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NON LINEAR FUNCTIONAL EQUATIONS

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The aim of this note is to study some problems in the theory of nonlinear functional equations; this subject has been studied by M. Krasnoselskii, F. Browder, J. L. Lions and others. We prove here some results on solvability of functional equations; these results generalize some known results.

1. We recall some definitions: let B be a real Banach space, B^* its dual and $\langle \cdot \rangle: B^* \times B \rightarrow \boldsymbol{R}$ the duality pairing of B^* and B. A nonlinear mapping $f: B \rightarrow B^*$ *defined everywhere* is said to be *monotone* if $\langle fx - fy, x - y\rangle \geqq 0$; we say that a mapping $f: B \rightarrow B^*$ is said to be *hemicontinuous* if it is continuous from line segments of B to the weak topology of B^*. $f: B \rightarrow B^*$ is called *demicontinuous* if it is continuous from the strong topology of B to the weak topology of B^*, i.e. $x_n \rightarrow x$ implies $\lim \langle fx_n, y\rangle = \langle fx, y\rangle$ for all $y \in B$. We remark that these definitions were introduced by F. E. Browder (cf. [1]). We prove the following result:

PROPOSITION 1. *Let B be a reflexive Banach space and let $f: B \rightarrow B^*$ be a monotone and hemicontinuous mapping; then f is demicontinuous.*

PROOF. Let $x_n \rightarrow x_0$; we first prove that $\|fx_n\| \leqq M$. Since f is monotone, we have $\langle fx_n - fx, x_n - x\rangle \geqq 0$ for all x; putting $y_n = fx_n/\|fx_n\|$, we have $\langle y_n - fx/\|fx_n\|, x_n - x\rangle \geqq 0$. Since $\|y_n\| = 1$ and B is reflexive, y_n converges weakly to y_0; thus $\langle y_0 - 0, x_0 - x\rangle \geqq 0$ for all x and hence $y_0 = 0$, a contradiction to $\|y_0\| = 1$. Hence the sequence (fx_n) has a weakly convergent subsequence $fx_{n_i} \rightharpoonup w$. Since $\langle fx - fx_{n_1}, x - x_{n_i}\rangle \geqq 0$, we have $\langle fx - w, x - x_0\rangle \geqq 0$. Let $x = x_0 + tu, t \geqq 0$; then $\langle f(x_0 + tu) - w, tv\rangle \geqq 0$ and since f is hemicontinuous, we have $\langle fx_0 - w, u\rangle \geqq 0$ for all u. Consequently $fx_0 = w$ and f is demicontinuous.

As a consequence, we have the following result:

COROLLARY 1. *Let B be a Banach space and, $f: B \rightarrow B^*$ be a mapping as in Proposition 1; if B is finite dimensional, then f is continuous.*

We now prove a result (cf., Theorem [3]) which is useful in the sequal (cf., [2], p. 314) and which is well-known to specialists:

PROPOSITION 2. *Let B be a finite dimensional Banach space and let $f: B \to B$ (we identify B with its dual), be a monotone and hemicontinuous mapping; if $\langle fx, x\rangle > 0$ for all $x \in B$ verifying $\|x\| = R$, then there exists a $\|x_0\| \in B$ such that $\|x_0\| < R$ and $fx_0 = 0$.*

PROOF. By Corollary 1, f is continuous; let $D = \{x \in B \mid \|x\| < R\}$ be the open ball. To prove $fx_0 = 0$, it is sufficient to prove that the continuous mapping $g = I - f$ has a fixed point $x_0 \in B$. But

$$\langle gx, x\rangle = \langle x - fx, x\rangle = \langle x, x\rangle - \langle fx, x\rangle < \|x\|^2$$

for $x \in \partial D$. Then the mapping $\bar{g}: B \to B$ defined by

$$\bar{g}(x) = \begin{cases} gx & \text{if } \|gx\| < R \\ \dfrac{Rgx}{\|gx\|} & \|g(x)\| \geqq R \end{cases}$$

is continuous and maps the closed ball $\bar{D}$ into itself; hence $\bar{g}$ has a fixed point $x_0 \in \bar{D}$ by Brower's "fixed point" theorem: clearly $\|x_0\| < R$; for if $\|x_0\| = R$, we have $\bar{g}(x_0) = x_0 = (Rgx_0)/\|gx_0\|$. Hence $\langle x_0, x_0\rangle = R\langle x_0, gx_0\rangle/\|gx_0\| < R\|x_0\|^2/\|gx_0\|$ i.e. $\|gx_0\| < R$, a contradiction to $\|gx_0\| = R$.

2. We say that a mapping $f: B \to B^*$ is *weakly coercive* if $\|fx\|_* \to \infty$ as $\|x\| \to \infty$ (here $\|\ \|_*$ denotes the norm in B^*). We prove the following result:

THEOREM 1. *Let B be a reflexive and separable Banach space and let $f: B \to B^*$ be a monotone mapping which is i) weakly coercive and ii) hemicontinuous; then f is onto.*

PROOF. Let (x_i) be a dense subset, i.e. a complete system and let W_k denote the subspace spanned by $\{x_1, \cdots, x_k\}$; we have $W_k \subset B$ and $\bigcup_{k \geqq 0} W_k$ is dense in B. We first prove the existence of approximate solutions $u_k \in W_k$ of the equation $fx = y$ satisfying $\langle fu_k, x\rangle = \langle h, x\rangle$ for every $x \in W_k$. We prove the following lemma:

LEMMA. *Let B be as in the Theorem 1; if $f: B \to B^*$ is monotone, hemicontinuous and satisfies (i) (weakly coercive), then $\langle fu, x\rangle = \langle h, x\rangle$ has at least one solution u_k in W_k.*

