Tôhoku Math. Journ.
28 (1976), 245-256.

ON TOTALLY RAMIFIED VALUES

TOSHIKO KUROKAWA AND KIKUJI MATSUMOTO

(Received February 1, 1975)

1. Introduction. Whenever the equation $f(z)=\alpha$ has no simple roots, a value α is called a totally ramified value of $f(z)$. In the Nevanlinna theory, it is well-known that every transcendental meromorphic function can have no more than four totally ramified values. Let E be a totally disconnected compact set in the z-plane and let R be its complementary domain. How many totally ramified values can functions $f(z)$ meromorphic in R with at least one essential singularity in E have? In this paper we shall give a result (Theorem) about this question and see as a corollary that there are perfect E's for which any $f(z)$ can have no more than four totally ramified values.

2. Lemmas. Here we shall give some lemmas. For a simply connected hyperbolic domain D we call $f(z)$ to be normal in D, if the family $\{f(s(z))\}$, where $\zeta = s(z)$ denotes an arbitrary one to one conformal mapping of D onto itself, is normal in the sense of Montel. For a multiply connected domain D with the universal covering surface $\tilde{D}$ being conformally equivalent to the unit disc we call $f(z)$ to be normal in D, if $f(z)$ is normal on $\tilde{D}$. As a sufficient condition for $f(z)$ to be normal, the following is well-known.

LEMMA 1. *Let $w = f(z)$ be meromorphic in $|z| < 1$. Let $\Delta_1, \Delta_2, \cdots, \Delta_q$ $(q \geqq 3)$ be q mutually disjoint closed Jordan domains on the Riemann w-sphere. Denote by m_j $(j = 1, 2, \cdots, q)$ the minimum of the numbers of sheets of islands of F above Δ_j, where F denotes the covering surface generated by $w = f(z)$. Suppose that*

$$\sum_{j=1}^{q}\left(1-\frac{1}{m_j}\right) > 2\,.$$

Then, $f(z)$ is normal in $|z| < 1$. (Cf. Noshiro [4] *pp.* 88-89)

The following lemma is a generalization of Picard's classical theorem.

LEMMA 2. (Lehto-Virtanen [1], Th. 9) *A meromorphic function cannot be normal in any neighbourhood of an isolated essential singularity.*

For a normal meromorphic function $f(z)$ in a domain D, we shall estimate the lengths or areas of the images of some curves or sets in D under $w = f(z)$. Let G be a doubly connected subdomain of D and let $\mu(G)$ denote the harmonic modulus of G. Then G is conformally equivalent to the annulus $G' = \{\zeta \mid 1 < |\zeta| < \exp \mu(G)\}$. For δ_0, $0 < \delta_0 < 1/2$, we consider $G'_{\delta_0} = \{\zeta \mid \exp \delta_0 \mu(G) < |\zeta| < \exp (1-\delta_0)\mu(G)\}$ and $L'_\delta = \{\zeta \mid |\zeta| = \exp \delta \mu(G)\}$, $\delta_0 \leqq \delta \leqq 1 - \delta_0$, and denote their images in G by G_{δ_0} and L_δ, respectively. Let $f(S)$ denote the Riemannian image of a set S in D under $w = f(z)$. The notations $A[f(\bar{G}_{\delta_0})]$ and $L[f(L_\delta)]$ will be used for the spherical area of $f(\bar{G}_{\delta_0})$, $\bar{G}_{\delta_0}$ being the closure of G_{δ_0}, and the spherical length of $f(L_\delta)$, respectively.

LEMMA 3. *Let $f(z)$ be normal meromorphic in a domain D, and $0 < \delta_0 < 1/2$. Then, for any doubly connected subdomain G of D with finite $\mu(G)$,*

$$A[f(\bar{G}_{\delta_0})] \leqq \frac{K}{\mu(G)}, \tag{1}$$

$$L[f(L_\delta)] \leqq \frac{K}{\mu(G)} \qquad (\delta_0 \leqq \delta \leqq 1 - \delta_0), \tag{2}$$

where K is a constant depending only on D, $f(z)$ and δ_0.

Denoting by $\chi(w_1, w_2)$ the spherical distance of w_1 and w_2 in the extended w-plane, we put $C(w_0, d) = \{w \mid \chi(w_0, w) < d\}$ $(d > 0)$.

LEMMA 4. (Carleson-Matsumoto). *Let $g(z)$ be meromorphic in an annulus $G: 1 \leqq |z| \leqq \exp \mu$ $(\mu > 0)$. If the image of G under $w = g(z)$ is contained in $C(w_0, d)$, $0 < d < \pi/2$, then the spherical diameter of the image of $|z| = \exp \mu/2$ under $w = g(z)$ is dominated by $A \exp(-\mu/2)$ whenever μ is sufficiently large $(\mu \geqq \mu_0)$, where A is a positive constant depending only on d.*

Moreover, if d is sufficiently small $(d < d_0)$, then $A < Bd$, where B is a positive constant. (Cf. Sario-Noshiro [6], *pp.* 128–129.)

