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ON SOME TYPES OF ISOPARAMETRIC HYPERSURFACES IN SPHERES II

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Introduction. This paper is a continuation of Part I [13]. In the first half of the present paper, we study the homogeneous isoparametric hypersurfaces in spheres. Every homogeneous hypersurface in a sphere is represented as an orbit of a linear isotropy group of a Riemannian symmetric space of rank 2, due to Hsiang-Lawson [8]. In §1, we study the linear isotropy representations of Riemannian symmetric spaces and their orbits in general. §2 and §3 are devoted to a study of the homogeneous isoparametric hypersurfaces, their classification and invariant polynomials. In §4 and §5, we construct explicitly the defining polynomial F for each homogeneous isoparametric hypersurface in a sphere, which was done by Cartan [3] in case $g = 3$.

In the second half, we prove that every closed isoparametric hypersurface in a sphere in case $g = 4$ and m_1 or $m_2 = 2$ is homogeneous. Cartan [4] indicated, without proof, that in case $g = 4$, every closed isoparametric hypersurface in a sphere with the same multiplicities is homogeneous. In case $m_1 = m_2 = 2$, we give a brief outline of its proof in §9.

In §6, we exhibit explicit forms of $\{p_\alpha, q_\alpha\}$ for some of the homogeneous examples. We see that, for a homogeneous isoparametric hypersurface with $g = 4$, $m_1 = 4$ and $m_2 = 3$, its defining polynomial $-F$ does not satisfy the condition (B) given in §6 of Part I. Thus one can conclude that our example constructed in Theorem 2 of Part I for $\boldsymbol{F} = \boldsymbol{H}$ and $r = 1$ is not homogeneous. Consequently, there are at least two types of isoparametric hypersurfaces in S^{15} with the same multiplicities; one is homogeneous, and the other is not. It seems to be an interesting problem to seek a local geometric *quantity* in order to distinguish them.

1. ***s*-representations.** In this section we shall consider the linear isotropy representations of Riemannian symmetric spaces and investigate the structures of orbits of such representations.

Let V be a Euclidean space, i.e., a finite dimensional real vector space equipped with an inner product $(\,,)$. The unit sphere in V centered at the origin 0 will be denoted by $S(V)$. $O(V)$ and $SO(V)$ denote the orthogonal group and the special orthogonal group of V respectively. That is,

$$O(V) = \{\sigma \in GL(V) \mid (\sigma x, \sigma y) = (x, y) \quad \text{for each} \quad x, y \in V\} ,$$
$$SO(V) = \{\sigma \in O(V) \mid \det \sigma = 1\} .$$

If $V = \boldsymbol{R}^N$ equipped with the standard inner product $(\,,)$, then $S(V)$, $O(V)$ and $SO(V)$ are the usual unit sphere S^{N-1}, the usual linear groups $O(N)$ and $SO(N)$ respectively. Consider an orthogonal representation $\rho\colon K \to SO(V)$ of a compact connected Lie group K on V. In this note a representation of a topological group will be always assumed to be continuous. Through the representation ρ, the group K acts on V and $S(V)$ as linear automorphisms and isometries respectively. These actions are effective if and only if ρ is faithful. ρ is said to be *of cohomogeneity* ν if the maximum of dimensions of K-orbits in V is equal to $\dim V - \nu$, or equivalently if the maximum of dimensions of K-orbits in $S(V)$ is equal to $\dim S(V) - \nu + 1$. Orthogonal representations $\rho\colon K \to SO(V)$ and $\rho'\colon K' \to SO(V')$ of compact connected Lie groups K and K' respectively, are said to be $\approx$*-equivalent* and denoted by $\rho \approx \rho'$, if there exist an isomorphism $\varphi\colon K \to K'$ and an isometry $\sigma\colon V \to V'$ such that $\sigma\rho(k) = \rho'(\varphi(k))\sigma$ for each $k \in K$.

An s-representation associated to a Lie algebra of rank ν, which will be defined in the following, is an example of a faithful orthogonal representation of cohomogeneity ν.

Let $\mathfrak{g}$ be a non-commutative real reductive algebraic Lie algebra without compact factors. Let θ be a Cartan involution of $\mathfrak{g}$. The Cartan decomposition associated to θ is given by

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} ,$$

where

$$\mathfrak{k} = \{x \in \mathfrak{g} \mid \theta x = x\} ,$$
$$\mathfrak{p} = \{x \in \mathfrak{g} \mid \theta x = -x\} .$$

Let $\mathrm{Ad}\,\mathfrak{g} \subset GL(\mathfrak{g})$ denote the adjoint group of $\mathfrak{g}$. Then the Lie algebra of $\mathrm{Ad}\,\mathfrak{g}$ is identified with the commutator subalgebra $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$ and $\mathfrak{k}$ is a maximal compact subalgebra of $[\mathfrak{g}, \mathfrak{g}]$. Let K denote the connected subgroup of $\mathrm{Ad}\,\mathfrak{g}$ generated by $\mathfrak{k}$. Maximal abelian subalgebras in $\mathfrak{p}$ are mutually conjugate under the action of K on $\mathfrak{p}$. The dimension ν of such subalgebras is the so-called $\boldsymbol{R}$-rank of $\mathfrak{g}$. In this note we call it simply the *rank* of $\mathfrak{g}$. Denoting by $\mathfrak{c}$ the center of $\mathfrak{g}$, we have a direct sum decomposition:

$$\mathfrak{g} = \mathfrak{c} \oplus ([\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{p}) .$$

The Killing form B of $\mathfrak{g}$ is positive definite on $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{p}$. We choose an inner product $(,)$ on $\mathfrak{p}$ such that (1) it coincides with a positive multiple of B on $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{p}$, i.e., there exists $c > 0$ such that $(x, y) = cB(x, y)$ for each $x, y \in [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{p}$, and (2) $(\mathfrak{c}, [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{p}) = \{0\}$. The vector space $\mathfrak{p}$ will be considered as a Euclidian space with this inner product. We define an orthogonal representation $\rho: K \to SO(\mathfrak{p})$ by

$$\rho(k) = k|\mathfrak{p} \qquad \text{for} \quad k \in K .$$

It is known (cf. Helgason [7]) that ρ is of cohomogeneity ν and that for $x \in \mathfrak{p}$, the equality $\dim K(x) = \dim \mathfrak{p} - \nu$ holds if and only if x is a regular element of $\mathfrak{p}$. Note that ρ is faithful in virtue of $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$. The representation ρ is called the *s-representation* associated to the triple $(\mathfrak{g}, \theta, (,))$, or simply an s-representation associated to $\mathfrak{g}$.

The $\approx$-equivalence class of ρ depends only on the isomorphism class of $\mathfrak{g}$. In fact, let $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic, and let $\rho: K \to SO(\mathfrak{p})$ and $\rho': K' \to SO(\mathfrak{p}')$ be s-representations associated to $(\mathfrak{g}, \theta, (,))$ and $(\mathfrak{g}', \theta', (,)')$ respectively. Choose an isomorphism $\alpha: \mathfrak{g} \to \mathfrak{g}'$ such that $\theta'\alpha = \alpha\theta$. We define an isomorphism $\varphi: K \to K'$ and a linear isomorphism $\tau: \mathfrak{p} \to \mathfrak{p}'$ by

$$\varphi(k) = \alpha k \alpha^{-1} \qquad \text{for} \quad k \in K ,$$

$$\tau x = \alpha x \qquad \text{for} \quad x \in \mathfrak{p} .$$

Then we have $\tau\rho(k) = \rho'(\varphi(k))\tau$ for each $k \in K$. Furthermore $\tau\mathfrak{c} = \mathfrak{c}'$, $\tau([\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{p}) = [\mathfrak{g}', \mathfrak{g}'] \cap \mathfrak{p}'$ and $B(x, y) = B'(\tau x, \tau y)$ for each $x, y \in [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{p}$, where $\mathfrak{c}, \mathfrak{c}'$ and B, B' denote the centers and the Killing forms of $\mathfrak{g}, \mathfrak{g}'$ respectively. It follows that we can find an isometry $\sigma: \mathfrak{p} \to \mathfrak{p}'$ satisfying $\sigma\rho(k) = \rho'(\varphi(k))\sigma$ for each $k \in K$, and hence $\rho \approx \rho'$.

PROPOSITION 1. *The s-representation defines an injective map of the set of isomorphism classes of non-commutative real reductive algebraic Lie algebras of rank ν without compact factors into the set of $\approx$-equivalence classes of faithful orthogonal representations of cohomogeneity ν.*

PROOF. Let $\rho: K \to SO(\mathfrak{p})$ and $\rho': K' \to SO(\mathfrak{p}')$ be s-representations associated to $(\mathfrak{g}, \theta, (,))$ and $(\mathfrak{g}', \theta', (,)')$ respectively. Assume $\rho \approx \rho'$, i.e., there exist an isomorphism $\varphi: K \to K'$ and an isometry $\sigma: \mathfrak{p} \to \mathfrak{p}'$ such that

$$(1) \qquad \sigma\rho(k) = \rho'(\varphi(k))\sigma \qquad \text{for each} \quad k \in K .$$

We have to prove that $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic. From the above argument, we may assume that $(,)$ and $(,)'$ coincide with the Killing forms on $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{p}$ and $[\mathfrak{g}', \mathfrak{g}'] \cap \mathfrak{p}'$ respectively. Denoting by $\varphi_*: \mathfrak{k} \to \mathfrak{k}'$ the differential of the isomorphism φ, we define a linear isomorphism $\alpha: \mathfrak{g} \to \mathfrak{g}'$ by

$$\alpha(x + y) = \varphi_* x + \sigma y \qquad \text{for } x \in \mathfrak{k},\ y \in \mathfrak{p}\,.$$

In virtue of (1) we have

$$\alpha(\operatorname{ad} x)y = \operatorname{ad}(\alpha x)\alpha y \qquad \text{for } x \in \mathfrak{k},\ y \in \mathfrak{p}\,. \tag{2}$$

It follows that α sends the center $\mathfrak{c}$ of $\mathfrak{g}$ onto the center $\mathfrak{c}'$ of $\mathfrak{g}'$. It suffices to show that α is a Lie algebra homomorphism. We extend the inner products $(\,,)$ and $(\,,)'$ to adjoint invariant symmetric non-degenerate bilinear forms $(\,,)$ and $(\,,)'$ on $\mathfrak{g}$ and $\mathfrak{g}'$ respectively, in such a way that they coincide with the Killing forms on $[\mathfrak{g}, \mathfrak{g}]$ and $[\mathfrak{g}', \mathfrak{g}']$ respectively.

(a) Let $x, y \in \mathfrak{k}$. We have

$$\alpha[x, y] = \varphi_*[x, y] = [\varphi_* x, \varphi_* y] = [\alpha x, \alpha y]\,.$$

(b) Let $x \in \mathfrak{k}$ and $y \in \mathfrak{p}$. By (2) we have

$$\alpha[x, y] = \alpha(\operatorname{ad} x)y = \operatorname{ad}(\alpha x)\alpha y = [\alpha x, \alpha y]\,.$$

(a) and (b) show that $\operatorname{ad}(\alpha x) = \alpha(\operatorname{ad} x)\alpha^{-1}$ for each $x \in \mathfrak{k}$, and hence

$$(x, y) = (\alpha x, \alpha y)' \qquad \text{for } x, y \in \mathfrak{k}. \tag{3}$$

(c) We show that $\alpha[x, y] = [\alpha x, \alpha y]$ for each $x, y \in \mathfrak{p}$. As we can see easily, we may assume $x, y \in [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{p}$. For each $z \in \mathfrak{k}$, we have by (3)

$$\begin{aligned}([\alpha x, \alpha y], \alpha z)' &= -(\alpha x, [\alpha z, \alpha y])' = -(\alpha x, \alpha[z, y])' \\ &= -(\sigma x, \sigma[z, y])' = -(x, [z, y]) = ([x, y], z) \\ &= (\alpha[x, y], \alpha z)'\,.\end{aligned}$$

This shows $[\alpha x, \alpha y] = \alpha[x, y]$. q.e.d.

Now we consider the structure of K-orbits of s-representations. In general, for a group G acting on a space X, we denote by $G\backslash X$ the space of G-orbits in X. Let $\rho: K \to SO(\mathfrak{p})$ be the s-representation of cohomogeneity ν associated to $(\mathfrak{g}, \theta, (\,,))$. We may assume without loss of generality that the inner product $(\,,)$ coincides with the Killing form on $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{p}$. We extend $(\,,)$ to an adjoint invariant symmetric non-degenerate bilinear form $(\,,)$ on $\mathfrak{g}$ in such a way that it coincides with the Killing form on $[\mathfrak{g}, \mathfrak{g}]$. The C-linear extensions of θ and $(\,,)$ to the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$, are also denoted by θ and $(\,,)$ respectively. Choose a maximal abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$ and extend it to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then we have a direct sum decomposition:

$$\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{a} \quad \text{where} \quad \mathfrak{b} = \mathfrak{h} \cap \mathfrak{k}\,.$$

We put $\mathfrak{h}_0 = \sqrt{-1}\mathfrak{b} + \mathfrak{a}$. Then the form $(\,,)$ is positive definite on $\mathfrak{h}_0$, and hence it defines a Euclidean space structure on $\mathfrak{h}_0$. The set $\tilde{\Sigma}$ of roots of $\mathfrak{g}^C$ relative to $\mathfrak{h}^C$, the complexification of $\mathfrak{h}$, is identified with a subset of $\mathfrak{h}_0$ by means of the duality defined by the inner product $(\ ,\ )$. Choose a lexicographic order $>$ on $\mathfrak{h}_0$ in such a way that if $\alpha \in \tilde{\Sigma} - \sqrt{-1}\mathfrak{b}$, $\alpha > 0$, then $\theta\alpha < 0$. Denoting by $\tilde{\Pi}$ the fundamental root system for $\tilde{\Sigma}$ with respect to the order $>$, we define a *positive Weyl chamber* $\mathscr{C}$ in $\mathfrak{a}$ by

$$\mathscr{C} = \{h \in \mathfrak{a} \mid (\alpha, h) > 0 \quad \text{for each} \quad \alpha \in \tilde{\Pi} - \sqrt{-1}\mathfrak{b}\} .$$

And then we set

$$\mathscr{C}^1 = \mathscr{C} \cap S(\mathfrak{p}) = \mathscr{C} \cap S(\mathfrak{a}) .$$

Making use of the group of particular rotations:

$$P = \{\sigma \in O(\mathfrak{h}_0) \mid \sigma\mathfrak{a} = \mathfrak{a},\ \sigma\tilde{\Sigma} = \tilde{\Sigma},\ \sigma\tilde{\Pi} = \tilde{\Pi}\} ,$$

we define a subgroup C of $O(\mathfrak{a})$ by

$$C = \{\sigma|_{\mathfrak{a}} \mid \sigma \in P\} .$$

Note that the group C leaves $\mathscr{C}^1$ invariant. The Weyl group $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$, where $N_K(\mathfrak{a})$ and $Z_K(\mathfrak{a})$ denote the normalizer and the centralizer of $\mathfrak{a}$ in K, is identified with a finite subgroup of $O(\mathfrak{a})$. It is known (cf. Helgason [7]) that the inclusions $\mathscr{C} \subset \mathfrak{a} \subset \mathfrak{p}$ induce the natural identifications

$$\bar{\mathscr{C}} = W\backslash\mathfrak{a} = K\backslash\mathfrak{p} \quad \text{and} \quad \bar{\mathscr{C}}^1 = W\backslash S(\mathfrak{a}) = K\backslash S(\mathfrak{p}) ,$$

where $-$ means the closure in $\mathfrak{a}$. Let $I(\mathfrak{p})$ and $I(\mathfrak{a})$ denote the algebra of K-invariant polynomial functions on $\mathfrak{p}$ and the one of W-invariant polynomial functions on $\mathfrak{a}$ respectively. Then it is known by Chevalley [5], Harish-Chandra (cf. Helgason [7]) that the restriction map of $I(\mathfrak{p})$ into $I(\mathfrak{a})$ is an isomorphism and that $I(\mathfrak{p})$ has ν algebraically independent homogeneous generators, say $I_1, \cdots, I_\nu$. The K-orbits in $\mathfrak{p}$ are described by means of $I_1, \cdots, I_\nu$ as follows (cf. Helgason [7], Kostant-Rallis [11]):

(A) *The correspondence*

$$x_0 \mapsto \begin{pmatrix} I_1(x_0) \\ \vdots \\ I_\nu(x_0) \end{pmatrix} \quad \textit{for} \quad x_0 \in \mathfrak{p}$$

of $\mathfrak{p}$ into $\boldsymbol{R}^\nu$ induces an injective map $K\backslash\mathfrak{p} \to \boldsymbol{R}^\nu$ in such a way that

$$K(x_0) = \{x \in \mathfrak{p} \mid I_i(x) = I_i(x_0) \quad \textit{for} \quad i = 1, \cdots, \nu\}$$

for each $x_0 \in \mathfrak{p}$. The ideal in the algebra of polynomial functions on $\mathfrak{p}$, consisting of all f such that $f|_{K(x_0)} = 0$, is a prime ideal generated by

$I_1 - I_1(x_0), \cdots, I_\nu - I_\nu(x_0)$, *and hence for each* $x_0 \in \mathfrak{p}$ $K(x_0)$ *is an irreducible algebraic variety in* $\mathfrak{p}$.

We can choose generators $\{I_i\}$ of $\boldsymbol{I}(\mathfrak{p})$ such that $I_1 = r^2$, where r is the usual radius function on $\mathfrak{p}$. In fact, let $x_1, \cdots, x_{\nu_1}$ where ν_1 is the dimension of the center $\mathfrak{c}$ of $\mathfrak{g}$, be an orthonormal coordinate system for $\mathfrak{c}$ and $I'_1, \cdots, I'_{\nu_2}$, where $\nu_2 = \dim([\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{a})$, be a system of homogeneous generators of the algebra of K-invariant polynomial functions on $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{p}$. Then $\{x_i\ (1 \leqq i \leqq \nu_1), I'_j\ (1 \leqq j \leqq \nu_2)\}$ form a system of generators of $\boldsymbol{I}(\mathfrak{p})$, considering them as polynomial functions on $\mathfrak{p}$. Since we can choose $\{I'_j\}$ in such a way that a generator of the lowest degree, say I'_1, coincides with the Killing form on $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{p}$, we can find generators $\{I_i\}$ of $\boldsymbol{I}(\mathfrak{p})$ such that $I_1 = \sum_i x_i^2 + I'_1 = r^2$. Hence, after the above choice of generators of $\boldsymbol{I}(\mathfrak{p})$, we have

(B) *The correspondence*

$$x_0 \mapsto \begin{pmatrix} I_2(x_0) \\ \vdots \\ I_\nu(x_0) \end{pmatrix} \quad \textit{for} \quad x_0 \in S(\mathfrak{p})$$

of $S(\mathfrak{p})$ *into* $\boldsymbol{R}^{\nu-1}$ *induces an injective map* $K\backslash S(\mathfrak{p}) \to \boldsymbol{R}^{\nu-1}$ *in such a way that*

$$K(x_0) = \{x \in S(\mathfrak{p}) \mid I_i(x) = I_i(x_0) \quad \textit{for} \quad i = 2, \cdots, \nu\}$$

for each $x_0 \in S(\mathfrak{p})$.

In particular we have

PROPOSITION 2. *Let* $\nu = 2$. *Take a homogeneous generator* F *of* $\boldsymbol{I}(\mathfrak{p})$ *other than* r^2. *Then the map* $x_0 \mapsto F(x_0)$ *of* $S(\mathfrak{p})$ *into* $\boldsymbol{R}$ *induces an injective map* $K\backslash S(\mathfrak{p}) \to \boldsymbol{R}$ *in such a way that*

$$K(x_0) = \{x \in S(\mathfrak{p}) \mid F(x) = F(x_0)\}$$

for each $x_0 \in S(\mathfrak{p})$. *Each* $K(x_0)$ *is an irreducible algebraic variety in* $\mathfrak{p}$. *Denoting by* $|W|$ *the order of the Weyl group* W, *and by* g *the degree of* F, *we have*

$$|W| = 2g\ ,$$

and the possibilities of g *are* 1, 2, 3, 4 *and* 6.