PROOF. Let (x'_k) be a dual base of (x_k), i.e. $\langle x'_j, x_i\rangle = \delta_{ij}$; the equation $\langle fu_k, x\rangle = \langle u, x\rangle$ is equivalent to $fu_k - h \in W_k^{\perp}$ where $W_k^{\perp} \subset B^*$ is the annihilation of W_k. Let $p_k: B^* \to W_k^{\perp}$ be the projection $p_k w = \sum_1^k a_i x'_i$, $a_i \in \boldsymbol{R}$; we have $p_k w = \sum_1^k < p_k w, x_i > x'_i$. The equation $\langle fu_k, x\rangle = \langle h, x\rangle$

can then be written as $p_k f u_k = p_k h$; clearly the mapping $\phi_k = p_k f\colon B \to W_k^{\perp}$ is a finite dimensional and we have $\langle \phi_k u_k, u_k\rangle = \langle p_k f u_k, u_k\rangle = \langle f u_k, u_k\rangle$. Let $H_t = tI + (1-t)(p_k f - p_k h)$, $0 \leqq t \leqq 1$, be a homotopy between the identity mapping I and $p_k f - p_k h$; we have

$$\begin{aligned}\langle H_t u_k, u_k\rangle &= \langle t u_k + (1-t)(p_k f u_k - p_k h), u_k\rangle \\ &> 0\end{aligned}$$

on a sphere of sufficiently large radius by i). Thus the equation $\langle f u_k, x\rangle = \langle h, x\rangle$ is solvable by Proposition 2 (cf., Krasnoselskii [2], p.314).

PROOF OF THEOREM. Clearly we have $\| u_k \| \leqq M$; for $|\langle f u_k, u_k\rangle| = |\langle h, u_k\rangle| \leqq \| h \|_* \| u_k \|$ implies the existence of M by i). Since B is reflexive, the sequence (u_k) is weakly relatively compact and hence converges weakly to a limit u_0; since $\langle f u_k - fx, u_k - x\rangle \geqq 0$, we have $\langle h - fx, u_k - x\rangle \geqq 0$. As $k \to \infty$, we obtain $\langle h - fx, u_0 - x\rangle \geqq 0$ for any $x \in B$.

Let $x_k \in B$ be a sequence such that $x_k \to x$; since f is demicontinuous (cf., Proposition 1 and Remark), we have $f x_k \rightharpoonup fx$ weakly and hence

$$\langle f x_k, x_k\rangle = \langle f x_k, x\rangle + \langle f x_k, x_k - x\rangle$$

tends to $\langle fx, x\rangle$ since $|\langle f x_k, x_k - x\rangle| \leqq \| f x_k \|_* \| x_k - x \| \to 0$. Put $x = u_0 - tz$ where $t > 0$; then $\langle h - f u_0, z\rangle \geqq 0$ as $t \to 0$. Hence $f u_0 = h$.

In many applications, it is desirable to calculate explicitly a solution of the equation $f(x) = 0$; in Theorem 1, the Galerkin approximation method shows that a solution can be obtained as a weak limit. In fact we prove the following:

THEOREM 2. *Let B be a separable and reflexive Banach space and let $f\colon B \to B^*$ be a weakly coercive and hemicontinuous mapping; if, moreover, f is strongly monotone, i.e. f satisfies*

$$\langle fy - fx, y - x\rangle \geqq \mu(\| y - x \|)\, \| x - y \|$$

where $\mu(t) \geqq 0$ for $t \geqq 0$ and is a continuous increasing function such that $\mu(0) = 0$ and $\lim_{t\to\infty} \mu(t) = \infty$; then the subsequence (u_{k_i}) in Theorem 1 converges strongly to u_0.

PROOF. Since $\langle f u_k - fx, u_k - x\rangle \geqq \mu(\| u_k - x \|)\, \| u_k - x \|$ we have, $\langle f u_0 - fx, u_0 - x\rangle \geqq \underline{\lim}_{k_i\to\infty} \mu(\| u_{k_i} - x \|)\, \| u_{k_i} - x \|$ as $k_i \to \infty$; since $u_{k_i} \rightharpoonup u_0$ weakly and the norm is weakly lower semicontinuous, we have $\| u_0 - x \| \leqq \underline{\lim}_{k_i\to\infty} \| u_{k_i} - x \|$. Therefore

$$\varliminf_{k_i\to\infty} \mu(\| u_{k_i} - x \|)\, \| u_{k_i} - x \| \geqq \mu(\| u_0 - x \|)\, \| u_0 - x \|$$

since μ is continuous increasing. Since $u_{k_i} \rightharpoonup u_0$ weakly and

$$\langle f u_{k_i} - f u_0, u_{k_i} - u_0 \rangle \geqq \mu(\| u_{k_i} - u_0 \|) \| u_{k_i} - u_0 \| ,$$

we see that $\lim_{k_i \to \infty} \mu(\| u_{k_i} - u_0 \|) \| u_{k_i} - u_0 \| = 0$. Consequently $\lim_{k_i \to \infty} \| u_{k_i} - u_0 \| = 0$ since $\mu(t) > 0$ for $t > 0$ and μ is continuous increasing.

REFERENCES

[1] F. E. BROWDER, Problèmes Non Linéaires, Séminaire de Mathématiques Supérieures, Université de Montréal, 1966.
[2] M. A. KRASNOSELSKII, Topological methods in the theory of nonlinear integral equations, Pergamon Press, New York, 1964.
[3] K. SRINIVASACHARYULU, Some Fixed Point Theorems II, Bulletin de la Soc. Royale des Sciences de Liège, 3-4 (1970), 93-95.

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