3. **Theorem.** Before stating our Theorem, we shall prepare some notations. Let E be a totally disconnected compact set in the z-plane and let R be its complementary domain. Let $\{R_n\}$ be a normal exhaustion of R with an additional condition that each component $R_{n,m}(m = 1, 2, \cdots, N(n))$ of $R_n - \bar{R}_{n-1}$ is doubly connected. If every $R_{n,m}$ branches off into at most ρ regions $R_{n+1,k}$, we say that the exhaustion $\{R_n\}$ branches off at most ρ times everywhere. Now let L be the length of Noshiro's graph associated with $\{R_n\}$, and let $u(z) + iv(z)$ be the con-

formal mapping of $R - \bar{R}_0$ with at most a countable number of suitable slits onto the strip $0 < u < L$, $0 < v < 2\pi$ on the $w = u + iv$-plane with at most a countable number of suitable slits. Let β_r be a level curve $\{z \mid u(z) = r\}$ $(0 < r < L)$ and let $\beta_{r,m}(1 \leqq m \leqq n(r))$ be the components of β_r. We consider the components of $R_n - \bar{R}_k$ with $n > k \geqq 0$, which we call R-chains, being divided by $\beta_{r,m}$, or having it as a boundary component, and denote by $\mu(\beta_{r,m})$ the harmonic modulus of the longest *doubly* connected R-chain among them, where we say an R-chain is longer than another if the former contains the latter. Put

$$\mu(r) = \min_{1 \leqq m \leqq n(r)} \mu(\beta_{r,m})$$

THEOREM. *Suppose that there exists a normal exhaustion $\{R_n\}$ of R which branches off at most ρ $(\rho \geqq 2)$ times everywhere and that*

$$\lim_{r \to L} \mu(r) = \infty \ . \tag{3}$$

Then,

(i) *in the case $\rho = 2$, every normal meromorphic function in R with at least one essential singularity in E can have no more than* 3 *totally ramified values, and*

(ii) *in the case $\rho > 2$, every meromorphic function in R with at least one essential singularity in E can have no more than $\rho + 1$ totally ramified values.*

Any meromorphic function $f(z)$ with more than four totally ramified values is normal. (This is the reason why the condition "normal" is not necessary in the case $\rho > 2$.) In fact, if $f(z)$ has totally ramified values w_j $(j = 1, 2, \cdots, q > 4)$ with m_j, the minimum of the multiplicities of w_j-points, then

$$\sum_{j=1}^{q} \left(1 - \frac{1}{m_j}\right) > 2 \ ,$$

since $m_j \geqq 2$ $(j = 1, 2, \cdots, q)$. Thus we see from Lemma 1 that $f(z)$ is normal. Therefore, we have the following interesting

COROLLARY 1. *If there exists a normal exhaustion $\{R_n\}$ of R which branches off at most ρ $(\rho \geqq 1)$ times everywhere and if*

$$\lim_{r \to L} \mu(r) = \infty \ ,$$

then every function $f(z)$ meromorphic in R with at least one essential singularity in E can have no more than $\max(4, \rho + 1)$ *totally ramified values.*

COROLLARY 2. *Let E be a Cantor set with successive ratios ξ_n satisfying*

$$\lim_{n\to\infty} \xi_n = 0 .$$

Then every function $f(z)$ meromorphic in R with at least one essential singularity in E can have no more than four totally ramified values.

4. **Proof of Lemma 3.** Since all lemmas except for Lemma 3 were quoted from somewhere, we need only to prove Lemma 3. We put $\mu = \mu(G)$ for simplicity in this section. Let $\rho(f(z))$ denote the spherical derivative of $f(z)$, and let $d\sigma_D$ denote the hyperbolic metric with respect to D. Then, since $f(z)$ is normal in D, we have

$$\rho(f(z))\,|\,dz\,| < C\, d\sigma_D(z) , \tag{4}$$

where C is a constant depending on D and $f(z)$ (Lehto-Virtanen [1], Th. 3). By the principle of hyperbolic measure, it holds

$$d\sigma_D(z) \leqq d\sigma_G(z) . \tag{5}$$

Since $d\sigma_G(z)$ and $\rho(f(z))\,|\,dz\,|$ are conformally invariant, we have

$$\rho(g(\zeta))\,|\,d\zeta\,| < C\, d\sigma_{G'}(\zeta) , \tag{6}$$

where $g(\zeta)$ is the composite function $f(z(\zeta))$ of $f(z)$ with $z = z(\zeta)$, the mapping function of G' conformally onto G. We shall estimate $d\sigma_{G'}(\zeta)$. Let $w = \varphi(\zeta)$ be the function composed

$$w = \tan \frac{\pi}{2\mu}\left(u - \frac{\mu}{2}\right) = \frac{e^{((\pi/\mu)u - \pi/2)i} - 1}{i(e^{((\pi/\mu)u-\pi/2)i} + 1)}$$

with

$$u = \log \zeta .$$

Then we have

$$d\sigma_{G'}(\zeta) = \frac{|\,dw\,|}{1 - |\,w\,|^2} = \frac{|\,\varphi'(\zeta)\,|}{1 - |\,\varphi(\zeta)\,|^2}\,|\,d\zeta\,| . \tag{7}$$