PROOF. The first and the second assertions follow from (B) and (A). The possibilities of Weyl groups W are

(a-1) $\dim \mathfrak{c} = 1$. W is of type $A_1 \times \{1\}$ (W acts on $\mathfrak{c}$ trivially). $|W| = 2$.

(a-2) $\mathfrak{g}$ is semi-simple, not simple. W is of type $A_1 \times A_1$. $|W| = 4$.

(b) $\mathfrak{g}$ is simple. W is of type A_2, B_2 or G_2. $|W|$ is 6, 8 or 12 respectively.

(In this note a Lie algebra is said to be simple if it is not commutative and has no non-trivial ideal.) On the other hand, it is known (cf. Bourbaki [2]) that in each case $2g$ coincides with $|W|$. This can be also derived from a theorem of Kostant [10] on exponents of Weyl groups, without use of the classification of Weyl groups. q.e.d

In general, for a Riemannian manifold $\bar{M}$ and a submanifold M of $\bar{M}$, we denote by $I(\bar{M}, M)$ the group of all isometries of $\bar{M}$ leaving M invariant, endowed with the topology induced from the one of the group of isometries of $\bar{M}$. $I_0(\bar{M}, M)$ denotes the identity component of $I(M, \bar{M})$.

For an automorphism α of $\mathfrak{g}$, the C-linear extension of α to $\mathfrak{g}^C$ will be also denoted by α. We denote by $\mathrm{Aut}(\mathfrak{g}, \mathfrak{k}, (\ ,\ ))$ the group of all automorphisms α of $\mathfrak{g}$ such that $\alpha\mathfrak{k} = \mathfrak{k}$ and $(\alpha x, \alpha y) = (x, y)$ for each $x, y \in \mathfrak{g}$. Similarly, $\mathrm{Aut}(\mathfrak{g}, \mathfrak{k}, \mathfrak{h}, \tilde{\Pi}, (\ ,\ ))$ denotes the group of all $\alpha \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{k}, (\ ,\ ))$ such that $\alpha\mathfrak{h} = \mathfrak{h}$ and $\alpha\tilde{\Pi} = \tilde{\Pi}$. It is known (Takeuchi [16]) that K is a normal subgroup of $\mathrm{Aut}(\mathfrak{g}, \mathfrak{k}, (\ ,\ ))$,

$\mathrm{Aut}(\mathfrak{g}, \mathfrak{k}, (\ ,\ )) = \mathrm{Aut}(\mathfrak{g}, \mathfrak{k}, \mathfrak{h}, \tilde{\Pi}, (\ ,\ ))K$ (semi-direct), and the restriction map $\mathrm{Aut}(\mathfrak{g}, \mathfrak{k}, \mathfrak{h}, \tilde{\Pi}, (\ ,\ )) \to P$ is a surjective homomorphism. Hence a surjective homomorphism $\gamma\colon \mathrm{Aut}(\mathfrak{g}, \mathfrak{k}, (\ ,\ )) \to C$ is defined by the composite of

$$\begin{aligned}\mathrm{Aut}(\mathfrak{g}, \mathfrak{k}, (\ ,\ )) &\to \mathrm{Aut}(\mathfrak{g}, \mathfrak{k}, (\ ,\ ))/K \\ &= \mathrm{Aut}(\mathfrak{g}, \mathfrak{k}, \mathfrak{h}, \tilde{\Pi}, (\ ,\ )) \to P \to C\,.\end{aligned}$$

Then we have

PROPOSITION 3. *For an element $x_0 \in \mathscr{C}^1$, put*

$$\mathrm{Aut}(\mathfrak{g}, \mathfrak{k}, (\ ,\ ))_{x_0} = \{\alpha \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{k}, (\ ,\ )) \mid \gamma(\alpha)x_0 = x_0\}\,.$$

Then the restriction $\alpha \mapsto \alpha|_{\mathfrak{p}}$ defines an injective homomorphism:

$$\mathrm{Aut}(\mathfrak{g}, \mathfrak{k}, (\ ,\ ))_{x_0} \to I(S(\mathfrak{p}), K(x_0))\,.$$

If furthermore $\rho(K) = I_0(S(\mathfrak{p}), K(x_0))$, then the above homomorphism is an isomorphism.

PROOF. Let $\alpha \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{k}, (\ ,\ ))_{x_0}$. By definition, $\alpha|_{\mathfrak{p}} \in O(\mathfrak{p})$, $\alpha K \alpha^{-1} = K$ and there exist $\beta \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{k}, \mathfrak{h}, \tilde{\Pi}, (\ ,\ ))$ and $k \in K$ such that $\alpha = k\beta$ and $\beta(x_0) = x_0$. We have

$$\alpha K(x_0) = \alpha K \alpha^{-1}\alpha(x_0) = Kk\beta(x_0) = K(x_0)\,,$$

and hence $\alpha|_{\mathfrak{p}} \in I(S(\mathfrak{p}), K(x_0))$. The injectivity follows from $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$. Assume $\rho(K) = I_0(S(\mathfrak{p}), K(x_0))$. Let $\sigma \in I(S(\mathfrak{p}), K(x_0))$. Then we have

$\sigma\rho\,(K)\sigma^{-1} = \rho(K)$ by the assumption. We define an automorphism φ of K by

$$\rho(\varphi(k)) = \sigma\rho(k)\sigma^{-1} \qquad \text{for} \quad k \in K$$

and denote by φ_* the differential of φ. Then, as we have seen in the proof of Prop. 1, the linear automorphism α of $\mathfrak{g}$ defined by

$$\alpha(x + y) = \varphi_* x + \sigma y \qquad \text{for} \quad x \in \mathfrak{k},\ y \in \mathfrak{p}$$

is an element of $\mathrm{Aut}\,(\mathfrak{g}, \mathfrak{k}, (\ ,\ ))$ satisfying $\alpha|_{\mathfrak{p}} = \sigma$. Let $\alpha(x_0) = k_1^{-1}(x_0)$ with $k_1 \in K$. Then $k_1\alpha$ fixes a regular element x_0 of $\mathfrak{a}$, and hence $k_1\alpha\mathfrak{a} = \mathfrak{a}$. Since both $\mathfrak{b}$ and $k_1\alpha\mathfrak{b}$ are Cartan subalgebras of the centralizer $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ of $\mathfrak{a}$ in $\mathfrak{k}$, we can choose $k_2 \in Z_K(\mathfrak{a})$ such that $k_2k_1\alpha\mathfrak{b} = \mathfrak{b}$. Choose $k_3 \in K$ such that $k_3\mathfrak{h} = \mathfrak{h}$ and $k_3k_2k_1\alpha\tilde{\Pi} = \tilde{\Pi}$. Since k_3 leaves the positive Weyl chamber $\mathscr{C}$ invariant, we have $k_3 \in Z_K(\mathfrak{a})$. By the construction, $\beta = k_3k_2k_1\alpha$ is in $\mathrm{Aut}\,(\mathfrak{g}, \mathfrak{k}, \mathfrak{h}, \tilde{\Pi}, (\ ,\ ))$ and $\beta(x_0) = x_0$. It follows that $\gamma(\alpha)x_0 = x_0$, and hence $\alpha \in \mathrm{Aut}\,(\mathfrak{g}, \mathfrak{k}, (\ ,\ ))_{x_0}$. This shows the surjectivity of the map $\alpha \mapsto \alpha|_{\mathfrak{p}}$.

q.e.d.

K-orbits M and M' in $S(\mathfrak{p})$ are said to be *equivalent* if an element of $O(\mathfrak{p})$ transforms M onto M'. A K-orbit M in $S(\mathfrak{p})$ is said to be *principal* if $\dim M = \dim \mathfrak{p} - \nu$. Then we have

PROPOSITION 4. *The correspondence $x_0 \mapsto K(x_0)$ for $x_0 \in \mathscr{C}^1$ induces a surjective map of $C\backslash\mathscr{C}^1$ onto the set of equivalence classes of principal K-orbits in $S(\mathfrak{p})$. If furthermore $\rho(K) = I_0(S(\mathfrak{p}), K(x_0))$ for each $x_0 \in \mathscr{C}^1$, then this map is bijective.*

PROOF. Let $x_0, x_0' \in \mathscr{C}^1$. Assume that there exists $\sigma \in C$ such that $\sigma x_0 = x_0'$. From the surjectivity of the homomorphism $\mathrm{Aut}\,(\mathfrak{g}, \mathfrak{k}, \mathfrak{h}, \tilde{\Pi}, (\ ,\ )) \to P$, it follows that σ can be extended to an automorphism $\alpha \in \mathrm{Aut}\,(\mathfrak{g}, \mathfrak{k}, (\ ,\ ))$. Then $\alpha K\alpha^{-1} = K$ and hence $K(x_0') = \alpha K\alpha^{-1}(x_0') = \alpha K(x_0)$ with $\alpha|_{\mathfrak{p}} \in O(\mathfrak{p})$. This shows the equivalence of $K(x_0)$ and $K(x_0')$. Hence our map is well defined. The surjectivity of the map follows from the natural identification: $\bar{\mathscr{C}}^1 = K\backslash S(\mathfrak{p})$. Suppose further that $\rho(K) = I_0(S(\mathfrak{p}), K(x_0))$ for each $x_0 \in \mathscr{C}^1$. Let $x_0, x_0' \in \mathscr{C}^1$. Assume that there exists $\sigma \in O(\mathfrak{p})$ such that $\sigma K(x_0) = K(x_0')$. Since $\sigma I(S(\mathfrak{p}), K(x_0))\sigma^{-1} = I(S(\mathfrak{p}), K(x_0'))$, we have $\sigma\rho(K)\sigma^{-1} = \rho(K)$. In the same way as in the proof of Prop. 3, we can choose $\alpha \in \mathrm{Aut}\,(\mathfrak{g}, \mathfrak{k}, (\ ,\ ))$ satisfying $\alpha|_{\mathfrak{p}} = \sigma$. Let $\alpha(x_0) = k_1^{-1}(x')$ with $k_1 \in K$. Since $k_1\alpha(x_0) = x_0'$ is an element of $\mathfrak{a}$, we can choose $k_2 \in K$ such that $k_2x_0' = x_0'$ and $k_2k_1\alpha\mathfrak{a} = \mathfrak{a}$. In the same way as in the proof of Prop. 3, we can choose $k_3 \in Z_K(\mathfrak{a})$ such that $\beta = k_3k_2k_1\alpha$ is in $\mathrm{Aut}\,(\mathfrak{g}, \mathfrak{k}, \mathfrak{h}, \tilde{\Pi}, (\ ,\ ))$ and $\beta(x_0) = x_0'$. It follows that x_0 and x_0' are in the same C-orbit in $\mathscr{C}^1$. This shows the injectivity of our map.

q.e.d.

2. Homogeneous hypersurfaces in spheres. In this section we shall reduce the classification of homogeneous hypersurfaces in spheres to the one of certain representations of compact connected Lie groups, and then state a theorem of Hsiang-Lawson giving the classification of such hypersurfaces.

Let S^{N-1} ($N \geqq 3$) be the unit sphere in an N-dimensional Euclidean space centered at the origin and M a connected locally closed $(N-1)$-dimensional submanifold in S^{N-1}. As in Introduction of Part I, M is said to be homogeneous if the group $I(S^{N-1}, M)$ acts transitively on M. In the sequel, a homogeneous connected locally closed $(N-2)$-dimensional submanifold in S^{N-1} will be called a *homogeneous hypersurface* in S^{N-1}. As in Introduction of Part I, hypersurfaces M in S^{N-1} and M' in $S^{N'-1}$ are said to be equivalent, if $N = N'$ and an element of $O(N)$ transforms M onto M'.

Let M be a homogeneous hypersurface in S^{N-1}, and $I(M)$ the Lie group of isometries of M with respect to the Riemannian metric of M induced from the one of S^{N-1}. Then the restriction $\lambda: I(S^{N-1}, M) \to I(M)$ is a continuous homomorphism. Let $K(M)$ denote the λ-image $\lambda I_0(S^{N-1}, M)$ of $I_0(S^{N-1}, M)$, endowed with the topology induced from the one of $I(M)$.

LEMMA 1. *Let M be a homogeneous hypersurface in S^{N-1}.*

(i) *The restriction $\lambda_0: I_0(S^{N-1}, M) \to K(M)$ is an isomorphism, and hence the inverse isomorphism of λ_0 defines a faithful orthogonal representation $\rho_M: K(M) \to SO(N)$ of the group $K(M)$.*

(ii) *M is compact, and hence $K(M)$ is a compact connected Lie group.*

PROOF. (i) The surjectivity of λ_0 follows from definition. Let $\sigma \in I_0(S^{N-1}, M)$ such that $\lambda_0(\sigma) = 1$. Take a point $x_0 \in M$. Without loss of generality we may assume that

$$x_0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad T_{x_0}(M) = \left\{ \begin{pmatrix} \xi \\ 0 \\ 0 \end{pmatrix} \middle| \xi \in \boldsymbol{R}^{N-2} \right\}.$$

The differential of σ at x_0 is the identity by the assumption: $\lambda_0(\sigma) = 1$. It follows that $\sigma \in SO(N)$ is of the form

$$\sigma = \left(\begin{array}{c|cc} 1_{N-2} & \multicolumn{2}{c}{0} \\ \hline 0 & \pm 1 & 0 \\ & 0 & 1 \end{array} \right),$$

and hence $\sigma = 1_N$. This shows the injectivity of λ_0.

(ii) Let $\bar{d}$ be the Riemannian distance of S^{N-1} and d the one of M with respect to the induced Riemannian metric. Note that the distance d is complete since $I(M)$ acts transitively on M. Take a point $x_0 \in M$, and choose $\varepsilon > 0$ such that each point $x \in S^{N-1}$ with $\bar{d}(x_0, x) < \varepsilon$ can be joined to x_0 by a unique geodesic in S^{N-1}. Put $U = \{x \in M \mid \bar{d}(x_0, x) < \varepsilon\}$. Then there exists a positive constant c such that $d(x_0, x) \leqq c\bar{d}(x_0, x)$ for each $x \in U$. Since $I(S^{N-1}, M)$ acts transitively on M, we have

$$d(x, y) \leqq c\bar{d}(x, y) \quad \text{for each} \quad x, y \in M \quad \text{with} \quad \bar{d}(x, y) < \varepsilon .$$

Now let $\{x_n\}_{n=1,2,\dots}$ be a sequence in M, converging in S^{N-1} to a point $s_0 \in S^{N-1}$. It follows from the above inequality that $\{x_n\}$ is a Cauchy sequence in M with respect to the complete distance d. Thus $\{x_n\}$ converges to a point $x_0 \in M$ and hence $s_0 = x_0 \in M$. This shows that M is closed in S^{N-1}. q.e.d.

For a homogeneous hypersurface M in S^{N-1}, the above faithful orthogonal representation ρ_M of the compact connected Lie group $K(M)$ is said to be *associated to* M. A faithful orthogonal representation $\rho: K \to SO(V)$ of cohomogeneity ν is said to be *maximal* if there is no faithful orthogonal representation $\rho': K' \to SO(V)$ of cohomogeneity ν such that K is a proper subgroup of K' and $\rho'(k) = \rho(k)$ for each $k \in K$.

LEMMA 2. *Let $\rho: K \to SO(N)$ be a maximal faithful orthogonal representation of cohomogeneity* 2, *and M an $(N-2)$-dimensional K-orbit in S^{N-1}. Then $\rho(K) = I_0(S^{N-1}, M)$.*

PROOF. We identify K with a compact subgroup of $SO(N)$ through the faithful representation ρ. Let $M = K(x_0)$ with $x_0 \in S^{N-1}$. Put $K' = I_0(S^{N-1}, M)$. Then the inclusion homomorphism $K' \to SO(N)$ is of cohomogeneity 2. In fact, if there would exist $y_0 \in S^{N-1}$ such that $\dim K'(y_0) = N-1$, then $K'(y_0) = S^{N-1}$ and $I_0(S^{N-1}, M)(x_0) = S^{N-1}$, which is a contradiction. It follows from the maximality of ρ that $K' = K$. This proves the lemma. q.e.d.

THEOREM 1. *For a homogeneous hypersurface M in S^{N-1}, the representation $\rho_M: K(M) \to SO(N)$ associated to M is a maximal faithful orthogonal representation of cohomogeneity* 2, *and M is an $(N-2)$-dimensional $K(M)$-orbit in S^{N-1}. If M and M' are equivalent, then ρ_M and $\rho_{M'}$ are $\approx$-equivalent. Conversely, any maximal faithful orthogonal representation of cohomogeneity* 2 *is obtained as the representation ρ_M associated to a homogeneous hypersurface M in a sphere.*

PROOF. Let $\rho_M: K(M) \to SO(N)$ be the representation associated to a homogeneous hypersurface M in S^{N-1}. The same argument as in the proof of Lemma 2 shows that ρ_M is of cohomogeneity 2. Let K' be a compact connected subgroup of $SO(N)$ containing $I_0(S^{N-1}, M)$ such that the maximum of dimensions of K'-orbits is equal to $N-2$. Then for each point $x \in M$, $K'(x) \supset I_0(S^{N-1}, M)(x) = M$, and hence $K'(x) = M$. This means $K' \subset I_0(S^{N-1}, M)$. Thus we have proved the maximality of ρ_M.

Assume that homogeneous hypersurfaces M and M' in S^{N-1} are equivalent, i.e., there exists $\sigma \in O(N)$ such that $\sigma M = M'$. Then the isomorphism $\varphi: I_0(S^{N-1}, M) \to I_0(S^{N-1}, M')$ defined by

$$\varphi(k) = \sigma k \sigma^{-1} \qquad \text{for} \quad k \in I_0(S^{N-1}, M)$$

satisfies $\sigma k = \varphi(k)\sigma$ for each $k \in I_0(S^{N-1}, M)$. This shows the $\approx$-equivalence of ρ_M and $\rho_{M'}$.

Let $\rho: K \to SO(N)$ be a maximal faithful orthogonal representation of cohomogeneity 2. Take an $(N-2)$-dimensional K-orbit M in S^{N-1}. Then by Lemma 2 we have $I_0(S^{N-1}, M) = K$, and hence $K = K(M)$ and $\rho = \rho_M$. This proves the last assertion. q.e.d.

In virtue of Theorem 1, the classification of equivalence classes of homogeneous hypersurfaces in spheres is reduced to the following two problems:

(I) *Classify $\approx$-equivalence classes of maximal faithful orthogonal representations of cohomogeneity* 2 *of compact connected Lie groups.*

(II) *Let $\rho: K \to SO(N)$ be a maximal faithful orthogonal representation of cohomogeneity* 2. *Classify equivalence classes of K-orbits in S^{N-1} of dimension $N-2$.*

We denote by $\mathfrak{o}(1, r)$ the Lie algebra of the Lorentz group for a quadratic form of signature $(1, r)$, i.e.,

$$\mathfrak{o}(1, r) = \{x \in \mathfrak{gl}(r+1, \boldsymbol{R}) \mid x'S + Sx = 0\} ,$$

where

$$S = \begin{pmatrix} 1 & \\ & -1_r \end{pmatrix} .$$

Then an answer to the problem (I) is given by the following theorem, which is due to Hsiang-Lawson.

