| \Phi(y, t)P(y)\Phi^{-1}(y, s) \| \leqq Ke^{-\alpha(t-s)} \qquad (t \geqq s) , \\ &\| \Phi(y, t)(I - P(y))\Phi^{-1}(y, s) \| \leqq Ke^{\alpha(t-s)} \qquad (s \geqq t) , \end{aligned} \tag{12}$$

for some positive constants K and α and for any t, $s \in T^+$ (or t, $s \in T$). Moreover, in the above we can replace "every" by "some".

This assertion can be verified in the same way as in the proofs of [4, 42D (p. 114) for T^+; 82F (p. 285) for T]. The latters are stated for the solutions of a linear system

$$\dot{x} = A(t)x \tag{13}$$

under the assumption

$$\sup_{t \in R} \int_t^{t+1} \| A(s) \| \, ds < \infty ,$$

which is required to guarantee that

$$\| X(t)X^{-1}(s) \| \leqq B(\tau) \quad \text{if} \quad |t - s| \leqq \tau , \tag{14}$$

where $B(\tau)$ is a constant associated with any fixed number $\tau > 0$ and $X(t)$ is a fundamental matrix of the system (13).

In our case, (14) corresponds to

$$\| \Phi(y, t) \| \leqq B(\tau)$$

for any $y \in Y$ and any $t \in T$, $-\tau \leqq t \leqq \tau$, and this relation can be proved as a simple consequence of the compactness of Y.

Now we are ready to prove Theorem 4.

PROOF OF THEOREM 4. Let $P(\sigma(y, t))$ be the one given in Lemma 1. By the relation (1) we have

$$\Phi(\sigma(y, \tau), t)P(\sigma(y, \tau))\Phi^{-1}(\sigma(y, \tau), s) = \Phi(y, t + \tau)P_0(y)\Phi^{-1}(y, \tau + s) ,$$

which shows that

$$\| \Phi(\sigma(y, \tau), t)P(\sigma(y, \tau))\Phi^{-1}(\sigma(y, \tau), s) \| \leqq Ke^{-\alpha(t-s)} \tag{15}$$

for all $t \geqq s \geqq -\tau$ by Lemma 2. Similarly we have

$$\| \Phi(\sigma(y, \tau), t)(I - P(\sigma(y, \tau)))\Phi^{-1}(\sigma(y, \tau), s) \| \leqq Ke^{\alpha(t-s)} \tag{16}$$

for all $s \geqq t \geqq -\tau$.

For any given $z \in \Lambda^+(y)$, choose a divergent sequence $\{\tau_k\}$, $\tau_k \in T^+$, so that $\sigma(y, \tau_k)$ converges to z. Since we have

$$\| P(\sigma(y, \tau)) \| \leqq K$$

by putting $t = s = \tau$ in the relation (12) and dim $X < \infty$, $\{\tau_k\}$ contains a subsequence $\{\tau_{k_j}\}$ for which $\{P(\sigma(y, \tau_{k_j}))\}$ converges to a limit, say $P(z)$. Clearly $P(z)$ is a projection operator.

Thus, from (15) and (16) we have

$$\| \Phi(z, t)P(z)\Phi^{-1}(z, s) \| \leqq Ke^{-\alpha(t-s)} \qquad (t \geqq s > -\infty)$$

and

$$\| \Phi(z, t)(I - P(z))\Phi^{-1}(z, s) \| \leqq Ke^{-\alpha(t-s)} \qquad (s \geqq t > -\infty) ,$$

respectively, which shows that the skew product flow $\pi = (\phi, \sigma)$ has an exponential dichotomy on T at z by using Lemma 2, again. This completes the proof.

Theorem 2 proves that there is a redundant in the statement of Favard's theorem [5, p. 88].

THEOREM 5. *A non-homogeneous linear almost periodic system*

$$\dot{x} = A(t)x + f(t) \tag{17}$$

has a unique almost periodic solution, if the homogeneous linear system

$$\dot{x} = B(t)x$$

is non-critical for any $B \in H(A)$.

In the original theorem in [5], Favard has assumed the existence of a bounded solution of (17) in addition to the assumption in Theorem 5. However, it is known that if the linear system

$$\dot{x} = A(t)x$$

has an exponential dichotomy on R, then (17) has a bounded solution (see [4, 103B (p. 344)] or [6, p. 138]).

REFERENCES

[1] J. K. HALE, Ordinary Differential Equations, Wiley-Interscience, New York, 1969.
[2] R. J. SACKER AND G. R. SELL, Existence of dichotomies and invariant splittings for linear differential systems I, J. Differential Eqs., 15 (1974), 429-458.
[3] W. A. COPPEL, Dichotomies and reducibility, J. Differential Eqs., 3 (1967), 500-521.
[4] J. L. MASSERA AND J. J. SCHÄFFER, Linear differential equations and function spaces, Academic Press, New York, 1966.
[5] J. FAVARD, Leçons sur les functions presque-périodiques, Gauthier-Villars, Paris, 1933.
[6] W. A. COPPEL, Stability and asymptotic behavior of differential equations, Heath, Boston 1965.

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