By a simple computation, we have

$$\frac{|\,\varphi'(\zeta)\,|}{1 - |\,\varphi(\zeta)\,|^2}\,|\,d\zeta\,| = \frac{\pi}{2\mu\,|\,\zeta\,|\sin\left(\dfrac{\pi}{\mu}\log|\,\zeta\,|\right)}\,|\,d\zeta\,| . \tag{8}$$

From (6), (7), (8) and the inequality

$$\sin\left(\frac{\pi}{\mu}\log|\,\zeta\,|\right) \geqq \sin \pi\delta_0 \qquad \text{for } \zeta \in G'_{\delta_0} ,$$

we have

$$\rho(g(\zeta)) < \frac{C\pi}{2\mu\,|\,\zeta\,|\,\sin\delta_0\pi} \qquad \text{for } \zeta \in G'_{\delta_0}\,,$$

$$\begin{aligned} A[f(\bar{G}_{\delta_0})] &= A[g(\bar{G}'_{\delta_0})] \\ &= \int_0^{2\pi}\int_{e^{\delta_0\mu}}^{e^{(1-\delta_0)\mu}} \{\rho(g(\zeta))\}^2\,|\,\zeta\,|\,d\,|\,\zeta\,|\,d\theta \\ &< \int_0^{2\pi}\int_{e^{\delta_0\mu}}^{e^{(1-\delta_0)\mu}} \left(\frac{C\pi}{2\mu\,|\,\zeta\,|\,\sin\delta_0\pi}\right)^2 |\,\zeta\,|\,d\,|\,\zeta\,|\,d\theta \\ &= \frac{C^2\pi^3(1-2\delta_0)}{2\mu(\sin\delta_0\mu)^2} \end{aligned}$$

and

$$\begin{aligned} L[f(L_\delta)] &= L[g(L'_\delta)] \\ &= \int_0^{2\pi} \rho(g(\zeta))\,|\,\zeta\,|\,d\theta \\ &< \int_0^{2\pi} \frac{C\pi}{2\mu\sin\delta_0\pi}d\theta \\ &= \frac{C\pi^2}{\mu\sin\delta_0\pi} \qquad (\delta_0 \leqq \delta \leqq 1-\delta_0)\,. \end{aligned}$$

Thus it is enough for us to put

$$K = \max\left(\frac{C^2\pi^3(1-2\delta_0)}{2(\sin\delta_0\pi)^2}\,, \quad \frac{C\pi^2}{\sin\delta_0\pi}\right).$$

Obviously K is a positive constant depending on D, $f(z)$ and δ_0. The proof is complete.

5. Proof of Theorem. Now we shall prove our Theorem. Let E be a totally disconnected compact set in the z-plane whose complementary domain R satisfies all the assumptions of Theorem. Suppose that there exists a normal meromorphic function $f(z)$ in R with at least one essential singularity z_0 in E and with more than $\rho+1$ totally ramified values $w_1, w_2, \cdots, w_q$, $q \geqq \rho+2$.

Put

$$d_1 = \min_{i \neq j} \chi(w_i, w_j)$$

and

$$(9) \quad \mu_1 = \max\left(\frac{4(\rho+1)K}{d_1}, \frac{3(\rho+1)K}{d_0}, \mu_0, 2\log 3(\rho+1)B, 4(\rho+1)K\right),$$

where d_0, μ_0, B and K are those given in § 2. By our assumption (3), there exists r_1, $0 < r_1 < L$, such that

$$\mu(r) > \mu_1 \qquad \text{for any } r > r_1 .$$

The level line $\beta_r = \{z \mid u(z) = r\}$ consists of a finite number of Jordan curves $\beta_{r,m}$ ($m = 1, 2, \cdots, n(r)$) and one of them, say $\beta_{r,1}$, encloses z_0. Let $D_{1,1}$ be the longest doubly connected R-chain containing this $\beta_{r,1}$. Then the harmonic modulus $\mu(D_{1,1})$ of $D_{1,1}$ is equal to $\mu(\beta_{r,1})$ ($\geqq \mu(r)$) and hence is larger than μ_1. We see $\mu(D_{1,1})$ is finite. In fact, if $\mu(D_{1,1}) = \infty$, one of the component of $\partial D_{1,1}$ must be the point z_0, so that the normal function $f(z)$ has an isolated essential singularity. This is contradictory to Lemma 2. Therefore $D_{1,1}$ must branch off. Suppose $D_{1,1}$ is a component of $R_n - \bar{R}_{n'}$ with $n > n'$, and branches off into at most ρ, say $Q(2)$, regions $R_{n+1,q}$ ($q = 1, 2, \cdots, Q(2)$). For each q, we consider the longest doubly connected R-chain $D_{2,q}$ containing $R_{n+1,q}$. They all have moduli greater than μ_1 and one of them, say $D_{2,1}$, separates z_0 from $D_{1,1}$. Its harmonic modulus $\mu(D_{2,1})$ is finite by the same reason as above. Hence $D_{2,1}$ is a component of the open set $R_{\tilde{n}} - \bar{R}_n$ for some $\tilde{n}$ and branches off into at most ρ, say $Q_{3,1}$, regions $R_{\tilde{n}+1,q'}$ ($q' = 1, 2, \cdots, Q_{3,1}$). For $q = 2, 3, \cdots, Q(2)$, if $\mu(D_{2,q}) = \infty$, one of the boundary components of $D_{2,q}$ is a point $z_{2,q}$ in E and $f(z)$ is meromorphic at $z_{2,q}$ by Lemma 2. If $\mu(D_{2,q}) < \infty$, we obtain at most ρ, say $Q_{3,q}$, R-chains $D_{3,q'}$ ($q' = \sum_{p=1}^{q-1} Q_{3,p} + r$ with $r = 1, 2, \cdots, Q_{3,q}$, where $Q_{3,p} = 0$ when $\mu(D_{2,p}) = \infty$) in the same manner as above. Thus we have at most ρ^2 R-chains $D_{3,q}$ such that their harmonic moduli are greater than μ_1, and one of them encloses z_0. Moreover, each of them branches off into at most ρ regions if the modulus is finite, or has a point in E as one of its boundary components at which $f(z)$ is meromorphic, if the modulus is infinite.