THEOREM 2. (i) *The following two families of Lie algebras exhaust the all non-commutative real reductive algebraic Lie algebras without compact factors such that the associated s-representations are maximal* *faithful orthogonal representations of cohomogeneity* 2;

(a) *Lie algebras isomorphic to*

(a-1) $\boldsymbol{R} \oplus \mathfrak{o}(1, s)$ $(s \geqq 2)$, *or*

(a-2) $\mathfrak{o}(1, r) \oplus \mathfrak{o}(1, s)$ $(s \geqq r \geqq 2)$.

(b) *Non-compact simple Lie algebras of rank* 2.

(ii) *The s-representation defines a bijective map from the set of isomorphism classes of Lie algebras in families* (*a*) *and* (*b*) *onto the set of* $\approx$*-equivalence classes of maximal faithful orthogonal representations of cohomogeneity* 2.

PROOF. (i) and the surjectivity of the map in (ii) were proved in Hsiang-Lawson [8]. The injectivity of this map follows from Prop. 1. q.e.d.

REMARK. An associated s-representation is reducible or irreducible, according to case (a) or case (b).

An answer to the problem (II) is given by (i) of the following theorem.

THEOREM 3. *Let* $\mathfrak{g}$ *be a non-commutative real reductive algebraic Lie algebra without compact factors such that an associated s-representation is a maximal faithful orthogonal representation of cohomogeneity* 2. *Let* $\rho: K \rightarrow SO(\mathfrak{p})$ *be an s-representation associated to* $\mathfrak{g}$ *such that the inner product* $(\ ,\ )$ *on* $\mathfrak{p}$ *coincides with the Killing form on* $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{p}$. *Let* $\mathscr{C}^1$, C *and* $\mathrm{Aut}(\mathfrak{g}, \mathfrak{k}, (\ ,\ ))_{x_0}$ *be as in* §1.

(i) *The correspondence* $x_0 \mapsto K(x_0)$ *for* $x_0 \in \mathscr{C}^1$ *induces a bijective map of* $C \backslash \mathscr{C}^1$ *onto the set of equivalence classes of principal K-orbits in* $S(\mathfrak{p})$.

(ii) *For each* $x_0 \in \mathscr{C}^1$, $\mathrm{Aut}(\mathfrak{g}, \mathfrak{k}, (\ ,\ ))_{x_0}$ *is isomorphic to* $I(S(\mathfrak{p}), K(x_0))$ *by the correspondence* $\alpha \mapsto \alpha|_{\mathfrak{p}}$.

PROOF. These are immediate consequences of Prop. 4, Prop. 3 and Lemma 2. q.e.d.

$g = (1/2)|W|$ and the group C are given as follows:

(a-1) $\mathfrak{g} = \boldsymbol{R} \oplus \mathfrak{o}(1, s)$ $(s \geqq 2)$. $g = 1$, $C \cong \boldsymbol{Z}_2$.

(a-2) $\mathfrak{g} = \mathfrak{o}(1, r) \oplus \mathfrak{o}(1, s)$ $(s \geqq r \geqq 2)$. $g = 2$,

$$C \cong \begin{cases} \boldsymbol{Z}_2 & r = s , \\ \{1\} & r < s . \end{cases}$$

(b) $\mathfrak{g}$ a non-compact simple Lie algebra of rank 2. $g = 3, 4$ or 6,

$$C \cong \begin{cases} \boldsymbol{Z}_2 & \text{if } W \text{ is of type } A_2\,, \\ \{1\} & \text{if } W \text{ is of type } B_2 \text{ or } G_2\,. \end{cases}$$

Each non-trivial element of C acts on the open arc $\mathscr{C}^1$ in the circle $S(\mathfrak{a})$ by the "symmetry" with respect to the middle point of $\mathscr{C}^1$.

3. Homogeneous isoparametric hypersurfaces in spheres. A maximal family $\mathscr{I} = \{M_t \mid t \in I\}$ of isoparametric hypersurfaces in a sphere is said to be a *maximal family of homogeneous isoparametric hypersurfaces* if each M_t is a homogeneous hypersurface. In this section, such families of hypersurfaces will be classified.

For a maximal faithful orthogonal representation $\rho: K \to SO(N)$ of cohomogeneity 2, the family of all $(N-2)$-dimensional K-orbits in S^{N-1} will be denoted by $\mathscr{I}_\rho$. We shall investigate the structure of such family $\mathscr{I}_\rho$. For this purpose, we consider a non-commutative real reductive algebraic Lie algebra $\mathfrak{g}$ without compact factors such that an associated s-representation is a maximal faithful orthogonal representation of cohomogeneity 2. Let $\rho: K \to SO(\mathfrak{p})$ be an s-representation associated to $\mathfrak{g}$. Choosing a maximal abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$, a Cartan subalgebra $\mathfrak{h} = \mathfrak{b} + \mathfrak{a}$ of $\mathfrak{g}$ containing $\mathfrak{a}$, and a lexicographic order $>$ on $\mathfrak{h}_0 = \sqrt{-1}\mathfrak{b} + \mathfrak{a}$, we define a positive Weyl chamber $\mathscr{C}$ in $\mathfrak{a}$ as in §1. Let h_0 denote the middle point of $\mathscr{C}^1 = \mathscr{C} \cap S(\mathfrak{a})$. Choose an $h_{\pi/2} \in S(\mathfrak{a})$ with $(h_0, h_{\pi/2}) = 0$ and fix it once and for all. We define a real parameter θ of $S(\mathfrak{a})$ by

$$h_\theta = \cos\theta h_0 + \sin\theta h_{\pi/2} \qquad \text{for} \quad \theta \in \boldsymbol{R}\,.$$

Then we have

$$\mathscr{C}^1 = \left\{h_\theta \,\middle|\, -\frac{\pi}{2g} < \theta < \frac{\pi}{2g}\right\},$$

where $2g$ is the order $|W|$ of the Weyl group W. Recall that the family $\mathscr{I}_\rho$ is given by

$$\mathscr{I}_\rho = \left\{K(h_\theta) \,\middle|\, -\frac{\pi}{2g} < \theta < \frac{\pi}{2g}\right\}.$$

Denoting by $\{\lambda_1, \lambda_2\}$ the dual basis of the basis $\{h_0, h_{\pi/2}\}$ for $\mathfrak{a}$, we define a homogeneous polynomial function F_0 on $\mathfrak{a}$ of degree g by

$$F_0 = \sum_{i=0}^{[(g-1)/2]} \binom{g}{2i+1} (-1)^i \lambda_1^{g-(2i+1)} \lambda_2^{2i+1}\,. \tag{3.1}$$

Then $F_0(h_\theta) = \sin g\theta$ for each $\theta \in \boldsymbol{R}$. It is easy to see that the Weyl group W is generated by elements w_1 and w_2, which act on $S(\mathfrak{a})$ by

$$w_1: h_\theta \mapsto h_{\pi/g-\theta} ,$$
$$w_2: h_\theta \mapsto h_{\theta+2\pi/g} .$$

It follows that F_0 is a W-invariant polynomial function on $\mathfrak{a}$. By the theorem of Harish-Chandra cited in §1, F_0 is extended uniquely to a K-invariant polynomial function F on $\mathfrak{p}$. By Prop. 2, each K-orbit $K(h_\theta)$ is an irreducible algebraic variety in $\mathfrak{p}$ satisfying

$$K(h_\theta) = \{x \in S(\mathfrak{p}) \mid F(x) = \sin g\theta\} .$$

Let $\tilde{\Sigma}$ and $\tilde{\Sigma}^+$ be the set of roots and the one of positive roots respectively, and $\tilde{\omega}: \mathfrak{h}_0 \to \mathfrak{a}$ the orthogonal projection. We define Σ, Σ^+, Σ_* and Σ_*^+ by

$$\Sigma = \tilde{\omega}(\tilde{\Sigma} - \sqrt{-1}\mathfrak{b}) , \qquad \Sigma^+ = \tilde{\omega}(\tilde{\Sigma}^+ - \sqrt{-1}\mathfrak{b}) ,$$
$$\Sigma_* = \left\{\gamma \in \Sigma \,\middle|\, \frac{1}{2}\gamma \notin \Sigma\right\} , \qquad \Sigma_*^+ = \Sigma^+ \cap \Sigma_* .$$

The cardinality of the set Σ_*^+ coincides with g. For $\gamma \in \mathfrak{a}$, we denote by $\mu(\gamma)$ the number of roots α of $\tilde{\Sigma} - \sqrt{-1}\mathfrak{b}$ such that $\tilde{\omega}(\alpha) = \gamma$. We put $m(\gamma) = \mu(\gamma) + \mu(2\gamma)$ for $\gamma \in \Sigma_*$. For each $\gamma \in \Sigma_*^+$, there exists uniquely $\theta(\gamma)$ with $-\pi/2 < \theta(\gamma) < \pi/2$ satisfying $(h_{\theta(\gamma)+\pi/2}, \gamma) = 0$. We number the roots in Σ_*^+ in such a way that $\theta(\gamma_1) < \cdots < \theta(\gamma_g)$. Then we have

$$\theta(\gamma_i) = \frac{\pi}{2g}(2i-1) - \frac{\pi}{2} \quad \text{for} \quad i = 1, \cdots, g .$$

We put $m_i = m(\gamma_i)$ and $\theta_i = \theta(\gamma_i)$ for $i = 1, \cdots, g$. Seeing that $m(\gamma) = m(-\gamma)$, $m(w\gamma) = m(\gamma)$ for $\gamma \in \Sigma_*$, $w \in W$, we have

$$m_1 = m_2 \quad \text{for odd} \quad g \geqq 3 ,$$
$$m_1 = m_3 = \cdots ,$$
$$m_2 = m_4 = \cdots .$$

Let $-\pi/(2g) < \theta < \pi/(2g)$. We define a unit normal vector field X_θ on $K(h_\theta)$ in $S(\mathfrak{p})$ by

$$X_\theta(kh_\theta) = k(-\sin\theta\, h_0 + \cos\theta\, h_{\pi/2}) \qquad \text{for} \quad k \in K ,$$

identifying a tangent space of $S(\mathfrak{p})$ with a subspace of $\mathfrak{p} \cdot X_\theta$ is well defined since the stabilizer in K of the point h_θ is the centralizer $Z_K(\mathfrak{a})$ of $\mathfrak{a}$ in K. It is known (Takagi-Takahashi [15]) that $K(h_\theta)$ has g distinct principal curvatures with respect to X_θ, which are given by

$$k_i(\theta) = \tan(\theta - \theta_i) \qquad \text{for} \quad i = 1, \cdots, g , \tag{3.2}$$

and that the multiplicity of $k_i(\theta)$ is equal to m_i for each i. Note that $k_1(\theta) > k_2(\theta) > \cdots > k_g(\theta)$. Denoting by Exp the exponential map of the normal bundle of $K(h_0)$ into $S(\mathfrak{p})$, we define a C^∞-map p_θ: $K(h_0) \longrightarrow S(\mathfrak{p})$ by

$$p_\theta(x) = \operatorname{Exp}(\theta X_0(x)) \qquad \text{for} \quad x \in K(h_0) .$$

For $x = kh_0$ with $k \in K$, we have

$$p_\theta(x) = \operatorname{Exp}(\theta k\, h_{\pi/2}) = k(\cos\theta\, h_0 + \sin\theta\, h_{\pi/2}) = kh_\theta ,$$

and hence p_θ is a diffeomorphism of $K(h_0)$ onto $K(h_\theta)$. Thus the family $\mathscr{I}_\rho$ consists of parallel hypersurfaces $K(h_\theta)$ of constant principal curvatures given by (3.2). It follows from Satz 2 in Münzner [12] that the restriction to $S(\mathfrak{p})$ of the polynomial F is an isoparametric function on $S(\mathfrak{p})$ and that F satisfies the differential equations of Münzner:

$$\text{(M)} \qquad \begin{cases} (dF, dF) = g^2 r^{2g-2} \\ \Delta F = c r^{g-2} , \end{cases}$$

where

$$c = \begin{cases} \dfrac{1}{2}(m_2 - m_1)g^2 & g \quad \text{even} , \\ 0 & g \quad \text{odd} . \end{cases}$$

Hence the family $\mathscr{I}_\rho$ is a maximal family of homogeneous isoparametric hypersurfaces in $S(\mathfrak{p})$. Furthermore if ρ and ρ' are $\approx$-equivalent maximal faithful orthogonal representations of cohomogeneity 2, then $\mathscr{I}_\rho$ and $\mathscr{I}_{\rho'}$ are equivalent families of isoparametric hypersurfaces. Thus, together with the theorems in §2, we have the following theorem.

THEOREM 4. (i) *Let $\mathscr{I} = \{M_t \mid t \in I\}$ be a maximal family of isopara metric hypersurfaces in a sphere. If one of M_t is homogeneous, then each M_t is homogeneous, i.e., $\mathscr{I}$ is a maximal family of homogeneous isoparametric hypersurfaces. In a maximal family $\mathscr{I} = \{M_t \mid t \in I\}$ of homogeneous isoparametric hypersurfaces in S^{N-1}, each M_t is an irreducible algebraic variety in $\boldsymbol{R}^N$.*

(ii) *The correspondence $\rho \mapsto \mathscr{I}_\rho$ induces a bijective map of the set of $\approx$-equivalence classes of maximal faithful orthogonal representations of cohomogeneity 2 onto the set of equivalence classes of maximal families of homogeneous isoparametric hypersurfaces in spheres.*

4. Defining polynomials for homogeneous hypersurfaces in spheres—I. In this and the next sections, we shall compute a polynomial function F on $\boldsymbol{R}^N$ satisfying the differential equations (M) for each maximal family of homogeneous isoparametric hypersurfaces in S^{N-1}.

As we have seen in §3, one of such polynomials is obtained by the following procedures: Take a non-commutative real reductive algebraic Lie algebra $\mathfrak{g}$ without compact factors such that an associated s-representation is a maximal faithful orthogonal representation of cohomogeneity 2. Take an associated s-representation $\rho: K \to SO(\mathfrak{p})$ and a maximal abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$. Choose an orthonormal coordinate system $\{\lambda_1, \lambda_2\}$ for $\mathfrak{a}$ such that the middle point h_0 of $\mathscr{C}^1$ satisfies $\lambda_1(h_0) = 1$ and $\lambda_2(h_0) = 0$. Define a polynomial F_0 on $\mathfrak{a}$ of degree $g = (1/2)|W|$ by the formula (3.1), and then extend it to a K-invariant polynomial F on $\mathfrak{p}$. Then F is a required polynomial. For $g = 1$ or 2, the construction of F is immediate; so we shall state only the results in these cases.

Case $g = 1$: F is constructed from $\mathfrak{g} = \boldsymbol{R} \oplus \mathfrak{o}(1, s)$ $(s \geqq 2)$. $m_1 = s - 1$. With respect to the standard orthonormal coordinate system $\{x_i\}$ for $\boldsymbol{R}^{s+1}$, F is given by

$$F = x_{s+1} .$$

Case $g = 2$: F is constructed from $\mathfrak{g} = \mathfrak{o}(1, r) \oplus \mathfrak{o}(1, s)$ $(2 \leqq r \leqq s)$. $m_1 = r - 1$ and $m_2 = s - 1$. With respect to the standard orthonormal coordinate system $\{x_i\}$ for $\boldsymbol{R}^{r+s}$, F is given by

$$F = x_1^2 + \cdots + x_r^2 - (x_{r+1}^2 + \cdots + x_{r+s}^2) .$$

Case $g = 3$: Let $\boldsymbol{F}$ be a division algebra over $\boldsymbol{R}$, i.e., $\boldsymbol{F} = \boldsymbol{R}, \boldsymbol{C}$, the real quaternion algebra $\boldsymbol{H}$ or the real Cayley algebra $\boldsymbol{K}$. A linear form $t(x)$ and a quadratic form $n(x)$ on $\boldsymbol{F}$ are defined by

$$t(x) = x + \bar{x} , \qquad n(x) = x\bar{x} \qquad \text{for} \quad x \in \boldsymbol{F} ,$$

where $x \mapsto \bar{x}$ denotes the canonical involution of $\boldsymbol{F}$. Let

$$H_3(\boldsymbol{F}) = \{u \in M_3(\boldsymbol{F}) \mid \bar{u}' = u\}$$

and define

$$u \circ v = \frac{1}{2}(uv + vu) \qquad \text{for} \quad u, v \in H_3(\boldsymbol{F}) .$$

Then $H_3(\boldsymbol{F})$ becomes a compact simple Jordan algebra with respect to the product $u \circ v$. An element

$$u = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \quad \xi_i \in \boldsymbol{R},\ x_i \in \boldsymbol{F} \tag{4.1}$$

of $H_3(\boldsymbol{F})$ will be denoted by

$$u = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + x_1 u_1 + x_2 u_2 + x_3 u_3 .$$

We define a cubic form N on $H_3(\boldsymbol{F})$, called the *norm* of the Jordan algebra $H_3(\boldsymbol{F})$, by

$$N(u) = \xi_1 \xi_2 \xi_3 - \sum_{i=1}^{3} \xi_i n(x_i) + t(x_1 x_2 x_3)$$

for the above u. The norm N is invariant by the group $\mathrm{Aut}\,(H_3(\boldsymbol{F}))$ of automorphisms of the algebra $H_3(\boldsymbol{F})$. We define an $\mathrm{Aut}\,(H_3(\boldsymbol{F}))$-invariant inner product $(\ ,\ )$ on $H_3(\boldsymbol{F})$ by

$$(u,\ v) = \frac{1}{2}\,\mathrm{Tr}\,(u \circ v) \qquad \text{for} \quad u,\ v \in H_3(\boldsymbol{F}) ,$$

and $\mathrm{Aut}\,(H_3(\boldsymbol{F}))$-invariant subspace $\mathfrak{p}$ of $H_3(\boldsymbol{F})$ by

$$\mathfrak{p} = \{u \in H_3(\boldsymbol{F}) \mid (u,\ \boldsymbol{R}1_3) = 0\} = \{u \in H_3(\boldsymbol{F}) \mid \mathrm{Tr}\, u = 0\} .$$

The inner product $(\ ,\ )$ defines a Euclidean space structure on $\mathfrak{p}$ of dimension $N = 3 \dim \boldsymbol{F} + 2$. For $u \in M_3(\boldsymbol{F})$ we define $T(u) \in \boldsymbol{F}$ by

$$T(u) = \begin{cases} t\,(\mathrm{Tr}\, u) & \boldsymbol{F} = \boldsymbol{H} , \\ \mathrm{Tr}\, u & \text{otherwise} , \end{cases}$$

and put

$$SH_3(\boldsymbol{F}) = \{u \in M_3(\boldsymbol{F});\ \bar{u}' = -u,\ T(u) = 0\} .$$

Injective linear maps $R\colon H_3(\boldsymbol{F}) \to \mathfrak{gl}(H_3(\boldsymbol{F}))$ and $D\colon SH_3(\boldsymbol{F}) \to \mathfrak{gl}(H_3(\boldsymbol{F}))$ are defined by

$$(4.2) \qquad \begin{cases} R(u)v = u \circ v = \dfrac{1}{2}(uv + vu) & \text{for} \quad u,\ v \in H_3(\boldsymbol{F}) , \\[2ex] D(u)v = \dfrac{1}{2}(uv - vu) & \text{for} \quad u \in SH_3(\boldsymbol{F}),\ v \in H_3(\boldsymbol{F}) . \end{cases}$$

Let $\mathfrak{k}$ denote the subalgebra of $\mathfrak{gl}(H_3(\boldsymbol{F}))$ generated by $D(SH_3(\boldsymbol{F}))$. Then $\mathfrak{k}$ is a compact simple Lie algebra of type B_1, A_2, C_3 or F_4 according to $\boldsymbol{F} = \boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}$ or $\boldsymbol{K}$. (See also the next section.) We have relations:

$$[D,\ R(u)] = R(D(u)) \qquad \text{for} \quad D \in \mathfrak{k},\ u \in H_3(\boldsymbol{F}) .$$

We identify $\mathfrak{p}$ with $R(\mathfrak{p})$ through the injective map R. Then

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

is a subalgebra of $\mathfrak{gl}(H_3(\boldsymbol{F}))$ and these Lie algebras exhaust non-compact simple Lie algebras of rank 2 with $g = 3$. Furthermore the above decomposition of $\mathfrak{g}$ is a Cartan decomposition, and the inner product $(\ ,\ )$ on $\mathfrak{p}$

is a positive multiple of the Killing form of $\mathfrak{g}$. The image $\rho(K)$ of the associated s-representation $\rho\colon K \to SO(\mathfrak{p})$ coincides with the restriction to $\mathfrak{p}$ of the identity component of the group $\mathrm{Aut}(H_3(\boldsymbol{F}))$. Thus $N|\mathfrak{p}$ is a homogeneous K-invariant polynomial on $\mathfrak{p}$ of degree 3. As for these properties of the Jordan algebra $H_3(\boldsymbol{F})$, we refer to Schafer [14].