Continuing inductively we obtain a set of R-chains $D_{p,q}$ with $p = 1, 2, \cdots$ and $q = 1, 2, \cdots, Q(p) = \sum_{r=1}^{Q(p-1)} Q_{p,r} \leqq \rho^{p-1}$, which has the following properties:

(10) $\displaystyle\bigcup_{p=1}^{\infty} \bigcup_{q=1}^{Q(p)} \bar{D}_{p,q} \supset \Omega$, where Ω denotes the intersection of R with the set bounded by the Jordan curve $\beta_{r,1}$,

(11) $$\mu(D_{p,q}) > \mu_1 \qquad (p = 1, 2, \cdots; q = 1, 2, \cdots, Q(p)) ,$$

and

(12) $D_{p,q}$ branches off into $Q_{p+1,q}$($\leqq \rho$) R-chains $D_{p+1,q'}$, if its harmonic modulus $\mu(D_{p,q})$ is finite ,

or

(12)′ $D_{p,q}$ has a point $z_{p,q} \in E$ as one of its boundary components and $f(z)$ is meromorphic at $z_{p,q}$, if $\mu(D_{p,q})$ is infinite.

Now each $D_{p,q}$ is conformally equivalent to the annulus $1 < |\zeta| < \exp \mu(D_{p,q})$. For $D_{p,q}$ with $\mu(D_{p,q}) < \infty$, we denote by $\beta_{p,q}$ the closed Jordan curve corresponding to $|\zeta| = \exp(1/2)\mu(D_{p,q})$. Such $D_{p,q}$ branches off into $Q_{p+1,q}(\leqq \rho)$ R-chains $D_{p+1,q'}$ ($q' = \sum_{r=1}^{q-1} Q_{p+1,r} + s$ with $s = 1, 2, \cdots, Q_{p+1,q}$). We shall denote by $\Delta_{p,q}$ the $(Q_{p+1,q} + 1)$-ply connected domain bounded by $\beta_{p,q}$ and $\beta_{p+1,q'}$'s, where $\beta_{p+1,q'} = z_{p+1,q'}$ when $\mu(D_{p+1,q'}) = \infty$. Taking a point $\zeta_{p,q}$ in $f(\beta_{p,q})$ and a point $\zeta_{p+1,q'}$ in $f(\beta_{p+1,q'})$ for each q', we consider spherical discs $C(\zeta_{p,q}, K/\mu_1)$ and $C(\zeta_{p+1,q'}, K/\mu_1)$'s, which contain $f(\beta_{p,q})$ and $f(\beta_{p+1,q'})$'s, respectively, because of (11) and by Lemma 3. We set $H = K/\mu_1$.

6. We shall study the image of $\Delta_{p,q}$ under $w = f(z)$. Let Δ be one of $\Delta_{p,q}$. Then Δ is an at most $(\rho+1)$-ply connected domain, whose boundary components are denoted by $\beta_1, \beta_2, \cdots, \beta_Q$, $Q \leqq \rho + 1$. As mentioned at the end of the preceding section, images $f(\beta_q)$ of these β_q ($q = 1, 2, \cdots, Q$) are contained in some discs C_q ($q = 1, 2, \cdots, Q$) with radius H. Therefore we can cover the set $\bigcup_{q=1}^{Q} C_q$ with at most Q closed discs G_i ($i = 1, 2, \cdots, m \leqq Q$) which are disjoint by pair and whose radii are less than $d_1/4$, since $QH \leqq (\rho + 1)H < d_1/4$, so that each G_i contains at most one totally ramified value of $f(z)$. Now $\nu(w, f, \Delta)$ denotes the number of w-points of $f(z)$ in Δ, multiplicities being taken into account. Then, obviously, $\nu(w, f, \Delta)$ is constant outside $\bigcup_{i=1}^{m} G_i$. We shall show that $\nu(w, f, \Delta) = 0$ outside $\bigcup_{i=1}^{m} G_i$, that is, we can cover $\bigcup_{q=1}^{Q} C_q$ with only one closed disc G with radius less than $d_1/4$ and the image $f(\Delta)$ of Δ is contained in G. Supposing contrary that $\nu(w, f, \Delta) > 0$ outside $\bigcup_{i=1}^{m} G_i$, we shall first prove two propositions.

PROPOSITION 1. *Let F denote a closed set which consists of a finite number of components and whose complement Ω is a domain. Suppose that F contains $\bigcup_{q=1}^{Q} f(\beta_q)$ and that there are two totally ramified values of $f(z)$, say w_1 and w_2, in Ω. Then there is a simple closed regular curve β in Δ whose image $f(\beta)$ lies on some simple arc joining w_1 and w_2 in Ω.*

PROOF. We call a value w a ramified value of $f(z)$ in Δ if $f(z)$ has a w-point with multiplicity > 1. We take a simple regular arc Γ joining w_1 to w_2 in Ω on which there are no ramified values of $f(z)$ in Δ except for the end points w_1 and w_2. The inverse image $f^{-1}(\Gamma)$ in Δ consists of a finite number of simple regular arcs $\{\gamma\}$ joining a w_1-point to a w_2-point in Δ. We note that, for any w_1-point or any w_2-point in Δ, there are at least two γ's having it as a common end point, because w_1-points and w_2-points are multiple points. Let a_1 be any one of the w_1-points in Δ and let $\gamma^{(1)} \in \{\gamma\}$ be an arc joining a_1 to one of the w_2-points, being denoted by b_1. As mentioned just above, there is another $\gamma^{(2)} \neq \gamma^{(1)}$ in $\{\gamma\}$ ending at b_1. We denote by a_2 the other end point of $\gamma^{(2)}$ which is a w_1-point. If $a_2 = a_1$, we may take $\gamma^{(1)} - \gamma^{(2)}$ as β. If $a_2 \neq a_1$, we consider the curve $\gamma^{(1)} - \gamma^{(2)} + \gamma^{(3)}$, $\gamma^{(3)}$ being an arc in $\{\gamma\}$ which differs from $\gamma^{(2)}$ and starts from a_2. Let b_2 denote the other end point of $\gamma^{(3)}$. If $b_2 = b_1$, the part $-\gamma^{(2)} + \gamma^{(3)}$ of our curve can be taken as β. If $b_2 \neq b_1$, there is an arc $\gamma^{(4)} \neq \gamma^{(3)}$ in $\{\gamma\}$ ending at b_2 and we consider the curve $\gamma^{(1)} - \gamma^{(2)} + \gamma^{(3)} - \gamma^{(4)}$. In general, assume that we had a curve $\gamma_n = \sum_{i=1}^{n} (-1)^{i-1}\gamma^{(i)}$ such that $\gamma^{(i)}$ and $\gamma^{(i+1)}$ have a w_1-point $a_{(i+2)/2}$ if i is even or a w_2-point $b_{(i+1)/2}$ if i is odd as a common end point and these $\{a_k\}$ and $\{b_l\}$ are distinct from each other. Then we extend γ_n by connecting an arc $(-1)^n\gamma^{(n+1)}$, $\gamma^{(n+1)} \in \{\gamma\}$, which has a w_1-point or a w_2-point as a common end point with $\gamma^{(n)}$. If the other end point of $\gamma^{(n+1)}$ is a w_1-point or a w_2-point being already passed by γ_n, then the last part of $\gamma_{n+1} = \sum_{i=1}^{n+1} (-1)^{i-1}\gamma^{(i)}$ starting from it can be taken as β. Otherwise we continue our construction. Since the number of w_1-points and w_2-points in Δ is finite, we always obtain a wanted curve β by the above construction.

PROPOSITION 2. *Let G be a closed disc disjoint from F. Suppose that G and F contain $f(\beta_1)$ and $\bigcup_{q=2}^{Q} f(\beta_q)$, respectively and that there are two totally ramified values of $f(z)$, say w_1 and w_2, such that $w_1 \in \Omega - G$ and $w_2 \in G$. Then either there is a simple closed regular curve β in Δ whose image $f(\beta)$ lies on some simple arc joining w_1 and w_2 in Ω, or there are two simple arcs β' and β'' in Δ joining a w_1-point in Δ to the boundary component β_1, whose images $f(\beta')$ and $f(\beta'')$ lie on some simple arc joining w_1 and w_2 in Ω.*

PROOF. In the present case, the inverse image $f^{-1}(\Gamma)$ in Δ consists of a finite number of simple regular arcs joining a w_1-point to a w_2-point, ones joining a w_1-point or a w_2-point to β_1 and ones joining a point on β_1 to another point on β_1. We pick up only the arcs of the former two kinds and denote them by $\{\gamma\}$. Using this $\{\gamma\}$, we construct a curve in the similar manner as above. Then either we obtain a wanted closed curve β, or, for some n, the arc $\gamma^{(n)}$ connected to γ_{n-1} is of the second kind, so that γ_n joins the w_1-point a_1 to β_1. In the latter case, we construct another curve begining with $\tilde{\gamma}^{(1)} \neq \gamma^{(1)}$ in $\{\gamma\}$ which starts from a_1, and obtain a β or a curve $\tilde{\gamma}_{\tilde{n}}$ for some $\tilde{n}$ joining a_1 to β_1, while

$\beta' = \gamma_n$ and $\beta'' = \tilde{\gamma}_n$ give a wanted pair.