Now we choose

$$\mathfrak{a} = \{\sum \xi_i e_i \mid \sum \xi_i = 0\}$$

as a maximal abelian subalgebra in $\mathfrak{p}$. A linear form $\sum \xi_i e_i \mapsto \xi_i$ on $\mathfrak{a}$ will be denoted by ξ_i. Such notations will be often used in the sequel. Then Σ is given by

$$\Sigma = \left\{\frac{1}{2}(\xi_i - \xi_j) \mid i, j = 1, 2, 3,\ i \neq j\right\}.$$

We introduce an order $>$ satisfying $\xi_1 < \xi_2 < \xi_3$. Then Σ_*^+ consists of 3 roots $\gamma_1 = (1/2)(\xi_2 - \xi_1)$, $\gamma_2 = (1/2)(\xi_3 - \xi_1)$ and $\gamma_3 = (1/2)(\xi_3 - \xi_2)$. We have $m_1 = m_2 = m_3 = \dim \boldsymbol{F}$. Linear forms

$$\lambda_1 = -\xi_1 - \frac{1}{2}\xi_2\,, \quad \lambda_2 = -\frac{\sqrt{3}}{2}\xi_2$$

give a required orthonormal coordinate system for $\mathfrak{a}$, and hence

$$F_0 = 3\lambda_1^2\lambda_2 - \lambda_2^3 = \frac{3\sqrt{3}}{2}\xi_1\xi_2\xi_3\,.$$

Thus

$$F(u) = \frac{3\sqrt{3}}{2}N(u) \qquad \text{for} \quad u \in \mathfrak{p}$$

is a required polynomial for $\mathfrak{g}$. These polynomials were given in Cartan [3].

Case $g = 4$:

(i) Let $\boldsymbol{F}$ be an associative division algebra over $\boldsymbol{R}$, i.e., $\boldsymbol{F} = \boldsymbol{R}, \boldsymbol{C}$ or $\boldsymbol{H}$, and r an integer such that $r \geqq 3$ for $\boldsymbol{F} = \boldsymbol{R}$ or $\boldsymbol{C}$ and $r \geqq 2$ for $\boldsymbol{F} = \boldsymbol{H}$. We consider a non-compact simple Lie algebra

$$\mathfrak{g} = \{A \in \mathfrak{gl}(r+2, \boldsymbol{F}) \mid T(A) = 0,\ \bar{A}'\Phi + \Phi A = 0\}\,,$$

where $T(A)$ is defined in the same way as in case $g = 3$ and

$$\Phi = \left(\begin{array}{c|c} 1_2 & \\ \hline & -1_r \end{array}\right).$$

The linear map $A \mapsto -\bar{A}'$ of $\mathfrak{g}$ is a Cartan involution of $\mathfrak{g}$. We denote by $M_{r,2}(\boldsymbol{F})$ the space of $r \times 2$ matrices with coefficients in $\boldsymbol{F}$, and define $\hat{X} \in M_{r+2}(\boldsymbol{F})$ for $X \in M_{r,2}(\boldsymbol{F})$ by

$$\hat{X} = \begin{pmatrix} 0 & \bar{X}' \\ X & 0 \end{pmatrix} .$$

Then (-1)-eigenspace $\mathfrak{p}$ of the above Cartan involution is given by

$$\mathfrak{p} = \{\hat{X} \mid X \in M_{r,2}(\boldsymbol{F})\} .$$

We define an inner product $(\ ,\ )$ on $\mathfrak{p}$ by

$$(\hat{X},\ \hat{Y}) = \frac{1}{2}\mathfrak{Re}\ \mathrm{Tr}\ \hat{X}\hat{Y} = \mathfrak{Re}\ \mathrm{Tr}\ \bar{X}'Y \quad \text{for } X,\ Y \in M_{r,2}(\boldsymbol{F}) .$$

It is a positive multiple of the Killing form of $\mathfrak{g}$. The associated s-representation $\rho: K \longrightarrow SO(\mathfrak{p})$ is lifted to a covering group $\tilde{K}$ of K as follows: Let

$$\tilde{K} = \begin{cases} SO(2) \times SO(r) & \boldsymbol{F} = \boldsymbol{R} , \\ S(U(2) \times U(r)) & \boldsymbol{F} = \boldsymbol{C} , \\ Sp(2) \times Sp(r) & \boldsymbol{F} = \boldsymbol{H} . \end{cases}$$

Define a homomorphism $\tilde{\rho}: \tilde{K} \longrightarrow SO(\mathfrak{p})$ by

$$\tilde{\rho}(k_1 \times k_2)\hat{X} = \widehat{k_2 X k_1^{-1}} \quad \text{for} \quad k_1 \times k_2 \in \tilde{K},\ X \in M_{r,2}(\boldsymbol{F}) .$$

Then there exists a covering homomorphism $\pi: \tilde{K} \longrightarrow K$ such that $\rho(\pi(k)) = \tilde{\rho}(k)$ for each $k \in \tilde{K}$. Denoting by $\{E_{ij}\}$ the standard basis of $M_n(\boldsymbol{F})$ over $\boldsymbol{F}$, we put

$$H(\xi_1,\ \xi_2) = \xi_1(E_{31} + E_{13}) + \xi_2(E_{42} + E_{24}) \qquad \text{for} \quad \xi_1,\ \xi_2 \in \boldsymbol{R} .$$

Then

$$\mathfrak{a} = \{H(\xi_1,\ \xi_2) \mid \xi_1,\ \xi_2 \in \boldsymbol{R}\}$$

is a maximal abelian subalgebra in $\mathfrak{p}$ and $\{\xi_1,\ \xi_2\}$ is an orthonormal coordinate system for $\mathfrak{a}$. We have

$$\Sigma = \{\pm(\xi_1 \pm \xi_2),\ \pm\xi_1,\ \pm\xi_2,\ \pm 2\xi_1,\ \pm 2\xi_2\} .$$

We introduce an order $>$ satisfying $\xi_1 > \xi_2 > 0$. Then Σ_*^+ consists of 4-roots

(4.3) $$\gamma_1 = \xi_1 - \xi_2,\ \gamma_2 = \xi_1,\ \gamma_3 = \xi_1 + \xi_2,\ \gamma_4 = \xi_2 ,$$

and

$$(m_1, m_2) = \begin{cases} (1, r-2) & \boldsymbol{F} = \boldsymbol{R}\,, \\ (2, 2r-3) & \boldsymbol{F} = \boldsymbol{C}\,, \\ (4, 4r-5) & \boldsymbol{F} = \boldsymbol{H}\,. \end{cases}$$

Linear forms

$$\lambda_1 = \frac{\sqrt{2+\sqrt{2}}}{2}\xi_1 + \frac{\sqrt{2-\sqrt{2}}}{2}\xi_2, \ \lambda_2 = -\frac{\sqrt{2-\sqrt{2}}}{2}\xi_1 + \frac{\sqrt{2+\sqrt{2}}}{2}\xi_2$$

constitute a required orthonormal coordinate system for $\mathfrak{a}$, and hence

$$F_0 = 4\lambda_1^3\lambda_2 - 4\lambda_1\lambda_2^3 = 3(\xi_1^2 + \xi_2^2)^2 - 4(\xi_1^2 + \xi_2^4)\,. \tag{4.4}$$

We define a polynomial F on $\mathfrak{p}$ by

$$F(Z) = \frac{3}{4}(\operatorname{Tr} Z^2)^2 - 2\operatorname{Tr}(Z^4) \qquad \text{for} \quad Z \in \mathfrak{p}\,.$$

Then F is invariant by $\tilde{K}$ and coincides with F_0 on $\mathfrak{a}$. Thus F is a required polynomial. The polynomial F for $\boldsymbol{F} = \boldsymbol{R}$ is equivalent to the polynomial F for $m_1 = 1$ given in Theorem 2, (ii) of Part I.

(ii) Let $1, i, j, k$ be the standard units of $\boldsymbol{H}$. We identify $\boldsymbol{C}$ with a subalgebra of $\boldsymbol{H}$ by the natural map $x + \sqrt{-1}y \mapsto x1 + yi$. This identification induces an identification $\mathfrak{gl}(n, \boldsymbol{C}) \subset \mathfrak{gl}(n, \boldsymbol{H})$. We consider a non-compact simple Lie algebra

$$\mathfrak{g} = \{A \in \mathfrak{gl}(5, \boldsymbol{H}) \mid \bar{A}'\Psi + \Psi A = 0\} \quad \text{where} \quad \Psi = \sqrt{-1}\, 1_5\,.$$

The linear map $A \mapsto -\bar{A}'$ of $\mathfrak{g}$ is a Cartan involution of $\mathfrak{g}$ and the associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is given by

$$\mathfrak{k} = \check{\mathfrak{u}}(5)\,,$$
$$\mathfrak{p} = \{jZ \mid Z \in M_5(\boldsymbol{C}),\ Z' = -Z\}\,.$$

We identify $\mathfrak{p}$ with the space of complex skew-symmetric matrices of degree 5 by the map $jZ \mapsto Z$.

Next let $\mathfrak{g} = \mathfrak{o}(5, \boldsymbol{C})$, considered as a real Lie algebra. The linear map $A \mapsto \bar{A}$ of $\mathfrak{g}$ is a Cartan involution of $\mathfrak{g}$ and the associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is given by

$$\mathfrak{k} = \mathfrak{o}(5)\,,$$
$$\mathfrak{p} = \sqrt{-1}\mathfrak{o}(5) = \{\sqrt{-1}Z \mid Z \in M_5(\boldsymbol{R}),\ Z' = -Z\}\,.$$

We identify also $\mathfrak{p}$ with the space of real skew-symmetric matrices of degree 5 by the map $\sqrt{-1}Z \mapsto Z$.

In the following, we shall consider the above two Lie algebras $\mathfrak{g}$

simultaneously. We define an inner product on

$$\mathfrak{p} = \{Z \in M_5(\boldsymbol{F}) \mid Z' = -Z\} \qquad \boldsymbol{F} = \boldsymbol{R} \text{ or } \boldsymbol{C},$$

by

$$(Z, W) = -\frac{1}{2}\Re\mathrm{e}\,\mathrm{Tr}\,(Z\bar{W}) \qquad \text{for } Z, W \in \mathfrak{p}.$$

It is a positive multiple of the Killing form of $\mathfrak{g}$. Let

$$\tilde{K} = \begin{cases} SO(5) & \boldsymbol{F} = \boldsymbol{R}, \\ U(5) & \boldsymbol{F} = \boldsymbol{C}. \end{cases}$$

Then the associated s-representation $\rho: K \to SO(\mathfrak{p})$ is covered by the homomorphism $\tilde{\rho}: \tilde{K} \to SO(\mathfrak{p})$ defined by

$$\tilde{\rho}(k)Z = \bar{k}Zk^{-1} \qquad \text{for } k \in \tilde{K},\ Z \in \mathfrak{p}.$$

We put

$$H(\xi_1, \xi_2) = \xi_1(E_{21} - E_{12}) + \xi_2(E_{43} - E_{34}) \qquad \text{for } \xi_1, \xi_2 \in \boldsymbol{R}.$$

Then

$$\mathfrak{a} = \{H(\xi_1, \xi_2) \mid \xi_1, \xi_2 \in \boldsymbol{R}\}$$

is a maximal abelian subalgebra in $\mathfrak{p}$ and $\{\xi_1, \xi_2\}$ is an orthonormal coordinate system for $\mathfrak{a}$. We introduce an order $>$ satisfying $\xi_1 > \xi_2 > 0$. Then Σ_*^+ consists of 4 roots of the same form as (4.3), and

$$(m_1, m_2) = \begin{cases} (2, 2) & \boldsymbol{F} = \boldsymbol{R}, \\ (4, 5) & \boldsymbol{F} = \boldsymbol{C}. \end{cases}$$

Hence F_0 has the same form as (4.4). We define a polynomial F on $\mathfrak{p}$ by

$$F(Z) = \frac{3}{4}(\mathrm{Tr}\, Z\bar{Z})^2 - 2\,\mathrm{Tr}\,(Z\bar{Z})^2 \qquad \text{for } Z \in \mathfrak{p}.$$

Then F is invariant by $\tilde{K}$ and coincides with F_0 on $\mathfrak{a}$, and hence F is a required polynomial.

(iii) It remains a non-compact simple Lie algebra of type EIII among non-compact simple Lie algebras of rank 2 with $g = 4$. The polynomial F for this Lie algebra will be computed in the next section.

Case $g = 6$:

Let $c_1, \cdots, c_7$ be the standard pure imaginary units of the real Cayley algebra $\boldsymbol{K}$. They satisfy the relations:

$$\begin{aligned} c_i c_{i+1} &= -c_{i+1}c_i = c_{i+3}, \quad c_{i+1}c_{i+3} = -c_{i+3}c_{i+1} = c_i, \\ c_{i+3}c_i &= -c_i c_{i+3} = c_{i+1}, \quad c_i^2 = -1 \quad \text{for } i \in \boldsymbol{Z}_7. \end{aligned}$$

A linear map of $\boldsymbol{K}$ will be represented by a matrix with respect to the basis $\{1, c_1, \cdots, c_7\}$ of $\boldsymbol{K}$. Then the group $\mathrm{Aut}(\boldsymbol{K})$ of automorphisms of the algebra $\boldsymbol{K}$ is a compact simply connected subgroup of $O(8)$ and the Lie algebra $\mathfrak{G}$ of $\mathrm{Aut}(\boldsymbol{K})$ is described as follows (cf. Borel-Hirzebruch [1]). Put

$$G_{ij} = E_{ij} - E_{ji} \qquad \text{for} \quad i, j = 1, \cdots, 7,\ i \neq j$$

and

$$\mathfrak{G}_i = \{\eta_1 G_{i+1,i+3} + \eta_2 G_{i+2,i+6} + \eta_3 G_{i+4,i+5} \mid \eta_i \in \boldsymbol{R}, \textstyle\sum \eta_i = 0\} \qquad \text{for} \quad i = 1, \cdots, 7 .$$

Then $\mathfrak{G}$ has a direct sum decomposition:

$$\mathfrak{G} = \sum_{i=1}^{7} \mathfrak{G}_i$$

with commutation relations:

$$[\mathfrak{G}_i, \mathfrak{G}_i] = \{0\}, \quad [\mathfrak{G}_i, \mathfrak{G}_{i+1}] = \mathfrak{G}_{i+3},$$
$$[\mathfrak{G}_{i+1}, \mathfrak{G}_{i+3}] = \mathfrak{G}_i, \quad [\mathfrak{G}_{i+3}, \mathfrak{G}_i] = \mathfrak{G}_{i+1}.$$

$\mathfrak{G}$ is a compact simple Lie algebra of type G_2. We put

$$\mathfrak{k} = \mathfrak{G}_3 + \mathfrak{G}_4 + \mathfrak{G}_6,$$
$$\mathfrak{p}_u = \mathfrak{G}_1 + \mathfrak{G}_2 + \mathfrak{G}_5 + \mathfrak{G}_7.$$

It follows from the above relations that $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}_u] \subset \mathfrak{p}_u$ and $[\mathfrak{p}_u, \mathfrak{p}_u] \subset \mathfrak{k}$. The connected subgroup of $\mathrm{Aut}(\boldsymbol{K})$ generated by $\mathfrak{k}$ is isomorphic to $SO(4)$. We define a real subalgebra $\mathfrak{g}$ of the complexification $\mathfrak{G}^C$ of $\mathfrak{G}$ by

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} \qquad \text{where} \quad \mathfrak{p} = \sqrt{-1}\mathfrak{p}_u .$$

Then $\mathfrak{g}$ is a non-compact simple Lie algebra of type GI and the above decomposition is a Cartan decomposition of $\mathfrak{g}$. We identify $\mathfrak{p}$ with $\mathfrak{p}_u$ by the map $\sqrt{-1}X \mapsto X$.

Next we consider $\mathfrak{g} = \mathfrak{G}^C$ as a real Lie algebra. As for $\mathfrak{g} = \mathfrak{o}(5, \boldsymbol{C})$ in case $g = 4$, (ii), we have a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ by $\mathfrak{k} = \mathfrak{G}$ and $\mathfrak{p} = \sqrt{-1}\mathfrak{G}$. We also identify $\mathfrak{p}$ with $\mathfrak{G}$ by the map $\sqrt{-1}X \mapsto X$.

The above two Lie algebras exhaust non-compact simple Lie algebras of rank 2 with $g = 6$. In the following, we shall consider these Lie algebras simultaneously. We define an inner product $(\ ,\ )$ on $\mathfrak{p} \subset \mathfrak{o}(8)$, which is a positive multiple of the Killing form of $\mathfrak{g}$, by

$$(X, Y) = -\frac{1}{2}\mathrm{Tr}(XY) \qquad \text{for} \quad X, Y \in \mathfrak{p} .$$

We put

$$H(\xi_1, \xi_2, \xi_3) = \xi_1 G_{24} + \xi_2 G_{37} + \xi_3 G_{56} \quad \text{for} \quad \xi_i \in \boldsymbol{R}, \sum \xi_i = 0 .$$

Then $(H(\xi_1, \xi_2, \xi_3), H(\xi_1, \xi_2, \xi_3)) = \xi_1^2 + \xi_2^2 + \xi_3^2$ and

$$\mathfrak{a} = \{H(\xi_1, \xi_2, \xi_3) \mid \xi_i \in \boldsymbol{R}, \sum \xi_i = 0\}$$

is a maximal abelian subalgebra in $\mathfrak{p}$. We introduce an order satisfying $0 > \xi_2 > \xi_3$. Then Σ_*^+ consists of 6 roots $\gamma_1 = -\xi_2, \gamma_2 = \xi_1 - \xi_2, \gamma_3 = \xi_1$, $\gamma_4 = \xi_1 - \xi_3, \gamma_5 = -\xi_3$ and $\gamma_6 = \xi_2 - \xi_3$. We have $m_1 = m_2 = 1$ or 2, according to $\mathfrak{g} = GI$ or $\mathfrak{G}^C$. Linear forms

$$\lambda_1 = \frac{\sqrt{3}+1}{2}\xi_1 + \frac{\sqrt{3}-1}{2}\xi_2 , \quad \lambda_2 = \frac{\sqrt{3}-1}{2}\xi_1 + \frac{\sqrt{3}+1}{2}\xi_2$$

define a required orthonormal coordinate system for $\mathfrak{a}$. A computation shows

$$\begin{aligned} F_0 &= 6\lambda_1^5\lambda_2 - 20\lambda_1^3\lambda_2^3 + 6\lambda_1\lambda_2^5 \\ &= 10(\xi_1^2 + \xi_2^2 + \xi_3^2)^2 - 36(\xi_1^6 + \xi_2^6 + \xi_3^6) . \end{aligned}$$

We define a polynomial F on $\mathfrak{p}$ by

$$F(X) = -\frac{5}{4}(\operatorname{Tr} X^2)^3 + 18 \operatorname{Tr}(X^6) \qquad \text{for} \quad X \in \mathfrak{p} .$$

Then F is invariant by the connected subgroup K of $\operatorname{Ad}\mathfrak{g}$ generated by $\mathfrak{k}$. Furthermore it coincides with F_0 on $\mathfrak{a}$. Thus F is a required polynomial.

5. Defining polynomials for homogeneous hypersurfaces in spheres—II. Let $\boldsymbol{K}$ be the real Cayley algebra and $c_0 = 1, c_1, \cdots, c_7$ the standard units of $\boldsymbol{K}$ as in the previous section. Let $x \mapsto \bar{x}$ be the canonical involution of $\boldsymbol{K}$, $(\, , \,)$ the canonical inner product on $\boldsymbol{K}$. We extend them $\boldsymbol{C}$-linearly to the complexified algebra $\boldsymbol{K}^C$ of $\boldsymbol{K}$ and denote them by the same notations $x \mapsto \bar{x}$ and $(\, , \,)$ respectively. Denoting by $x \mapsto \tilde{x}$ the complex conjugation of $\boldsymbol{K}^C$ with respect to $\boldsymbol{K}$, we define a hermitian inner product $\langle\!\langle \, , \, \rangle\!\rangle$ on $\boldsymbol{K}^C$ by

$$\langle\!\langle x, y \rangle\!\rangle = (x, \tilde{y}) \qquad \text{for} \quad x, y \in \boldsymbol{K}^C .$$

This satisfies

$$\langle\!\langle x, y \rangle\!\rangle = \langle\!\langle \bar{x}, \bar{y} \rangle\!\rangle \qquad \text{for} \quad x, y \in \boldsymbol{K}^C .$$

In general, a complex vector space V, considered as a real vector space, will be denoted by $V_{\boldsymbol{R}}$. We define an inner product $((\, , \,))$ on $(\boldsymbol{K}^C)_{\boldsymbol{R}}$ by

$$((x, y)) = \mathfrak{Re}\, \langle\!\langle x, y \rangle\!\rangle \qquad \text{for} \quad x, y \in \boldsymbol{K}^C$$

and denote the associated norm by $\| \ \|$.

Let $H_3(\boldsymbol{K})$ be the compact simple Jordan algebra defined in §4 and $(\ ,\ )$ the inner product on $H_3(\boldsymbol{K})$ defined there. We extend the form $(\ ,\ )$ $\boldsymbol{C}$-linearly to the complexified Jordan algebra $H_3(\boldsymbol{K})^C$ and denote it by the same notation $(\ ,\ )$. It satisfies

$$(u \circ v,\, w) = (v,\, u \circ w) \qquad \text{for} \quad u,\, v,\, w \in H_3(\boldsymbol{K})^C . \tag{5.1}$$

$H_3(\boldsymbol{K})^C$ is canonically identified with

$$H_3(\boldsymbol{K}^C) = \{u \in M_3(\boldsymbol{K}^C) \,|\, \bar{u}' = u\} .$$

In the same way, the complexification $SH_3(\boldsymbol{K})^C$ of the space $SH_3(\boldsymbol{K})$ defined in §4, is identified with

$$SH_3(\boldsymbol{K}^C) = \{u \in M_3(\boldsymbol{K}^C) \,|\, \bar{u}' = -u,\ \mathrm{Tr}\, u = 0\} .$$

We also define a hermitian inner product $\langle\!\langle\ ,\ \rangle\!\rangle$ on $H_3(\boldsymbol{K})^C$ by

$$\langle\!\langle u,\, v \rangle\!\rangle = (u,\, \tilde{v}) \qquad \text{for} \quad u,\, v \in H_3(\boldsymbol{K})^C ,$$

denoting by $u \mapsto \tilde{u}$ the complex conjugation of $H_3(\boldsymbol{K})^C$ with respect to $H_3(\boldsymbol{K})$. An element $u \in H_3(\boldsymbol{K})^C$ of the form (4.1), with $\xi_i \in \boldsymbol{C}$, $x_i \in \boldsymbol{K}^C$, is denoted by

$$u = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + x_1 u_1 + x_2 u_2 + x_3 u_3 ,$$

and an element $u \in SH_3(\boldsymbol{K})^C$ of the form

$$u = \begin{pmatrix} z_1 & x_3 & -\bar{x}_2 \\ -\bar{x}_3 & z_2 & x_1 \\ x_2 & -\bar{x}_1 & z_3 \end{pmatrix} \quad z_i,\, x_i \in \boldsymbol{K}^C,\ \bar{z}_i = -z_i,\ \sum z_i = 0$$

is denoted by

$$u = z_1 e_1 + z_2 e_2 + z_3 e_3 + x_1 \bar{u}_1 + x_2 \bar{u}_2 + x_3 \bar{u}_3 .$$

We identify the Lie algebra $\mathfrak{gl}(H_3(\boldsymbol{K}))$ of $\boldsymbol{R}$-linear maps of $H_3(\boldsymbol{K})$ with a real subalgebra of the Lie algebra $\mathfrak{gl}(H_3(\boldsymbol{K})^C)$ of $\boldsymbol{C}$-linear maps of $H_3(\boldsymbol{K})^C$. $R(u) \in \mathfrak{gl}(H_3(\boldsymbol{K})^C)$ for $u \in H_3(\boldsymbol{K})^C$ and $D(u) \in \mathfrak{gl}(H_3(\boldsymbol{K})^C)$ for $u \in SH_3(\boldsymbol{K})^C$ are defined by the same formula as (4.2). Let $\mathfrak{D}_0$ denote the subalgebra of $\mathfrak{gl}(H_3(\boldsymbol{K}))$ generated by the set $\{D(\sum z_i e_i) \,|\, z_i \in \boldsymbol{K},\ \bar{z}_i = -z_i,\ \sum z_i = 0\}$, and let

$$\begin{aligned} \mathfrak{D}_i &= \{D(x\bar{u}_i) \,|\, x \in \boldsymbol{K}\} && \text{for} \quad i = 1, 2, 3 , \\ \mathfrak{R}_0 &= \{R(\textstyle\sum \xi_i e_i) \,|\, \xi_i \in \boldsymbol{R},\ \sum \xi_i = 0\} , && \\ \mathfrak{R}_i &= \{R(x u_i) \,|\, x \in \boldsymbol{K}\} && \text{for} \quad i = 1, 2, 3 . \end{aligned}$$

We put

$$\mathfrak{D} = \mathfrak{D}_0 + \mathfrak{D}_1 + \mathfrak{D}_2 + \mathfrak{D}_3 ,$$
$$\mathfrak{R} = \mathfrak{R}_0 + \mathfrak{R}_1 + \mathfrak{R}_2 + \mathfrak{R}_3 .$$

Then $\mathfrak{D}$ is a subalgebra of $\mathfrak{gl}(H_3(\boldsymbol{K}))$ and a compact simple Lie algebra of type F_4. Denoting by $\mathfrak{D}^C$ and $\mathfrak{R}^C$ the complexifications of $\mathfrak{D}$ and $\mathfrak{R}$ respectively, we put

$$\mathfrak{g}^C = \mathfrak{D}^C + \mathfrak{R}^C .$$

Then $\mathfrak{g}^C$ is a subalgebra of $\mathfrak{gl}(H_3(\boldsymbol{K})^C)$ and a complex simple Lie algebra of type E_6. The inclusion $\varphi\colon \mathfrak{g}^C \subset \mathfrak{gl}(H_3(\boldsymbol{K})^C)$ is a 27-dimensional irreducible representation of $\mathfrak{g}^C$. We define a real form $\mathfrak{g}$ of $\mathfrak{g}^C$ by

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} ,$$

where

$$\mathfrak{k} = \mathfrak{D}_0 + \mathfrak{D}_1 + \sqrt{-1}\mathfrak{R}_0 + \sqrt{-1}\mathfrak{R}_1 ,$$
$$\mathfrak{p} = \sqrt{-1}\mathfrak{D}_2 + \sqrt{-1}\mathfrak{D}_3 + \mathfrak{R}_2 + \mathfrak{R}_3 .$$

Then $\mathfrak{g}$ is a non-compact simple Lie algebra of type EIII and the above decomposition is a Cartan decomposition of $\mathfrak{g}$. Note that the hermitian inner product $\langle\!\langle\ ,\ \rangle\!\rangle$ on $H_3(\boldsymbol{K})^C$ is invariant by the compact dual $\mathfrak{g}_u = \mathfrak{k} + \sqrt{-1}\mathfrak{p}$ of $\mathfrak{g}$ in virtue of (5.1). $\mathfrak{k}$ is isomorphic to $\mathfrak{o}(2) \oplus \mathfrak{o}(8)$ and

$$[\mathfrak{k}, \mathfrak{k}] = \mathfrak{D}_0 + \mathfrak{D}_1 + \sqrt{-1}\boldsymbol{R}R(e_2 - e_3) + \sqrt{-1}\mathfrak{R}_1$$

is isomorphic to $\mathfrak{o}(8)$. We put

$$Z = \frac{2}{3}R(2e_1 - e_2 - e_3) .$$

Then the center of $\mathfrak{k}$ is spanned by $\sqrt{-1}Z$. The eigenvalues of $\operatorname{ad} Z$ on $\mathfrak{g}^C$ are $0, 1$ and -1 and the complexification $\mathfrak{p}^C$ of $\mathfrak{p}$ is decomposed into the direct sum:

$$\mathfrak{p}^C = \mathfrak{p}^+ + \mathfrak{p}^-$$

of the eigenspaces $\mathfrak{p}^\pm$ for ± 1 of $\operatorname{ad} Z$. We define subspaces V_1, V_2 and V_3 of $H_3(\boldsymbol{K})^C$ by

$$V_1 = \{\xi_1 e_1 \mid \xi_1 \in \boldsymbol{C}\} , \quad \dim V_1 = 1 ,$$
$$V_2 = \{x_2 u_2 + x_3 u_3 \mid x_2, x_3 \in \boldsymbol{K}^C\} , \quad \dim V_2 = 16 ,$$
$$V_3 = \{\xi_2 e_2 + \xi_3 e_3 + x_1 u_1 \mid \xi_2, \xi_3 \in \boldsymbol{C}, x_1 \in \boldsymbol{K}^C\} , \quad \dim V_3 = 10 .$$

Then we have an orthogonal (with respect to $\langle\!\langle\ ,\ \rangle\!\rangle$) direct sum decomposition:

$$H_3(\boldsymbol{K})^C = V_1 + V_2 + V_3$$

of $H_3(\boldsymbol{K})^C$. Each V_i is a $\mathfrak{k}$-invariant $\mathfrak{k}$-irreducible subspace of $H_3(\boldsymbol{K})^C$. We have

$$\varphi(Z)|V_1 = \frac{4}{3}1_{V_1},\ \varphi(Z)|V_2 = \frac{1}{3}1_{V_2},\ \varphi(Z)|V_3 = -\frac{2}{3}1_{V_3}\ .$$

Since $\varphi(Z)\varphi(X)u = \varphi([Z, X])u + \varphi(X)\varphi(Z)u = \varphi(X)u + \varphi(X)\varphi(Z)u$ for each $X \in \mathfrak{p}^+$ and $u \in H_3(\boldsymbol{K})^C$, we have $\varphi(X)V_1 = \{0\}$, $\varphi(X)V_2 \subset V_1$ and $\varphi(X)V_3 \subset V_2$ for each $X \in \mathfrak{p}^+$. Hence each $X \in \mathfrak{p}^+$ has a unique decomposition:

(5.2) $\varphi(X) = X_{12} + X_{23}$ with $X_{12} \in \mathrm{Hom}\,(V_2, V_1)$, $X_{23} \in \mathrm{Hom}\,(V_3, V_2)$,

where $\mathrm{Hom}\,(V_i, V_j)$ denotes the space of linear maps of V_i into V_j. As for these properties of the representation φ, we refer to Schafer [14], Ise [9].

Now let $\tilde{G}^C$ denote the simply connected complex Lie group with the Lie algebra $\mathfrak{g}^C$, $\tilde{K}$ the connected subgroup of $\tilde{G}^C$ generated by $\mathfrak{k}$. The extension of φ to $\tilde{G}^C$ will be also denoted by $\varphi: \tilde{G} \to GL(H_3(\boldsymbol{K})^C)$. The connected subgroup of $\mathrm{Ad}\,\mathfrak{g}$ generated by $\mathfrak{k}$ is denoted by K. Making use of the decomposition (5.2), we define a polynomial function F_1 on $(\mathfrak{p}^+)_R$ of degree 2 by

$$F_1(X) = \frac{1}{2}\,\mathrm{Tr}\,(X_{12}X_{12}^*) \qquad \text{for} \quad X \in \mathfrak{p}^+\ ,$$

where $X_{12}^* \in \mathrm{Hom}\,(V_1, V_2)$ is the adjoint operator of $X_{12} \in \mathrm{Hom}\,(V_2, V_1)$ with respect to the hermitian inner product $\langle\!\langle\ ,\ \rangle\!\rangle$. It follows from the $\mathfrak{k}$-invariance of $\langle\!\langle\ ,\ \rangle\!\rangle$ that for $k \in K$, $X \in \mathfrak{p}^+$ we have

$$\begin{aligned} F_1(kX) &= \frac{1}{2}\,\mathrm{Tr}\,(\varphi(\tilde{k})X_{12}\varphi(\tilde{k})^{-1}(\varphi(\tilde{k})X_{12}\varphi(\tilde{k})^{-1})^*) \\ &= \frac{1}{2}\,\mathrm{Tr}\,(\varphi(\tilde{k})X_{12}X_{12}^*\varphi(\tilde{k})^{-1}) = F_1(X), \end{aligned}$$

where $\tilde{k}$ is an element of $\tilde{K}$ such that $\mathrm{Ad}\,\tilde{k} = k$. Thus F_1 is a K-invariant polynomial on $(\mathfrak{p}^+)_R$ of degree 2. In the similar way we define

$$F_2(X) = \mathrm{Tr}\,((X_{12}X_{23})(X_{12}X_{23})^*) \qquad \text{for} \quad X \in \mathfrak{p}^+\ .$$

Then it is verified in the same way that F_2 is also a K-invariant polynomial on $(\mathfrak{p}^+)_R$ of degree 4. It will be shown later that the linear map $X \mapsto X_{12}$ of $\mathfrak{p}^+$ into $\mathrm{Hom}\,(V_2, V_1)$ is injective. Let $((\ ,\ ))$ be a K-invariant inner product on $(\mathfrak{p}^+)_R$ such that $((X, X)) = F_1(X)$ for each $X \in \mathfrak{p}^+$. We define a K-equivariant linear isomorphism $\psi: (\mathfrak{p}^+)_R \to \mathfrak{p}$ by

$$\psi(X) = \frac{1}{\sqrt{2}}(X + \tilde{X}) \qquad \text{for} \quad X \in \mathfrak{p}^+ ,$$

where $X \mapsto \tilde{X}$ denotes the complex conjugation of $\mathfrak{g}^C$ with respect to $\mathfrak{g}$. Making use of the map ψ, we define an inner product (,) on $\mathfrak{p}$ by

$$(X, Y) = ((\psi^{-1}X, \psi^{-1}Y)) \qquad \text{for } X, Y \in \mathfrak{p} .$$

It is K-invariant and hence a positive multiple of the Killing form of $\mathfrak{g}$.

Now we shall compute explicitly the polynomials F_1 and F_2. First we give below a list of necessary commutation rules for $\mathfrak{g}^C$. In the following list, $x, y \in \boldsymbol{K}^C$ and $\xi_1, \xi_2, \xi_3 \in \boldsymbol{C}$ with $\sum \xi_i = 0$. In formulae (1) $\sim$ (6), (i, j, k) is a cyclic permutation of (1, 2, 3). In formulae (7) and (8), $i = 1, 2,$ or 3.

(1) $[R(xu_i), R(yu_j)] = -(1/2)D(\overline{xy}\,\bar{u}_k)$,

(2) $[R(xu_i), D(y\bar{u}_j)] = [D(x\bar{u}_i), R(yu_j)] = (1/2)R(\overline{xy}\,u_k)$,

(3) $[D(x\bar{u}_i), D(y\bar{u}_j)] = -(1/2)D(\overline{xy}\,\bar{u}_k)$,

(4) $[D(x\bar{u}_i), R(yu_i)] = (x, y)R(e_j - e_k)$,

(5) $[R(\sum \xi_l e_l), R(xu_i)] = (1/2)(\xi_j - \xi_k)D(x\bar{u}_i)$,

(6) $[R(\sum \xi_l e_l), D(x\bar{u}_i)] = (1/2)(\xi_j - \xi_k)R(xu_i)$,

(7) $[R(xu_i), [R(xu_i), R(yu_i)]] = R(((x, x)y - (x, y)x)u_i)$,

(8) $[D(x\bar{u}_i), [D(x\bar{u}_i), D(y\bar{u}_i)]] = D(((x, y)x - (x, x)y)\bar{u}_i)$,

(9) $[\mathfrak{R}_0^C, \mathfrak{R}_0^C + \mathfrak{D}_0^C] = \{0\}$.

We put

$$X(x, y) = D(x\bar{u}_2) - R(xu_2) + D(y\bar{u}_3) + R(yu_3) \qquad \text{for} \quad x \times y \in \boldsymbol{K}^C \times \boldsymbol{K}^C .$$

Then from (5), (6) and (9) it follows that

$$\mathfrak{p}^+ = \{X(x, y) \,|\, x \times y \in \boldsymbol{K}^C \times \boldsymbol{K}^C\} .$$

The inner product ((,)) and the norm $\|\ \|$ on $(\boldsymbol{K}^C)_R$ are extended to $(\boldsymbol{K}^C)_R \times (\boldsymbol{K}^C)_R$ in the natural way, which will be also denoted by ((,)) and $\|\ \|$ respectively. Identifying $\boldsymbol{C}^8$ with $\boldsymbol{K}^C$ by the standard basis $\{c_0, c_1, \cdots, c_7\}$ of $\boldsymbol{K}^C$, we denote for $x \in \boldsymbol{K}^C$ by B_x the matrix of the linear map $y \mapsto \overline{xy}$ of $\boldsymbol{K}^C$. Then the linear map $y \mapsto \overline{yx}$ of $\boldsymbol{K}^C$ is represented by the matrix B_x'. In fact,

$$(y, B_x'z) = (B_xy, z) = (\overline{yx}, z) = (\bar{y}, zx) = (y, \overline{zx})$$

for each $x, y, z \in \boldsymbol{K}^C$. We put $f_i = \sqrt{2}\, e_i$ for $i = 1, 2, 3$. Then $\{f_1\}$, $\{c_0u_2, c_1u_2, \cdots, c_7u_2, c_0u_3, c_1u_3, \cdots, c_7u_3\}$ and $\{f_2, f_3, c_0u_1, c_1u_1, \cdots, c_7u_1\}$ are orthonormal basis with respect to $\langle\!\langle\ ,\ \rangle\!\rangle$ for V_1, V_2 and V_3 respectively. We shall represent a linear map in Hom(V_2, V_1) etc. by a matrix with respect to these basis and identify it with its matricial representation. Note that then $X_{12}^* = \bar{X}_{12}'$ and $X_{23}^* = \bar{X}_{23}'$. Now for

$$u = \xi_1 f_1 + \xi_2 f_2 + \xi_3 f_3 + x_1 u_1 + x_2 u_2 + x_3 u_3 \in H_3(\boldsymbol{K})^C$$

and $X = X(x, y) \in \mathfrak{p}^+$, we have

$$\begin{aligned}\varphi(X)u = \{-(\sqrt{2}\,x, x_2) + (\sqrt{2}\,y, x_3)\}f_1 + (-\sqrt{2}\,\xi_3 x + \overline{y\bar{x}_1})u_2 \\ + (-\overline{x_1 x} + \sqrt{2}\,\xi_2 y)u_3\,,\end{aligned}$$

and hence

$$\begin{aligned} X_{12} &= (-\sqrt{2}\,x', \sqrt{2}\,y')\,, \\ X_{23} &= \begin{pmatrix} 0 & -\sqrt{2}\,x & B_y \\ \sqrt{2}\,y & 0 & -B_x' \end{pmatrix}, \\ X_{12}X_{23} &= (2y'y, 2x'x, -\sqrt{2}\,(x'B_y + y'B_x'))\,. \end{aligned}$$

In particular, the linear map $X \mapsto X_{12}$ of $\mathfrak{p}^+$ into $\mathrm{Hom}\,(V_2, V_1)$ is injective. It follows that

$$F_1(X) = \frac{1}{2}X_{12}\bar{X}_{12}' = \|x\|^2 + \|y\|^2 = \|x \times y\|^2 \qquad \text{for} \quad X = X(x, y)\,, \tag{5.3}$$

and

$$\begin{aligned} F_2(X) &= (X_{12}X_{23})(\overline{X_{12}X_{23}})' = 4|(y, y)|^2 + 4|(x, x)|^2 + 2\|B_y'x + B_x y\|^2 \\ &= 4(|(x, x)|^2 + |(y, y)|^2) + 2\|\overline{xy} + \overline{xy}\|^2\,, \end{aligned}$$

and hence

$$F_2(X) = 4(|(x, x)|^2 + |(y, y)|^2) + 8\|xy\|^2 \quad \text{for } X = X(x, y)\,. \tag{5.4}$$

(5.3) shows that the linear isomorphism $x \times y \mapsto X(x, y)$ of $(\boldsymbol{K}^C)_R \times (\boldsymbol{K}^C)_R$ onto $(\mathfrak{p}^+)_R$ is an isometry with respect to the inner products $((\ ,\ ))$.