7. Since each G_i ($i = 1, 2, \cdots, m$) contains at most one totally ramified values of $f(z)$, the following two cases are possible.

Case (i). The complement $\Omega^{(1)}$ of $F^{(1)} = \bigcup_{i=1}^{m} G_i$ contains not less than two totally ramified values,
or

Case (ii). The number m of $\{G_i\}$ is equal to $\rho + 1$ and there is just one totally ramified value in $\Omega^{(1)}$. In this case, each G_i ($i = 1, 2, \cdots, m = \rho + 1$) coincides with some $\bar{C}_q$ ($q = 1, 2, \cdots, Q = \rho + 1$) and they contain images $f(\beta_q)$ ($q = 1, 2, \cdots, Q = \rho + 1$) one by one.

Case (i). We take any two, say w_1 and w_2, among the totally ramified values contained in $\Omega^{(1)}$. By Proposition 1, there is a closed curve β, which we denote by $\beta^{(1)}$, in Δ, whose image $f(\beta^{(1)})$ lies on a curve $\Gamma^{(1)}$ joining w_1 and w_2 in $\Omega^{(1)}$. The curve $\beta^{(1)}$ divides Δ into two domains. We denote by Δ_1 one with connectivity not greater than that of the other. Then Δ_1 is bounded by $\beta^{(1)}$ and some of $\{\beta_q\}$, say $\beta_1, \beta_2, \cdots, \beta_{Q^{(1)}}$, where $Q^{(1)} \leqq [(\rho+1)/2]$. Hence the image of $\partial\Delta_1$ is covered with some of $\{G_i\}$, say $G_1, G_2, \cdots, G_{m^{(1)}}(m^{(1)} \leqq Q^{(1)})$, and $\Gamma^{(1)}$. We denote by $\Omega^{(2)}$ the complementary domain of $F^{(2)} = (\bigcup_{i=1}^{m^{(1)}} G_i) \cup \Gamma^{(1)}$. Since

$$\left[\frac{\rho+1}{2}\right] \leqq \rho - 2 \qquad \text{for } \rho \geqq 4\,,$$

we see that $\Omega^{(2)}$ contains at least two totally ramified values, say w_3 and w_4, distinct from w_1 and w_2, excepting the following two cases.

Case (i)-(a). The case that $\rho = 2$ (which implies $m^{(1)} = Q^{(1)} \leqq 1$), $m^{(1)} = 1$ and $G_1(\supset f(\beta_1))$ contains w_3 or w_4, say w_4.

Case (i)-(b). The case that $\rho = 3$ (which implies $m^{(1)} = Q^{(1)} \leqq 2$) $m^{(1)} = 2$ and $G_1(\supset f(\beta_1))$ and $G_2(\supset f(\beta_2))$ contain two values among w_3, w_4 and w_5, say w_4 and w_5, one by one.

By Proposition 1, there is a closed curve $\beta^{(2)}$ in Δ_1 whose image $f(\beta^{(2)})$ lies on a curve $\Gamma^{(2)}$ joining w_3 and w_4 in $\Omega^{(2)}$. The curve $\beta^{(2)}$ divides Δ_1 into two domains. We denote by Δ_2 one not having $\beta^{(1)}$ in its boundary. The image of $\partial\Delta_2$ is covered with some of $\{G_i\}_{i=1}^{m^{(1)}}$ and $\Gamma^{(2)}$ so that $\Omega^{(3)}$, the complementary domain of the sum $F^{(3)}$ of these G_i's and $\Gamma^{(2)}$, contains w_1 and w_2 and there is a closed curve $\beta^{(3)}$ in Δ_2, again by Proposition 1, whose image $f(\beta^{(3)})$ lies on $\Gamma^{(1)}$. Thus, repeating the above argument again and again, we obtain a set of closed curves $\{\beta^{(n)}\}_{n=1}^{\infty}$ in Δ such that they are disjoint by pair and that the image $f(\beta^{(n)})$ covers $\Gamma^{(1)}$ if n is odd or $\Gamma^{(2)}$ if n is even. This is impossible.

In the excepted cases, we use Proposition 2 under the setting that

$F^{(2)} = \Gamma^{(1)}$ in the case (a) or $F^{(2)} = \Gamma^{(1)} \cup G_2$ in the case (b) and $G = G_1$. Then we see that there is a closed curve $\beta^{(2)}$ or a pair of arcs $\beta'^{(2)}$ and $\beta''^{(2)}$ joining a w_3-point to the boundary component β_1 of Δ_1 whose images lie on a curve $\Gamma^{(2)}$ joining w_3 and w_4 in $\Omega^{(2)}$, the complement of $F^{(2)}$. The curve $\beta^{(2)}$ or $-\beta'^{(2)} + \beta''^{(2)}$ divides Δ_1 into two domains. We denote by Δ_2 one not having $\beta^{(1)}$ in its boundary. The image of $\partial\Delta_2$ is covered with $F^{(3)} = \Gamma^{(2)} \cup G_1$ in the case (a), or with $F^{(3)} = \Gamma^{(2)} \cup G_1$ or the sum of $\Gamma^{(2)} \cup G_1$ and G_2 in the case (b), so that the complementary domain $\Omega^{(3)}$ of $F^{(3)}$ contains w_1 and w_2 and there is a closed curve $\beta^{(3)}$ in Δ_2 whose image $f(\beta^{(3)})$ lie on $\Gamma^{(1)}$. Repeating the same argument again and again, we obtain a set of curves $\{\beta^{(2n-1)}, \beta^{(2n)}$ or $-\beta'^{(2n)} + \beta''^{(2n)}\}_{n=1}^{\infty}$ in Δ such that they are disjoint by pair, and that $f(\beta^{(2n-1)})$ cover $\Gamma^{(1)}$ and $f(\beta^{(2n)})$ or $f(-\beta'^{(2n)} + \beta''^{(2n)})$ cover $\Gamma^{(2)}$ or its part joining w_3 to ∂G_1, respectively. This is impossible.

Case (ii). We may assume $G_i = \bar{C}_i$ contains w_i $(i = 1, 2, \cdots, \rho + 1)$ and $w_{\rho+2}$ is contained in the complement of $\bigcup_{i=1}^{\rho+1} G_i$. Setting $F^{(1)} = \bigcup_{i=1}^{\rho} G_i$ and $G = G_{\rho+1}$ we see from Proposition 2 that there is a closed curve $\beta^{(1)}$ or a pair of arcs $\beta'^{(1)}$ and $\beta''^{(1)}$ joining a $w_{\rho+2}$-point to $\beta_{\rho+1}$ in Δ whose images lie on a curve $\Gamma^{(1)}$ joining $w_{\rho+1}$ and $w_{\rho+2}$ in $\Omega^{(1)}$, the complement of $F^{(1)}$. The curve $\beta^{(1)}$ or $-\beta'^{(1)} + \beta''^{(1)}$ divides Δ into two domains. We denote by Δ_1 one with connectivity not greater than that of the other. Then $f(\partial\Delta_1)$ is covered with some of $\{G_i\}$, say $G_1, \cdots, G_{m^{(1)}} (m^{(1)} \leqq \rho - 1)$ and $\Gamma^{(1)} \cup G_{\rho+1}$. If $m^{(1)} < \rho - 1$, then $w_{\rho-1}$ and w_ρ are contained in $\Omega^{(2)}$, the complement of $F^{(2)} = (\bigcup_{i=1}^{m^{(1)}} G_i) \cup (\Gamma^{(1)} \cup G_{\rho+1})$, and we use Proposition 1. If $m^{(1)} = \rho - 1$, then $w_{\rho-1}$ and w_ρ are contained in $\Omega^{(2)}$, the complement of $F^{(2)} = (\bigcup_{i=1}^{\rho-2} G_i) \cup (\Gamma^{(1)} \cup G_{\rho+1})$, and we use Proposition 2 under the setting $G = G_{\rho-1}$. Then we see that there is a closed curve $\beta^{(2)}$ or a pair of arcs $\beta'^{(2)}$ and $\beta''^{(2)}$ joining a w_ρ-point to $\beta_{\rho-1}$ in Δ whose images lie on a curve $\Gamma^{(2)}$ joining $w_{\rho-1}$ and w_ρ in $\Omega^{(2)}$. The curve $\beta^{(2)}$ or $-\beta'^{(2)} + \beta''^{(2)}$ divides Δ_1 into two domains, one of which has not $\beta^{(1)}$ or $-\beta'^{(1)} + \beta''^{(1)}$ in its boundary. We denote it by Δ_2. Then $f(\partial\Delta_2)$ is covered with some of $\{G_1, G_2, \cdots, G_{m^{(1)}}, G_{\rho+1}\}$ and $\Gamma^{(2)}$ when $m^{(1)} < \rho - 1$ or with some of $\{G_1, G_2, \cdots, G_{\rho-2}, G_{\rho+1}\}$ and $\Gamma^{(2)} \cup G_{\rho-1}$ when $m^{(1)} = \rho - 1$. If the covering of $f(\partial\Delta_2)$ does not contain $G_{\rho+1}$, the complementary domain $\Omega^{(3)}$ of their sum $F^{(3)}$ contains $w_{\rho+1}$ and $w_{\rho+2}$ and we use Proposition 1. If it contains $G_{\rho+1}$, we take their sum deleting $G_{\rho+1}$ and denote it by $F^{(3)}$. The complement $\Omega^{(3)}$ of $F^{(3)}$ contains $w_{\rho+1}$ and $w_{\rho+2}$ and we use Proposition 2 setting $G = G_{\rho+1}$. Thus there is a closed curve $\beta^{(3)}$ or a pair of arcs $\beta'^{(3)}$ and $\beta''^{(3)}$ in Δ_2 whose images lie on $\Gamma^{(1)}$. Repeating the above argument, we obtain a set of curves $\{\beta^{(n)}$ or $-\beta'^{(n)} + \beta''^{(n)}\}$ in Δ such that they are disjoint by pair and that $f(\beta^{(n)})$ or $f(-\beta'^{(n)}+\beta''^{(n)})$ cover $\Gamma^{(1)}$ or its part joining $w_{\rho+2}$ to $\partial G_{\rho+1}$ for any odd n. This is absurd.