Next we shall find a maximal abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$ and then compute the root system Σ on $\mathfrak{a}$. For $x \times y \in \boldsymbol{K}^C \times \boldsymbol{K}^C$, we have

$$\widetilde{X(x, y)} = -D(\tilde{x}\bar{u}_2) - R(\tilde{x}u_2) - D(\tilde{y}\bar{u}_3) + R(\tilde{y}u_3)\,,$$

and hence

$$\begin{aligned}\psi(X(x, y)) = \sqrt{2}\,\{\sqrt{-1}D((\mathfrak{Im}\,x)\bar{u}_2) - R((\mathfrak{Re}\,x)u_2) \\ + \sqrt{-1}D((\mathfrak{Im}\,y)\bar{u}_3) + R((\mathfrak{Re}\,y)u_3)\}\,.\end{aligned}$$

We define $X_1, X_2 \in \mathfrak{p}^+$ with $((X_i, X_j)) = \delta_{ij}$ by

$$X_1 = \frac{1}{\sqrt{2}} X(c_1 + \sqrt{-1}c_2, 0) , \quad X_2 = \frac{1}{\sqrt{2}} X(0, c_2 + \sqrt{-1}c_1) ,$$

and then define $H_1, H_2 \in \mathfrak{p}$ with $(H_i, H_j) = \delta_{ij}$ by

$$\begin{aligned} H_1 &= \psi(X_1) = \sqrt{-1}D(c_2\bar{u}_2) - R(c_1u_2) , \\ H_2 &= \psi(X_2) = \sqrt{-1}D(c_1\bar{u}_3) + R(c_2u_3) . \end{aligned}$$

Then we have by (1), (2) and (4)

$$\begin{aligned} [H_1, H_2] &= \{-[D(c_2\bar{u}_2), D(c_1\bar{u}_3)] - [R(c_1u_2), R(c_2u_3)]\} \\ &\quad + \sqrt{-1}\{[D(c_2\bar{u}_2), R(c_2u_3)] - [R(c_1u_2), D(c_1\bar{u}_3)]\} \\ &= \left\{\frac{1}{2}D(\overline{c_2c_1}\,\bar{u}_1) + \frac{1}{2}D(\overline{c_1c_2}\,\bar{u}_1)\right\} + \sqrt{-1}\left\{\frac{1}{2}R(\bar{c}_2^2u_1) - \frac{1}{2}R(\bar{c}_1^2u_1)\right\} \\ &= 0 . \end{aligned}$$

Hence, if we put

$$H(\xi_1, \xi_2) = \xi_1H_1 + \xi_2H_2 \qquad \text{for} \quad \xi_1, \xi_2 \in \boldsymbol{R} ,$$

and

$$\mathfrak{a} = \{H(\xi_1, \xi_2) \mid \xi_1, \xi_2 \in \boldsymbol{R}\} ,$$

then $\mathfrak{a}$ is a maximal abelian subalgebra in $\mathfrak{p}$[1] and $\{\xi_1, \xi_2\}$ is an orthonormal coordinate system for $\mathfrak{a}$. We define $Y_1, Y_2 \in \mathfrak{p}$ by

$$Y_1 = \sqrt{-1}D(c_3\bar{u}_2) + R(c_6u_2) , \quad Y_2 = \sqrt{-1}D(c_1\bar{u}_2) + R(c_2u_2) .$$

We shall show equalities:

$$\text{(5.5)} \qquad \begin{cases} [H(\xi_1, \xi_2), [H(\xi_1, \xi_2), Y_1]] = \xi_1^2Y_1 \\ [H(\xi_1, \xi_2), [H(\xi_1, \xi_2), Y_2]] = (2\xi_1)^2Y_2 . \end{cases}$$

Then it will follow that

$$\Sigma = \{\pm(\xi_1 \pm \xi_2), \pm\xi_1, \pm\xi_2, \pm2\xi_1, \pm2\xi_2\} ,$$

since it is known (Harish-Chandra [6]) that for a non-compact simple Lie algebra $\mathfrak{g}$ of hermitian type of rank ν, the root system Σ is written as $\{\pm(\eta_i \pm \eta_j)\ (1 \leqq i < j \leqq \nu), \pm\eta_i, \pm2\eta_i\ (1 \leqq i \leqq \nu)\}$ by mutually orthogonal linear forms $\eta_1, \cdots, \eta_\nu$ of the same length. For the proof of (5.5), it suffices to show the following equalities:

(i) $[H_1[H_1, Y_1]] = Y_1$, $[H_1, [H_1, Y_2]] = 4Y_2$,

(ii) $[H_2, Y_1] = 0$, $[H_2, Y_2] = 0$.

Proof of (i). Let $x, y \in \boldsymbol{K}$. We have

[1] The construction of this maximal abelian subalgebra $\mathfrak{a}$ is due to M. Ise.

$$\begin{aligned}
&[H_1, \sqrt{-1}D(x\bar{u}_2) + R(yu_2)]\\
&\quad = [\sqrt{-1}D(c_2\bar{u}_2) - R(c_1u_2), \sqrt{-1}D(x\bar{u}_2) + R(yu_2)]\\
&\quad = \{-[D(c_2\bar{u}_2), D(x\bar{u}_2)] - [R(c_1u_2), R(yu_2)]\} + \sqrt{-1}\{[D(c_2\bar{u}_2),\\
&\qquad R(yu_2)] - [R(c_1u_2), D(x\bar{u}_2)]\},
\end{aligned}$$

and hence

$$\begin{aligned}
&[H_1, [H_1, \sqrt{-1}D(x\bar{u}_2) + R(yu_2)]]\\
&\quad = \{-[D(c_2\bar{u}_2), [D(c_2\bar{u}_2), R(yu_2)]] + [D(c_2\bar{u}_2), [R(c_1u_2), D(x\bar{u}_2)]]\\
&\qquad + [R(c_1u_2), [D(c_2\bar{u}_2), D(x\bar{u}_2)]] + [R(c_1u_2, [R(c_1u_2), R(yu_2)]]\}\\
&\qquad + \sqrt{-1}\{-[D(c_2\bar{u}_2), [D(c_2\bar{u}_2), D(x\bar{u}_2)]] - [D(c_2\bar{u}_2), [R(c_1u_2), R(yu_2)]]\\
&\qquad - [R(c_1u_2), [D(c_2\bar{u}_2), R(yu_2)]] + [R(c_1u_2), [R(c_1u_2), D(x\bar{u}_2)]]\}.
\end{aligned}$$

We compute each term of the right hand side using (4)~(8):

$$\begin{aligned}
&-[D(c_2\bar{u}_2), [D(c_2\bar{u}_2), R(yu_2)]] = -(c_2, y)[D(c_2\bar{u}_2), R(e_3 - e_1)]\\
&\qquad\qquad = (c_2, y)R(c_2u_2).\\
&[D(c_2\bar{u}_2), [R(c_1u_2), D(x\bar{u}_2)] = -(x, c_1)[D(c_2\bar{u}_2), R(e_3 - e_1)]\\
&\qquad\qquad = (x, c_1)R(c_2u_2).\\
&[R(c_1u_2), [D(c_2\bar{u}_2), D(x\bar{u}_2)]] = [[R(c_1u_2), D(c_2\bar{u}_2)], D(x\bar{u}_2)]\\
&\qquad + [D(c_2\bar{u}_2), [R(c_1u_2), D(x\bar{u}_2)]]\\
&\quad = -(x, c_1)[D(c_1\bar{u}_2), R(e_3 - e_1)] = (x, c_1)R(c_1u_2).\\
&[R(c_1u_2), [R(c_1u_2), R(yu_2)]] = R((y - (c_1, y)c_1)u_2).\\
&-[D(c_2\bar{u}_2), [D(c_2\bar{u}_2), D(x\bar{u}_2)]] = D((x - (c_2, x)c_2)\bar{u}_2).\\
&-[D(c_2\bar{u}_2), [R(c_1u_2), R(yu_2)]] = -[[D(c_2\bar{u}_2), R(c_1u_2)], R(yu_2)]\\
&\qquad\qquad - [R(c_1u_2), [D(c_2\bar{u}_2), R(yu_2)]]\\
&\quad = -(c_2, y)[R(c_1u_2), R(e_3 - e_1)] = (c_2, y)D(c_1\bar{u}_2).\\
&-[R(c_1u_2), [D(c_2\bar{u}_2), R(yu_2)]] = -(c_2, y)[R(c_1u_2), R(e_3 - e_1)]\\
&\qquad\qquad = (c_2, y)D(c_1\bar{u}_2).\\
&[R(c_1u_2), [R(c_1u_2), D(x\bar{u}_2)]] = -(x, c_1)[R(c_1u_2), R(e_3 - e_1)]\\
&\qquad\qquad = (x, c_1)D(c_1\bar{u}_2).
\end{aligned}$$

Thus we have

$$[H_1, [H_1, \sqrt{-1}D(x\bar{u}_2) + R(yu_2)]] = \sqrt{-1}D(a\bar{u}_2) + R(bu_2),$$

where

$$\begin{aligned}
a &= x - (c_2, x)c_2 + (2(c_2, y) + (x, c_1))c_1,\\
b &= y - (c_1, y)c_1 + ((c_2, y) + 2(x, c_1))c_2.
\end{aligned}$$

Now we have $a = c_3$, $b = c_6$ for $x = c_3$, $y = c_6$ and $a = 4c_1$, $b = 4c_2$ for $x = c_1$, $y = c_2$. This shows the equalities (i).

Proof of (ii). Let $x, y \in \boldsymbol{K}$. We have by (1), (2) and (3)

$$\begin{aligned}
[H_2, \sqrt{-1}D(x\bar{u}_2)+R(yu_2)] &= [\sqrt{-1}D(c_1\bar{u}_3)+R(c_2u_3), \sqrt{-1}D(x\bar{u}_2)+R(yu_2)] \\
&= \{-[D(c_1\bar{u}_3), D(x\bar{u}_2)]+[R(c_2u_3), R(yu_2)]\} + \sqrt{-1}\{[D(c_1\bar{u}_3), R(yu_2)] \\
&\quad + [R(c_2u_3), D(x\bar{u}_2)]\} \\
&= \left\{-\frac{1}{2}D(\overline{xc_1}\,\bar{u}_1) + \frac{1}{2}D(\overline{yc_2}\,\bar{u}_1)\right\} + \sqrt{-1}\left\{-\frac{1}{2}R(\overline{yc_1}\,u_1) - \frac{1}{2}R(\overline{xc_2}\,u_1)\right\} \\
&= -\frac{1}{2}D(\overline{yc_2 - xc_1}\,\bar{u}_1) - \frac{\sqrt{-1}}{2}R(\overline{yc_1 + xc_2}\,u_1) .
\end{aligned}$$

Now we have $yc_2 - xc_1 = yc_1 + xc_2 = 0$ for each of the pairs $(x, y) = (c_3, c_6)$ and $(x, y) = (c_1, c_2)$. This proves the equalities (ii).

Now in the same way as in §4, case $g = 4$, (i), the polynomial F_0 is given by the formula (4.4). Note that (4.4) is also written as

$$F_0 = 8\xi_1^2\xi_2^2 - (\xi_1^2 + \xi_2^2)^2 .$$

The required polynomial F is a K-invariant polynomial on $\mathfrak{p}$ such that $F|_{\mathfrak{a}} = F_0$. Passing to $(\mathfrak{p}^+)_R$ through the K-equivariant isometry ψ: $(\mathfrak{p}^+)_R \to \mathfrak{p}$, the required F is a K-invariant polynomial on $(\mathfrak{p}^+)_R$ such that

$$F(\xi_1X_1 + \xi_2X_2) = 8\xi_1^2\xi_2^2 - (\xi_1^2 + \xi_2^2)^2 \qquad \text{for } \xi_1, \xi_2 \in \boldsymbol{R} . \tag{5.6}$$

We define a K-invariant polynomial F on $(\mathfrak{p}^+)_R$ by $F = (1/2)F_2 - F_1^2$. Then F satisfies (5.6). In fact, we have $\xi_1X_1 + \xi_2X_2 = X(x, y)$ where

$$x = \frac{\xi_1}{\sqrt{2}}(c_1 + \sqrt{-1}c_2) , \quad y = \frac{\xi_2}{\sqrt{2}}(c_2 + \sqrt{-1}c_1) .$$

We have $(x, x) = (y, y) = 0$ and

$$xy = \frac{1}{2}\xi_1\xi_2\{(c_1c_2 - c_2c_1) + \sqrt{-1}(c_1^2 + c_2^2)\} = \xi_1\xi_2(c_4 - \sqrt{-1}c_0) ,$$

and hence $||xy||^2 = 2\xi_1^2\xi_2^2$. Now (5.6) follows from (5.3) and (5.4).

Under the identification of $(\boldsymbol{K}^C)_R \times (\boldsymbol{K}^C)_R$ with $(\mathfrak{p}^+)_R$ through the isometry $x \times y \mapsto X(x, y)$, the polynomial F is given by

$$F(x \times y) = 2(|(x, x)|^2 + |(y, y)|^2) + 4||xy||^2 - (||x||^2 + ||y||^2)^2 \qquad \text{for } x \times y \in \boldsymbol{K}^C \times \boldsymbol{K}^C .$$

6. Examples of $\{p_\alpha, q_\alpha\}$. In this section, we compute explicit forms of $\{p_\alpha, q_\alpha\}$ for some of the homogeneous examples in order to determine all isoparametric hypersurfaces in spheres in the case where $g = 4$ and

m_1 or $m_2 = 2$. We consider the examples given in §4 in case $g = 4$.

(i) $\boldsymbol{F} = \boldsymbol{R}, \boldsymbol{C}$ or $\boldsymbol{H}$, $\boldsymbol{R}^N = M_{r,2}(\boldsymbol{F})$. The polynomial F is given by

$$F(X) = \frac{3}{4}\{\mathrm{Tr}\,(\hat{X}^2)\}^2 - 2\,\mathrm{Tr}\,(\hat{X}^4)$$

where

$$\hat{X} = \begin{pmatrix} 0 & X' \\ X & 0 \end{pmatrix}.$$

First we compute $\{p_\alpha, q_\alpha\}$ in case $\boldsymbol{F} = \boldsymbol{H}$. Set

$$X = \begin{pmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_r & b_r \end{pmatrix}, \quad A = \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix}.$$

Then we have

$$\begin{aligned} F(X) = {} & 6\|A\|^2\|B\|^2 - \|A\|^4 - \|B\|^4 \\ & - 2\{(\bar{A}'B)(\bar{B}'A) + (\bar{B}'A)(\bar{A}'B)\} \\ & + \sum_{i=1}^{r} \{a_i(\bar{A}'B)\bar{b}_i + b_i(\bar{B}'A)\bar{a}_i\}\,. \end{aligned} \tag{6.1}$$

Let e be the point in $\boldsymbol{R}^N$ given by

$$\begin{cases} a_1 = \dfrac{1}{\sqrt{2}}\,, & a_i = 0 \quad \text{for} \quad i \neq 1\,, \\ b_2 = \dfrac{1}{\sqrt{2}}\,, & b_i = 0 \quad \text{for} \quad i \neq 2\,. \end{cases}$$

e satisfies $F(e) = 1$ and $\|e\| = 1$. Taking e as a reference point, we expand F as in §3 of Part I. Set

$$\alpha = \begin{pmatrix} a_3 \\ \vdots \\ a_r \end{pmatrix}, \quad \beta = \begin{pmatrix} b_3 \\ \vdots \\ b_r \end{pmatrix}$$

and

$$\bar{\alpha}'\beta = R + Ii + Jj + Kk$$

where R, I, J and K are real numbers. For a_1, a_2, b_1 and b_2, we give the following orthonormal transformation. Set

$$\begin{aligned} a_l &= x_l + x_{l,1}i + x_{l,2}j + x_{l,3}k\,, \\ b_l &= y_l + y_{l,1}i + y_{l,2}j + y_{l,3}k \end{aligned}$$

for $l = 1$ and 2, and also set

$$\begin{aligned} \sqrt{2}\,z &= x_1 + y_2\,, & \sqrt{2}\,w_0 &= x_1 - y_2\,,\\ \sqrt{2}\,z_1 &= x_2 - y_1\,, & \sqrt{2}\,w_1 &= x_2 + y_1\,,\\ \sqrt{2}\,z_2 &= x_{2,1} + y_{1,1}\,, & \sqrt{2}\,w_2 &= x_{2,1} - y_{1,1}\,,\\ \sqrt{2}\,z_3 &= x_{2,2} + y_{1,2}\,, & \sqrt{2}\,w_3 &= x_{2,2} - y_{1,2}\,,\\ \sqrt{2}\,z_4 &= x_{2,3} + y_{1,3}\,, & \sqrt{2}\,w_4 &= x_{2,3} - y_{1,3}\,. \end{aligned}$$

One can verify that z and w_α's satisfy the required conditions in §3 of Part I. To give $\{p_\alpha, q_\alpha\}$ we put

$$\begin{aligned} \sqrt{2}\,s_1 &= x_{1,1} + y_{2,1}\,, & \sqrt{2}\,t_1 &= x_{1,1} - y_{2,1}\,,\\ \sqrt{2}\,s_2 &= x_{1,2} + y_{2,2}\,, & \sqrt{2}\,t_2 &= x_{1,2} - y_{2,2}\,,\\ \sqrt{2}\,s_3 &= x_{1,3} + y_{2,3}\,, & \sqrt{2}\,t_3 &= x_{1,3} - y_{2,3}\,. \end{aligned}$$

Then we have

$$\tag{6.2} \begin{cases} p_0 = \|\beta\|^2 - \|\alpha\|^2 - 2(s_1t_1 + s_2t_2 + s_3t_3)\,,\\ p_1 = -2\{R + z_2s_1 + z_3s_2 + z_4s_3\}\,,\\ p_2 = 2\{I + z_1s_1 + z_3t_3 - z_4t_2\}\,,\\ p_3 = 2\{J + z_1s_2 - z_2t_3 + z_4t_1\}\,,\\ p_4 = 2\{K + z_1s_3 + z_2t_2 - z_3t_1\}\,, \end{cases}$$

and

$$\tag{6.3} \begin{cases} q_0 = 2\{z_1R - z_2I - z_3J - z_4K\}\,,\\ q_1 = 2\{t_1I + t_2J + t_3K\} + (\|\beta\|^2 - \|\alpha\|^2)z_1\,,\\ q_2 = 2\{t_1R - s_3J + s_2K\} + (\|\beta\|^2 - \|\alpha\|^2)z_2\,,\\ q_3 = 2\{t_2R + s_3I - s_1K\} + (\|\beta\|^2 - \|\alpha\|^2)z_3\,,\\ q_4 = 2\{t_3R - s_2I + s_1J\} + (\|\beta\|^2 - \|\alpha\|^2)z_4\,. \end{cases}$$

The case $F = C$ can be easily obtained from the above. We have

$$\tag{6.4} \begin{cases} p_0 = \|\beta\|^2 - \|\alpha\|^2 - 2s_1t_1\,,\\ p_1 = -2(R + z_2s_1)\,,\\ p_2 = 2(I + z_1s_1)\,, \end{cases}$$

and

$$\tag{6.5} \begin{cases} q_0 = 2(z_1R - z_2I)\,,\\ q_1 = 2t_1I + (\|\beta\|^2 - \|\alpha\|^2)z_1\,,\\ q_2 = 2t_1R + (\|\beta\|^2 - \|\alpha\|^2)z_2\,. \end{cases}$$

(i)′ $\boldsymbol{F}=\boldsymbol{H}$, $\boldsymbol{R}^N = M_{2,2}(\boldsymbol{H})$ $(r=2)$. For $-F$ instead of F, we examine the conditions (A) and (B) of Part I. $-F$ gives a homogeneous example with multiplicities $m_1 = 3$ and $m_2 = 4$ (unique up to $O(N)$-equivalence).

Let e be the point in S^{N-1} given by

$$a_1 = 1,\ a_2 = b_1 = b_2 = 0\ .$$

By (6.1), we have

$$-F(e) = 1\ .$$

Taking e as a reference point we expand $-F$. Put

$$\begin{aligned} a_1 &= z + z_1 i + z_2 j + z_3 k\ , \\ b_2 &= w_0 + w_1 i + w_2 j + w_3 k\ . \end{aligned}$$

One can verify that z and w_α's satisfy the required conditions. Put

$$\begin{aligned} a_2 &= x_0 + x_1 i + x_2 j + x_3 k\ , \\ b_1 &= y_0 + y_1 i + y_2 j + y_3 k\ . \end{aligned}$$

We have

$$\tag{6.6} \begin{cases} p_0 = 2(x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3)\ , \\ p_1 = 2(x_0y_1 + x_1y_0 - x_2y_3 - x_3y_2)\ , \\ p_2 = 2(x_0y_2 + x_2y_0 - x_3y_1 + x_1y_3)\ , \\ p_3 = 2(x_0y_3 + x_3y_0 - x_1y_2 + x_2y_1)\ . \end{cases}$$

A direct computation shows that our $\{p_\alpha\}$ satisfies the condition (A). Also we have

$$\tag{6.7} \begin{aligned} \frac{1}{2}q_0 = {} & z_1(x_0y_1 + x_1y_0 + x_2y_3 - x_3y_2) \\ & + z_2(x_0y_2 - x_1y_3 + x_2y_0 + x_3y_1) \\ & + z_3(x_0y_3 + x_1y_2 - x_2y_1 + x_3y_0)\ . \end{aligned}$$

From (6.6) and (6.7), we see also that the condition (B) is not satisfied in this case. Note that the condition (B) is independent on the choice of coordinates $\{z_k\}$ and $\{w_\alpha\}$ if the condition (A) holds.

Our example constructed in Theorem 2 of Part I for $\boldsymbol{F}=\boldsymbol{H}$ and $r=1$ gives a family of isoparametric hypersurfaces with multiplicities $m_1 = 3$ and $m_2 = 4$, and its defining polynomial satisfies the conditions (A) and (B). In view of Remarks 2 and 3 in §3 of Part I, we can conclude that the above example is not homogeneous.

(ii) $\boldsymbol{F}=\boldsymbol{R}$ or $\boldsymbol{C}$, $\boldsymbol{R}^N = \{Z \in M_5(\boldsymbol{F}) \mid Z = -Z'\}$. The polynomial F is defined by

$$F(Z) = \frac{3}{4}\{\mathrm{Tr}\,(Z\bar{Z})\}^2 - 2\,\mathrm{Tr}\,((Z\bar{Z})^2)\,.$$

We compute $\{p_\alpha, q_\alpha\}$ for $-F$ in case $F = \boldsymbol{R}$. Set

$$Z = (a_{ij})\,, \qquad Z_i = \begin{pmatrix} a_{i1} \\ \vdots \\ a_{i5} \end{pmatrix}$$

for $Z \in \boldsymbol{R}^N$. We have

$$\text{(6.8)} \qquad -F(Z) = \frac{5}{4}\sum_i \|Z_i\|^4 - \frac{3}{2}\sum_{k<j}\|Z_k\|^2\|Z_j\|^2 + 4\sum_{k<j}(Z_k^t Z_j)^2\,.$$

Let e be the point in $\boldsymbol{R}^N$ given by

$$\begin{cases} a_{12} = -a_{21} = 1\,, \\ a_{ij} = 0 \qquad\qquad \text{otherwise}\,. \end{cases}$$

We take e as a reference point. $-F$ has the following expansion with respect to $z = a_{12}$:

$$\text{(6.9)} \qquad \begin{aligned} -F = {} & a_{12}^4 \\ & + a_{12}^2\{2(a_{13}^2 + a_{14}^2 + a_{15}^2 + a_{23}^2 + a_{24}^2 + a_{25}^2) \\ & \quad - 6(a_{34}^2 + a_{35}^2 + a_{45}^2)\} \\ & + 16a_{12}\{a_{34}(a_{24}a_{13} - a_{23}a_{14}) \\ & \quad + a_{35}(a_{25}a_{13} - a_{23}a_{15}) \\ & \quad + a_{45}(a_{25}a_{14} - a_{24}a_{15})\} \\ & + G\,, \end{aligned}$$

where G does not contain a_{12}.
From (6.9), we see that $\{a_{34}, a_{35}, a_{45}\}$ and $\{a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, a_{25}\}$ are required orthonormal coordinate systems for W and Y respectively. Put

$$w_0 = a_{34},\ w_1 = a_{35},\ w_2 = a_{45}\,.$$

We have

$$\text{(6.10)} \qquad \begin{cases} p_0 = 2(a_{24}a_{13} - a_{23}a_{14})\,, \\ p_1 = 2(a_{25}a_{13} - a_{23}a_{15})\,, \\ p_2 = 2(a_{25}a_{14} - a_{24}a_{15})\,. \end{cases}$$

Computing G, we conclude

$$\text{(6.11)} \qquad q_0 = q_1 = q_2 = 0\,.$$

7. Case $m_1 = 2$. The rest of our paper is mainly devoted to prove the following.

THEOREM 5. *Let M be a closed isoparametric hypersurface in a sphere with 4 distinct principal curvatures. If $m_1 = 2$ or $m_2 = 2$, then M is homogeneous.*

We shall establish the theorem by classifying all the homogeneous polynomials of degree 4 satisfying the differential equations (M) with $m_1 = 2$ or $m_2 = 2$. We may assume $m_1 = 2$. Let F be a homogeneous polynomial of degree 4 on $\boldsymbol{R}^N$ satisfying (M). As in Part I, decomposing $\boldsymbol{R}^N$, we associate $\{p_\alpha\}$ and $\{q_\alpha\}$ to F. From the results of Part I, it suffices to show that our $\{p_\alpha, q_\alpha\}$ coincide with the ones associated to some of homogeneous examples.

We prepare a few lemmas and matricial notations in this section, and then deal with the case where $m_1 = 2$ and $m_2 \geqq 3$ in §8 and the case where $m_1 = m_2 = 2$ in §9. Following the notations in §5 of Part I, we prove

LEMMA 3. *Let α and β be two non zero distinct indices. If $Lp_{\alpha,0} = L'p_{\beta,0}$ for some non zero constants L and L', then $m_1 = m_2$.*

PROOF. Suppose $m_2 > m_1$. We have $a_\alpha a'_\alpha + 2b_\alpha b'_\alpha = 1$ from $(4\text{-}1)_\alpha$. This shows

$$||xa_\alpha|| \leqq ||x||$$

for any vector $x = (x_1, \cdots, x_{m_2})$, where $||\ \ ||$ indicates the length of a vector. Since $\operatorname{rank}(b_\alpha) \leqq m_1 < m_2$, there exists a non zero vector x such that $xb_\alpha = 0$. Then we have

$$||xa_\alpha|| = ||x|| \neq 0 .$$

Our assumption implies $La_\alpha = L'a_\beta$, and hence

$$||x|| = ||xa_\alpha|| = \frac{|L'|}{|L|}||xa_\beta|| .$$

Since $(4\text{-}1)_\beta$ implies $||xa_\beta|| \leqq ||x||$, we have

$$|L'| \geqq |L| .$$

Similarly we have $|L| \geqq |L'|$, and hence

$$|L| = |L'| .$$

Thus we have $p_{\beta,0} = \pm p_{\alpha,0}$, or equivalently, $a_\beta = \pm a_\alpha$. Substituting in $a_\alpha a'_\alpha + 2b_\alpha b'_\alpha = 1$ and $a_\beta a'_\beta + 2b_\beta b'_\beta = 1$, we get

$$\pm a_\beta a'_\alpha + 2b_\alpha b'_\alpha = 1 ,$$
$$\pm a_\alpha a'_\beta + 2b_\beta b'_\beta = 1 .$$

Consider $(4\text{-}3)_{0\alpha\beta}$. We have

$$a_\beta a'_\alpha + a_\alpha a'_\beta + 2(b_\beta b'_\alpha + b_\alpha b'_\beta) = 0 .$$

Using the above two equations, we obtain

$$b_\beta b'_\beta + b_\alpha b'_\alpha \pm (b_\beta b'_\alpha + b_\alpha b'_\beta) = 1 ,$$

that is,

$$(b_\beta - b_\alpha)(b'_\beta - b'_\alpha) = 1$$

or

$$(b_\beta + b_\alpha)(b'_\beta + b'_\alpha) = 1 .$$

This is a contradiction, since rank $(b_\beta \pm b_\alpha)$ is at most m_1. q.e.d.

LEMMA 4. *Assume* $m_1 = 2$. *If* $p_{1,1} = p_{2,1} = 0$, *then* $m_2 \leqq 2$.

PROOF. Suppose $p_{1,1} = p_{2,1} = 0$. Then the condition (A) in §6 of Part I is satisfied. We see that $q_\alpha = q_{\alpha,1}$, that is, each q_α is linear with respect to z_1, z_2. We put

$$q_\alpha = f_\alpha z_1 + g_\alpha z_2$$

for $\alpha = 0, 1, 2$. Consider the following matrix

$$S = \begin{pmatrix} p_0 & p_1 & p_2 \\ f_0 & f_1 & f_2 \\ g_0 & g_1 & g_2 \end{pmatrix} .$$

We claim $SS' = G1$, where 1 denotes the identity matrix of degree 3 and $G = \sum p_\alpha^2$. Recall the equations (3-7) and (5-8) of Part I. From $\sum p_\alpha q_\alpha = 0$, we have

$$\sum p_\alpha f_\alpha = 0 , \quad \sum p_\alpha g_\alpha = 0 .$$

From $\sum q_\alpha^2 = G(\sum z_j^2)$, we have

$$\sum f_\alpha^2 = \sum g_\alpha^2 = G , \quad \sum f_\alpha g_\alpha = 0 .$$

They proves $SS' = G1$. Taking their determinants, we have

$$(\det S)^2 = G^3 .$$

Thus G can be expressed as

$$G = H^2$$

by a suitable quadratic form H. For each α, we have

$$\langle p_\alpha, G \rangle = 2H \langle p_\alpha, H \rangle .$$

Since $\langle p_\alpha, p_\beta \rangle = 0$ for distinct α, β by Lemma 17 of Part I, we see

$$\langle p_\alpha, G \rangle = 2p_\alpha \langle p_\alpha, p_\alpha \rangle .$$

Again using Lemma 17, we obtain

$$p_\alpha \langle p_0, p_0 \rangle = H \langle p_\alpha, H \rangle$$

for any α. The quadratic form $\langle p_0, p_0 \rangle = 4(\sum u_i^2 + \sum v_i^2)$ is irreducible. Assume $m_2 \geqq 2$. Then each p_α is also irreducible. Thus, we see that H is a constant multiple of p_α or $\langle p_0, p_0 \rangle$. In view of Lemma 3, we can conclude that $H = c\langle p_0, p_0 \rangle$ for some constant c. One can see easily $c = \pm 1/4$. Finally we obtain

$$G = \sum p_\alpha^2 = (\sum u_i^2 + \sum v_i^2)^2 ,$$

or equivalently,

$$p_1^2 + p_2^2 = 4\,(\sum u_i^2)(\sum v_i^2) .$$

In this equation, we set $u_2 = \cdots = u_{m_2} = 0$. Since p_1 and p_2 are linear combinations of $\{u_i v_j\}$, we can write

$$p_i \,|_{u_2 = \cdots = u_{m_2} = 0} = u_1 h_i$$

where h_i is a linear function in $v_1, \cdots, v_{m_2}$. We have

$$h_1^2 + h_2^2 = 4(\sum v_i^2) .$$

The left hand side of this equation is of rank at most 2 as a quadratic form. This proves $m_2 \leqq 2$. q.e.d.

From now on we assume $m_1 = 2$. We use the following matricial notations. For p_1, we omit the index $\alpha = 1$, so that

$$p_1 \sim \begin{pmatrix} 0 & a & b \\ a' & 0 & c \\ b' & c' & 0 \end{pmatrix}$$

where $'$ indicates the transpose of a matrix. For p_2, we use the capital letters, so that

$$p_2 \sim \begin{pmatrix} 0 & A & B \\ A' & 0 & C \\ B' & C' & 0 \end{pmatrix} .$$

For each submatrix, say a, the (i, j)-element of a is denoted by a_{ij} unless otherwise stated.

We summarize here the conditions (4–1) $\sim$ (4–3) of Part I. $(4\text{–}1)_1$ and $(4\text{–}2)_{1,0}$ are equivalent to

$$
\text{(I)} \qquad \begin{cases} aa' + 2bb' = 1,\ a'a + 2cc' = 1,\ b'b = c'c\ , \\ bc'a' + acb' = 0,\ cb'a + a'bc' = 0,\ c'a'b + b'ac = 0\ . \end{cases}
$$

Similarly we have (I'), replacing a, b and c in (I) by A, B and C. The condition $(4\text{-}3)_{0,1,2}$ is expressed as

$$
\text{(III)} \qquad \begin{cases} (Aa' + aA') + 2(Bb' + bB') = 0\ , \\ (A'a + a'A) + 2(Cc' + cC') = 0\ , \\ B'b + b'B = C'c + c'C\ . \end{cases}
$$

The condition $(4\text{-}2)_{2,1}$ decomposes into the following 6 conditions.

$\mathrm{II}_{(1,1)}$ $\qquad Acb' + Bc'a' + aCb'$ is skew-symmetric ,

$\mathrm{II}_{(2,2)}$ $\qquad cb'A + a'Bc' + Cb'a$ is skew-symmetric ,

$\mathrm{II}_{(3,3)}$ $\qquad b'Ac + c'a'B + b'aC$ is skew-symmetric ,

$\mathrm{II}_{(1,2)}$
$$
\begin{aligned}
&(aa' + bb')A + A(a'a + cc') + aA'a \\
&\quad + bB'a + Bb'a + aCc' + acC' = A\ ,
\end{aligned}
$$

$\mathrm{II}_{(1,3)}$
$$
\begin{aligned}
&(aa' + bb')B + B(b'b + c'c) + bB'b \\
&\quad + Aa'b + aA'b + bc'C + bC'c = B\ ,
\end{aligned}
$$

$\mathrm{II}_{(2,3)}$
$$
\begin{aligned}
&(a'a + cc')C + C(b'b + c'c) + cC'c \\
&\quad + a'Ac + A'ac + cb'B + cB'b = C\ .
\end{aligned}
$$

In the above equations, interchanging the small letters with the capital letters, we obtain the conditions equivalent to $(4\text{-}2)_{1,2}$, which will be denoted by $\mathrm{II}'_{(i,j)}$ respectively.

In the case where $m_1 = 2$ and $m_2 \geqq 3$, we see $p_{1,0} \neq 0$ and $p_{2,0} \neq 0$. In fact, we have

$$
\begin{aligned}
6 \leqq 2m_2 = \operatorname{rank} p_1 &\leqq \operatorname{rank} p_{1,0} + \operatorname{rank} p_{1,1} \\
&\leqq \operatorname{rank} p_{1,0} + 4\ ,
\end{aligned}
$$

and hence $\operatorname{rank} p_{1,0} \geqq 2$. Similarly we have $\operatorname{rank} p_{2,0} \geqq 2$.

LEMMA 5. *Assume $m_1 = 2$ and $m_2 \geqq 3$. Then $p_{1,0}$ and $p_{2,0}$ have no common linear factor.*

PROOF. Suppose $p_{1,0}$ and $p_{2,0}$ have a common linear factor. If a quadratic form is not irreducible, then its rank $\leqq 2$. Thus, from the above remark, we have

$$
\operatorname{rank} p_{1,0} = \operatorname{rank} p_{2,0} = 2\ ,
$$

and $m_2 = 3$.

First we shall show that by a suitable choice of coordinates p_1 has the following representation:

$$a=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad b=c=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0\\ 1 & 0\\ 0 & 1\end{pmatrix}.$$

Since $p_{1,0}$ is of rank 2, i.e., a is of rank 1, we can choose $\{u_i\}$ and $\{v_i\}$ so that

$$p_{1,0}=2\lambda u_1 v_1 ,$$

with $\lambda>0$. Then the condition $aa'+2bb'=1$ implies that we have $\lambda=1$ and $b_{11}=b_{12}=0$ and the matrix

$$\sqrt{2}\begin{pmatrix}b_{21} & b_{22}\\ b_{31} & b_{32}\end{pmatrix}$$

is an orthogonal matrix. We transform $\{u_2, u_3\}$ into $\{u_2', u_3'\}$ by

$$(u_2', u_3')=(u_2, u_3)\sqrt{2}\begin{pmatrix}b_{21} & b_{22}\\ b_{31} & b_{32}\end{pmatrix}.$$

Similarly the condition $a'a+2cc'=1$ implies that we have $c_{11}=c_{12}=0$ and the matrix

$$\sqrt{2}\begin{pmatrix}c_{21} & c_{22}\\ c_{31} & c_{32}\end{pmatrix}$$

is orthogonal. Transforming $\{v_2, v_3\}$ into $\{v_2', v_3'\}$ similarly, we obtain

$$p_{1,1}=\frac{2}{\sqrt{2}}\{(u_2'+v_2')z_1+(u_3'+v_3')z_2\} ,$$

which proves our first claim.