8. We conclude now:

(13) For every $\Delta_{p,q}$ $(p=1,2,\cdots; q=1,2,\cdots,Q(p))$ there is a spherical closed disc with the spherical radius $(\rho+1)H$ containing its image $f(\Delta_{p,q})$.

Next consider $\beta_{p,q}$ for $p\geqq 2$. The domain $\Delta_{p,q}$ and some $\Delta_{p-1,q'}$ have $\beta_{p,q}$ as the common boundary and

$$D_{p,q}\subset \Delta_{p,q}\cup\beta_{p,q}\cup\Delta_{p-1,q'} .$$

In view of (13) the images of $\Delta_{p,q}\cup\beta_{p,q}\cup\Delta_{p-1,q'}$ and consequently of $D_{p,q}$ are contained in a spherical closed disc with spherical radius $2(\rho+1)H$. By applying Lemma 4 to $D_{p,q}$ for $d=3(\rho+1)H<d_0$, we see that the diameter of $f(\beta_{p,q})$ is less than $3(\rho+1)HB\exp(-(1/2)\mu_1)<H$. For $p\geqq 2$, each boundary component of $\Delta_{p,q}$ thus has an image with diameter less than H. By the same reasoning as above we infer

(14) For $p\geqq 2$ the image of every $\Delta_{p,q}$ is contained in a spherical closed disc with spherical radius $((\rho+1)H)/2$.

By induction we deduce for every n:

(15) For $p\geqq n$ the image of every $\Delta_{p,q}$ is contained in a spherical closed disc with spherical radius $((\rho+1)H)/2^{n-1}$.

Let Ω' be the intersection of R and the domain bounded by the Jordan curve $\beta_{1,1}$ and let z^* be a point of $\beta_{1,1}$. Then it follows from the property (10) of $\{D_{p,q}\}$ that

$$\Omega'\subset\bigcup_{p=1}^{\infty}\bigcup_{q=1}^{Q(p)}\bar{\Delta}_{p,q}$$

and consequently for any $z'\in\Omega'$ there is a $\Delta_{p',q'}$ whose closure contain z'. From (15) we have for a chain $\{\Delta_{p,q_p}\}_{p=1}^{p'}$ of $\Delta_{p,q}$ joining $\Delta_{1,1}$ and $\Delta_{p',q'}$,

$$\begin{aligned}\chi(f(z'),f(z^*)) &\leqq \sum_{p=1}^{p'}\operatorname{diam} f(\Delta_{p,q_p})\\ &\leqq 2\sum_{p=1}^{\infty}\frac{\rho+1}{2^{p-1}}H=2(\rho+1)H<\frac{1}{2} ,\end{aligned}$$

where the diameter is in terms of spherical metric. By means of a linear transformation we conclude that $f(z)$ is bounded in Ω'. On the other hand, on applying the criterion of Pfluger [5]-Mori [3] to the annular domains $\{D_{p,q}\}$, we see easily that the part E' of E contained in the region bounded by $\beta_{1,1}$ has a complement of class O_{AB}. Hence each point of E' must be a removable singularity of a bounded function $f(z)$. This contradicts our assumption that $z_0 \in E'$ is an essential singularity of $f(z)$. Thus we conclude that $f(z)$ cannot have more than $\rho + 1$ totally ramified values. This completes the proof of Theorem.

9. Any meromorphic function cannot be normal in any neighbourhood of an isolated essential singularity, so that it cannot have more than four totally ramified values there. On the other hand, there are sets E with the conditions of Theorem for $\rho = 2, 3$, whose complementary domain permits normal meromorphic functions with E as the set of singularities notwithstanding that they cannot have more than four totally ramified values (see [2]).

Problem. Is there any E whose complementary domain permits no normal or no exceptionally ramified meromorphic functions with E as the set of singularities? Here a meromorphic function $f(z)$ is called exceptionally ramified, if $f(z)$ has totally ramified values w_j with m_j, the minimum of the multiplicities of w_j-points, such that

$$\sum\left(1 - \frac{1}{m_j}\right) > 2\,.$$

REFERENCES

[1] O. LEHTO AND K. I. VIRTANEN, Boundary behaviour and normal meromorphic functions, Acta Math., 97 (1957), 47-65.

[2] K. MATSUMOTO, On exceptional values of meromorphic functions with the set of singularities of capacity zero, Nagoya Math. J., Vol. 18 (1961), 171-191.

[3] A. MORI, A note on unramified abelian covering surfaces of a closed Riemann surface, J. Math. Soc. Japan 6 (1954), 162-176.

[4] K. NOSHIRO, Cluster sets, Springer-Verlag, 1960.

[5] A. PFLUGER, Sur l'existence de fonctions non constantes, analytiques, uniformes et bornees sur une surface de Riemann ouverte, C. R. Acad. Sci. Paris 230 (1950), 166-168.

[6] L. SARIO AND K. NOSHIRO, Value distribution theory, D. Van Nostrand, 1966.

MIE UNIVERSITY
AND
NAGOYA UNIVERSITY