We decompose the matrices A, B and C as follows;

$$A=\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}, \qquad B=\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}, \qquad C=\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}$$

where α_{22}, β_2 and γ_2 are 2×2 matrices. $p_{2,0}$ must be divisible by u_1 or v_1. First assume that $p_{2,0}$ is divisible by u_1. Then we have $\alpha_{21}=0$ and $\alpha_{22}=0$. From the first two equations of III, we have

$$\alpha_{11}=0,\ \beta_1=0,\ \beta_2+\beta_2'=0,\ \gamma_2+\gamma_2'=0 \quad\text{and}\quad \alpha_{12}+\sqrt{2}\gamma_1=0 .$$

From the condition I′, we obtain

$$\alpha_{12}\alpha_{12}'=1,\ 2\beta_2\beta_2'=1,\ \gamma_1\gamma_1'=1,\ \gamma_1\gamma_2'=0,\ \alpha_{12}'\alpha_{12}+2\gamma_2\gamma_2'=1 \quad\text{and}$$
$$\beta_2\gamma_2'\alpha_{12}'=0 .$$

Now $2\beta_2\beta_2' = 1$ implies that β_2 is non-singular, and hence we have

$$\gamma_2'\alpha_{12}' = 0 ,$$

or equivalently, $\alpha_{12}\gamma_2 = 0$. $\gamma_2 + \gamma_2' = 0$ implies that $\gamma_2 = 0$ or γ_2 is non-singular. Suppose $\gamma_2 = 0$. Then we have $\alpha_{12}'\alpha_{12} = 1$. This is a contradiction since rank $\alpha_{12} \leqq 1$. Suppose γ_2 is non-singular. Then we have $\alpha_{12} = 0$, and hence $A = 0$. This is again a contradiction since $p_{2,0} \neq 0$.

The case where $p_{2,0}$ is divisible by v_1 leads also a contradiction similarly. q.e.d.

REMARK. In the case $m_1 = 2$ and $m_2 \geqq 3$, we see that $p_{1,0}$ and $p_{2,0}$ have no common factors. This follows from Lemmas 3 and 5.

8. **Case $m_1 = 2$ and $m_2 \geqq 3$.** In this section, we consider the case where $m_1 = 2$ and $m_2 \geqq 3$. We shall show first that, after a suitable choice of coordinates, p_0, p_1, p_2, $q_{1,0}$ and $q_{2,0}$ coincide with the ones given in §6 for the example (i) in case $g = 4$, and then that they determine uniquely the rest of terms.

First note that $p_{1,0} \neq 0$ and $p_{2,0} \neq 0$ and they have no common factors. In the equation (3-7): $\sum p_\alpha q_\alpha = 0$, setting $z_1 = z_2 = 0$, we obtain

$$p_{1,0}q_{1,0} + p_{2,0}q_{2,0} = 0 .$$

Therefore there exists a linear function h on $U \oplus V$ such that

$$q_{1,0} = hp_{2,0} , \quad q_{2,0} = -hp_{1,0} . \tag{1}$$

We decompose h as

$$h = \lambda - \mu \tag{2}$$

where λ and μ are linear functions on U and V respectively. Set $z_1 = z_2 = 0$ in the equation (3-8): $16 \sum q_\alpha^2 = 16 (\sum y_j^2) G - \langle G, G \rangle$. Since we have

$$\langle p_0, p_0 \rangle = 4(\sum u_i^2 + \sum v_i^2) ,$$
$$\langle p_0, p_1 \rangle |_{z_k=0} = \langle p_0, p_2 \rangle |_{z_k=0} = 0 ,$$

we get

$$\begin{aligned} &4h^2(p_{1,0}^2 + p_{2,0}^2) \\ &\quad = 4(\sum u_i^2 + \sum v_i^2)(p_{1,0}^2 + p_{2,0}^2) \\ &\qquad - \{p_{1,0}^2\langle p_1, p_1\rangle|_{z_k=0} + p_{2,0}^2\langle p_2, p_2\rangle|_{z_k=0} + 2p_{1,0}p_{2,0}\langle p_1, p_2\rangle|_{z_k=0}\} , \end{aligned}$$

or equivalently,

$$\begin{aligned} &p_{1,0}^2\{4(\sum u_i^2 + \sum v_i^2) - \langle p_1, p_1\rangle|_{z_k=0} - 4h^2\} \\ &\quad + p_{2,0}^2\{4(\sum u_i^2 + \sum v_i^2) - \langle p_2, p_2\rangle|_{z_k=0} - 4h^2\} \\ &\quad = 2p_{1,0}p_{2,0}\langle p_1, p_2\rangle|_{z_k=0} . \end{aligned}$$

Since $p_{1,0}$ and $p_{2,0}$ have no common factors, we can find constants L and L' such that

$$4(\sum u_i^2 + \sum v_i^2) - \langle p_1, p_1\rangle|_{z_k=0} - 4h^2 = Lp_{2,0}\,, \tag{3}$$

$$4(\sum u_i^2 + \sum v_i^2) - \langle p_2, p_2\rangle|_{z_k=0} - 4h^2 = L'p_{1,0}\,, \tag{4}$$

$$\langle p_1, p_2\rangle|_{z_k=0} = Lp_{1,0} + L'p_{2,0}\,. \tag{5}$$

Note that we have

$$\langle p_1, p_1\rangle|_{z_k=0} = 4(u_1, \cdots, u_{m_2}, v_1, \cdots, v_{m_2})\begin{pmatrix} aa' + bb' & bc' \\ cb' & a'a + cc' \end{pmatrix}\begin{pmatrix} u_1 \\ \vdots \\ u_{m_2} \\ v_1 \\ \vdots \\ v_{m_2} \end{pmatrix}.$$

In (3), we set $v_1 = \cdots = v_{m_2} = 0$, and we obtain

$$\sum u_i^2 = (u_1, \cdots, u_{m_2})(aa' + bb')\begin{pmatrix} u_1 \\ \vdots \\ u_{m_2} \end{pmatrix} + \lambda^2\,.$$

Similarly we obtain

$$\sum v_i^2 = (v_1, \cdots, v_{m_2})(a'a + cc')\begin{pmatrix} v_1 \\ \vdots \\ v_{m_2} \end{pmatrix} + \mu^2\,.$$

On the other hand, $aa' + 2bb' = 1$ and $a'a + 2cc' = 1$ in (I) give us

$$(u_1, \cdots, u_{m_2})bb'\begin{pmatrix} u_1 \\ \vdots \\ u_{m_2} \end{pmatrix} = \lambda^2\,, \tag{6}$$

$$(v_1, \cdots, v_{m_2})cc'\begin{pmatrix} v_1 \\ \vdots \\ v_{m_2} \end{pmatrix} = \mu^2\,. \tag{7}$$

The similar argument for the equation (4) gives us

$$(u_1, \cdots, u_{m_2})BB'\begin{pmatrix} u_1 \\ \vdots \\ u_{m_2} \end{pmatrix} = \lambda^2\,, \tag{8}$$

$$(9)\qquad (v_1, \cdots, v_{m_2})CC'\begin{pmatrix} v_1 \\ \vdots \\ v_{m_2} \end{pmatrix} = \mu^2 .$$

Now suppose $\lambda = 0$. By (6) and (8), we have $b = 0$ and $B = 0$. Since $b'b = c'c$ and $B'B = C'C$, we see $c = 0$ and $C = 0$. Thus we have $p_{1,1} = 0$ and $p_{2,1} = 0$. This contradicts $m_2 \geqq 3$ in view of Lemma 4. Therefore we have $\lambda \neq 0$, and similarly $\mu \neq 0$. And consequently the matrices b, c, B and C are all of rank 1 from (6) $\sim$ (9).

In (5), set $v_1 = \cdots = v_{m_2} = 0$, and next $u_1 = \cdots = u_{m_2} = 0$. Thereby we obtain

$$aA' + bB' + Aa' + Bb' = 0 ,$$

and

$$a'A + cC' + A'a + Cc' = 0 .$$

On the other hand, by (III), we know

$$aA' + Aa' + 2(Bb' + bB') = 0 ,$$
$$a'A + A'a + 2(Cc' + cC') = 0 .$$

Combining these together, we obtain

$$(10)\qquad \begin{cases} Bb' + bB' = 0 , & Cc' + cC' = 0 , \\ Aa' + aA' = 0 , & A'a + a'A = 0 . \end{cases}$$

Hereafter in this section, m_2 is denoted simply by m. We choose coordinates $\{u_i\}$ and $\{v_i\}$ so that

$$(11)\qquad \lambda = \varepsilon u_m \quad \text{and} \quad \mu = \delta v_m$$

with $\varepsilon > 0$. Now (6) $\sim$ (9) imply that b, c, B and C are of the following type:

$$\begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \times & \times \end{pmatrix} .$$

We choose $\{z_1, z_2\}$ so that

$$b = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \lambda_0 & 0 \end{pmatrix}$$

with $\lambda_0 < 0$.
From $b'b = c'c$, we can write

$$c = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \mu_0 & 0 \end{pmatrix}$$

with $\mu_0^2 = \lambda_0^2$. Suppose $\mu_0 > 0$. Then we take $-v_m$ instead of v_m so that μ_0 is transformed to $-\mu_0$. Thus we can assume

$$\lambda_0 = \mu_0 .$$

From (10), (8) and (9), it follows that we can write B and C as

$$B = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & \lambda_1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & \mu_1 \end{pmatrix}$$

with $\lambda_1^2 = \mu_1^2 = \lambda_0^2$.

Consider the matrix a. $bc'a' + acb' = 0$ and $cb'a + a'bc' = 0$ in (I) show that $a_{im} = a_{mj} = 0$ for all i, j. In view of $aa' + 2bb' = 1$, one sees that a suitable orthogonal transformation on $\{u_1, \cdots, u_{m-1}\}$ gives us

$$-a = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{pmatrix}$$

and that we have $\lambda_0 = -1/\sqrt{2}$. Consider the matrix A. $AA' + 2BB' = 1$ in (I′), $Aa' + aA' = 0$ and $A'a + a'A = 0$ in (10) show that A is of the form

$$A = \begin{pmatrix} & & 0 \\ & \alpha & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

with $\alpha + \alpha' = 0$ and $\alpha\alpha' = 1$, where 1 denotes the identity matrix of degree $m-1$. Therefore, $m-1$ must be even. Let $2l = m-1$. One can transform, keeping the matrix a fixed, α to the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where 1 denotes the identity matrix of degree l.

From $B'b + b'B = C'c + c'C$ in (III), we have $\lambda_1 = \mu_1$. Thus $\lambda_1 = \mu_1 = \pm 1/\sqrt{2}$. Now suppose $\lambda_1 = \mu_1 = -1/\sqrt{2}$. Then we take $-z_2$ instead of z_2, so that $\lambda_1 = \mu_1$ changes the signature. Thus, we can assume

$$\lambda_1 = \mu_1 = \frac{1}{\sqrt{2}} .$$

By the above choice of coordinates, we get finally

$$(12)\quad \begin{cases} -a = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{pmatrix}, \quad b = c = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \\ A = \left(\begin{array}{ccc|ccc|c} & & & 1 & & & 0 \\ & & & & \ddots & & \vdots \\ & & & & & 1 & \vdots \\ \hline -1 & & & & & & \vdots \\ & \ddots & & & & & \vdots \\ & & -1 & & & & 0 \\ \hline 0 & & & & \cdots & 0 & 0 \end{array}\right), \quad B = C = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}. \end{cases}$$

Substituting these in (3), we see that $L = 0$ and

$$(13)\qquad h = \frac{1}{\sqrt{2}}(u_m - v_m) ,$$

because of our choice (11). Set

$$t_1 = \frac{1}{\sqrt{2}}(u_m - v_m),\ s_1 = -\frac{1}{\sqrt{2}}(u_m + v_m) .$$

Finally we get

$$(14)\qquad \begin{cases} p_0 = \sum_{j=1}^{l} (u_j^2 + u_{l+j}^2 - v_j^2 - v_{l+j}^2) - 2s_1t_1 , \\ p_1 = -2\sum_{j=1}^{l} (u_jv_j + u_{l+j}v_{l+j}) - 2z_1s_1 , \\ p_2 = 2\sum_{j=1}^{l} (u_jv_{l+j} - u_{l+j}v_j) - 2z_2s_1 , \\ q_{1,0} = t_1p_{2,0},\ q_{2,0} = -t_1p_{1,0} . \end{cases}$$

We compare (14) with (6.4) and (6.5). *Interchange* z_1 *and* z_2 and put

$$b_{2+j} = u_{l+j} + \sqrt{-1}\, u_j ,$$
$$a_{2+j} = v_{l+j} + \sqrt{-1}\, v_j$$

for $j = 1, \cdots, l$. One can verify our first assertion on p_0, p_1, p_2, $q_{1,0}$ and $q_{2,0}$ for $r = l + 2$.

We come to the second step. We claim that p_0, p_1, p_2, $q_{1,0}$ and $q_{2,0}$ determine uniquely the rest of terms. First note that we have

$$q_{1,2} = q_{2,2} = 0 . \tag{15}$$

In fact, from (6.5), we have (15) for the homogeneous example (i). Consider the equation (3-8):

$$16 (\textstyle\sum q_\alpha^2) = 16 (\sum y_j^2) G - \langle G, G\rangle .$$

For the homogeneous example (i), the left hand side of (3-8) has no terms of degree 4 with respect to z_1, z_2. Since our p_0, p_1, p_2 coincide with the ones corresponding to the homogeneous example (i), we can conclude $q_{1,2} = q_{2,2} = 0$.

We put

$$q_0 = f_{0,1} z_1 + f_{0,2} z_2 ,$$
$$q_{1,1} = f_{1,1} z_1 + f_{1,2} z_2 ,$$
$$q_{2,1} = f_{2,1} z_1 + f_{2,2} z_2 .$$

We claim

$$f_{1,2} = f_{2,1} = 0 , \quad \frac{\partial f_{1,1}}{\partial s_1} = \frac{\partial f_{2,2}}{\partial s_1} = 0 . \tag{16}$$

In fact, from $\langle p_1, q_1\rangle = 0$ in (3-4), we have

$$\langle p_{1,0}, q_{1,0}\rangle + (\langle p_{1,1}, q_{1,0}\rangle + \langle p_{1,0}, q_{1,1}\rangle) + \langle p_{1,1}, q_{1,1}\rangle = 0 .$$

This is equivalent to

$$\langle p_{1,0}, q_{1,0}\rangle + \langle p_{1,1}, q_{1,1}\rangle_{\{z_k\}} = 0 , \tag{17}$$

$$\langle p_{1,1}, q_{1,0}\rangle + \langle p_{1,0}, q_{1,1}\rangle = 0 , \tag{18}$$

$$\langle p_{1,1}, q_{1,1}\rangle_{\{u_i, v_i\}} = 0 . \tag{19}$$

Substitute $p_{1,1} = -2z_2 s_1$, $q_{1,1} = f_{1,1} z_1 + f_{1,2} z_2$ in (19). We obtain

$$\langle p_{1,1}, q_{1,1}\rangle_{\{u_i, v_i\}} = -2\frac{\partial f_{1,1}}{\partial s_1} z_1 z_2 - 2\frac{\partial f_{1,2}}{\partial s_1} z_2^2 = 0 ,$$

and hence

$$\frac{\partial f_{1,1}}{\partial s_1} = 0 , \quad \frac{\partial f_{1,2}}{\partial s_1} = 0 .$$

Since $p_{1,0} = -2R$, $q_{1,0} = 2t_1 I$, we have

$$\langle p_{1,0}, q_{1,0} \rangle = -4t_1 \langle R, I \rangle .$$

A direct computation shows $\langle R, I \rangle = 0$, and hence

$$\langle p_{1,0}, q_{1,0} \rangle = 0 .$$

From (17), we have $\langle p_{1,1}, q_{1,1} \rangle_{\{z_k\}} = 0$. Since $p_{1,1} = -2z_2 s_1$,

$$\langle p_{1,1}, q_{1,1} \rangle_{\{z_k\}} = -2s_1 \frac{\partial q_{1,1}}{\partial z_2} = -2s_1 f_{1,2} = 0 ,$$

which shows $f_{1,2} = 0$. The similar argument for p_2 and q_2 completes our claim (16).

Consider (3-8): $\sum p_\alpha q_\alpha = 0$. We have

$$p_0 q_0 + (p_{1,0} q_{1,1} + p_{1,1} q_{1,0}) + (p_{2,0} q_{2,1} + p_{2,1} q_{2,0}) = 0 ,$$

and hence

$$p_0 f_{0,1} + p_{1,0} f_{1,1} + 2s_1 q_{2,0} = 0 ,$$
$$p_0 f_{0,2} + (-2s_1) q_{1,0} + p_{2,0} f_{2,2} = 0 .$$

Equivalently, we have

$$p_0 f_{0,1} = R(2f_{1,1} - 4s_1 t_1) ,$$
$$p_0 f_{0,2} = I(4s_1 t_1 - 2f_{2,2}) .$$

Since p_0 is irreducible, we can write

$$2f_{1,1} - 4s_1 t_1 = c_1 p_0 ,$$
$$4s_1 t_1 - 2f_{2,2} = c_2 p_0 .$$

Apply $\partial/\partial s_1$ to the above two equations. In view of (16), we obtain $c_1 = 2$, $c_2 = -2$. Thus we have

$$f_{1,1} = \|\beta\|^2 - \|\alpha\|^2 ,$$
$$f_{2,2} = \|\beta\|^2 - \|\alpha\|^2 ,$$
$$f_{0,1} = 2R ,$$
$$f_{0,2} = -2I .$$

Our second assertion is now proved.

9. Case $m_1 = m_2 = 2$. As mentioned in the introduction, this case is already indicated by Cartan without proof. We give here an outline of our proof. We use the notations given in §7. Note that a, b, c, A, B and C are all 2×2 matrices in this case. We write I for the identity matrix of degree 2, J for $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and θ for $1/\sqrt{2}$.

LEMMA 6. *Let a, b and c be matrices satisfying (I) in §7. Then by a suitable choice of coordinates $\{a, b, c\}$ can be represented as:*

(i) *case* $\operatorname{rank} a = 0, \quad a = 0, b = I, c = \theta J$;

(ii) *case* $\operatorname{rank} a = 1, \quad a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, b = c = \begin{pmatrix} 0 & 0 \\ 0 & \theta \end{pmatrix}$;

(iii) *case* $\operatorname{rank} a = 2$ *and* $p_{1,1} = 0, \quad a = I, b = c = 0$;

(iv) *case* $\operatorname{rank} a = 2$ *and* $p_{1,1} \neq 0, \quad a = \xi I, b = \eta I, c = \eta J$ *with* $\xi^2 + 2\eta^2 = 1$.

LEMMA 7. *In the case* (ii) *of Lemma* 6, *there exists no p_2 satisfying* (III), (II) *and* (I′).

Lemmas 6 and 7 can be verified by elementary but long calculations. From Lemmas 6 and 7, one can see that $\{p_1, p_2\}$ can be classified, interchanging w_1 and w_2 if necessary, into the following 5 cases;

(A) $p_{1,0} \neq 0, p_{2,0} \neq 0, p_{1,1} \neq 0$,

(B_1) $p_{1,0} = 0, p_{2,0} = 0$,

(B_2) $p_{1,0} = 0, p_{2,1} = 0$,

(B_3) $p_{1,0} = 0, p_{2,0} \neq 0, p_{2,1} \neq 0$,

(C) $p_{1,1} = 0, p_{2,1} = 0$.

LEMMA 8. *By a suitable choice of coordinates $\{w_1, w_2\}$, the case (A) can be reduced to the case (B_1) or (B_2).*

LEMMA 9. *In the case (B_1), by a suitable choice of coordinates, our $\{p_\alpha, q_\alpha\}$ coincide with those of $-F$, where F is the polynomial of the example* (ii) *in case $g = 4$ and $\boldsymbol{F} = \boldsymbol{R}$ in* §4.

One can prove this lemma, using the explicit forms (6.10) and (6.11) of $\{p_\alpha, q_\alpha\}$ associated to the above $-F$.

LEMMA 10. *In the cases (B_2) and (B_3), there exist no $\{q_\alpha\}$ satisfying* (3-4) $\sim$ (3-10) *of Part I.*

LEMMA 11. *The case (C) can be reduced to the case (B_1).*

More precisely, $\{p_\alpha, q_\alpha\}$ in the case (C) correspond to those of the polynomial F of the homogeneous example (ii). One can compute $\{p_\alpha, q_\alpha\}$ of F from those of $-F$.

The preceding lemmas complete our classification in case $m_1 = m_2 = 2$, and hence every closed isoparametric hypersurface in a sphere in this case is homogeneous.

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