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TOTALLY REAL SUBMANIFOLDS OF COMPLEX SPACE FORMS, I

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Introduction. Let $\bar{M}$ be a real $2m$-dimensional Kaehler manifold with almost complex structure J. An n-dimensional Riemannian manifold M isometrically immersed in $\bar{M}$ is called a *totally real* submanifold of $\bar{M}$ if $T_x(M) \perp JT_x(M)$ for each $x \in M$, where $T_x(M)$ denotes the tangent space to M at x. Here we have identified $T_x(M)$ with its image under the differential of the immersion because our computation is local. If X is a tangent vector of M at x, then JX is a normal vector to M. Thus we see that $n \leqq m$. When $n = m$, totally real submanifolds have many interesting properties studied by different authors (see [1], [2], [4], [5], [6] and [8]).

The purpose of the present paper is to study an n-dimensional totally real submanifold M of a real $2n$-dimensional complex space form $\bar{M}$ satisfying certain conditions on the second fundamental form of M. In §1 we state some fundamental formulas for totally real submanifolds of a Kaehler manifold. In §2 we prepare some lemmas which we need in the sequel. In §3 we prove the basic properties of an n-dimensional totally real submanifold M of a real $2n$-dimensional Kaehler manifold $\bar{M}$ with constant holomorphic sectional curvature under the condition that M has the parallel mean curvature vector. In the last section we prove our main theorems which give a characterization of a compact n-dimensional flat totally real submanifold $S^1(r_1) \times S^1(r_2) \times \cdots \times S^1(r_n)$ in the flat Kaehler manifold C^n of real dimension $2n$.

1. Preliminaries. Let $\bar{M}$ be a Kaehler manifold of real dimension $2n$ (complex dimension n) and M be an n-dimensional totally real submanifold of $\bar{M}$. We choose a local field of orthonormal frames $e_1, \cdots, e_{2n}$ in $\bar{M}$ in such a way that, restricted to M, the vectors $e_1, \cdots, e_n$ are tangent to M, and hence remaining vectors $e_{n+1}, \cdots, e_{2n}$ are normal to M. Unless stated otherwise, we use the following conventions that the ranges of indices are respectively:

$$1 \leqq A, B, C, \cdots \leqq 2n\ , \quad 1 \leqq i, j, k, \cdots \leqq n$$
$$n + 1 \leqq a, b, c, \cdots \leqq 2n\ ,$$

and that when a letter appears twice in any term as a subscript and a superscript, it is understood that this letter is summed over its range. Let $w^1, \cdots, w^{2n}$ be the field of dual frames with respect to the above frame field of $\bar{M}$. Then the structure equations of $\bar{M}$ are given by

$$dw^A = -w^A_B \wedge w^B , \quad w^A_B + w^B_A = 0 , \tag{1.1}$$

$$dw^A_B = -w^A_C \wedge w^C_B + \Phi^A_B , \quad \Phi^A_B = \frac{1}{2} K^A_{BCD} w^C \wedge w^D . \tag{1.2}$$

Restricting these forms to M, we have

$$w^a = 0 , \tag{1.3}$$

$$dw^i = -w^i_j \wedge w^j , \quad w^i_j + w^j_i = 0 , \tag{1.4}$$

$$dw^i_j = -w^i_k \wedge w^k_j + \Omega^i_j , \quad \Omega^i_j = \frac{1}{2} R^i_{jkl} w^k \wedge w^l . \tag{1.5}$$

Since $0 = dw^a = -w^a_i \wedge w^i$, by Cartan's lemma, we can write

$$w^a_i = h^a_{ij} w^j , \quad h^a_{ij} = h^a_{ji} , \tag{1.6}$$

and the Gauss-equation is given by

$$R^i_{jkl} = K^i_{jkl} + \sum_a (h^a_{ik} h^a_{jl} - h^a_{il} h^a_{jk}) . \tag{1.7}$$

Moreover we have the following equations:

$$dw^a_b = -w^a_c \wedge w^c_b + \Omega^a_b , \quad \Omega^a_b = \frac{1}{2} R^a_{bkl} w^k \wedge w^l , \tag{1.8}$$

$$R^a_{bkl} = K^a_{bkl} + \sum_i (h^a_{ik} h^b_{il} - h^a_{il} h^b_{ik}) . \tag{1.9}$$

The forms (w^i_j) define the Riemannian connection of M and the forms (w^a_b) define a connection in the normal bundle $T(M)^\perp$. We call $h^a_{ij} w^i w^j e_a$ the second fundamental form of M and sometimes the second fundamental form is denoted by its components h^a_{ij}. If the second fundamental form is identically zero, then M is said to be *totally geodesic.* If the second fundamental form is of the form $\delta_{ij}(\sum_k h^a_{kk} e_a)/n$, then M is said to be *totally umbilical,* where δ_{ij} denotes the Kronecker delta. We call $(\sum_k h^a_{kk} e_a)/n$ the mean curvature vector of M and M is said to be *minimal* if its mean curvature vector vanishes identically. We say that M has the *parallel mean curvature vector* if the mean curvature vector is parallel with respect to the connection in the normal bundle. We define the covariant derivative h^a_{ijk} of h^a_{ij} by setting

$$h^a_{ijk} w^k = dh^a_{ij} - h^a_{il} w^l_j - h^a_{lj} w^l_i + h^b_{ij} w^a_b . \tag{1.10}$$

If $h^a_{ijk} = 0$ for all a, i, j and k, the second fundamental form of M is said to be *parallel*. The Laplacian Δh^a_{ij} of h^a_{ij} is defined as

$$\Delta h^a_{ij} = \sum_k h^a_{ijkk} , \tag{1.11}$$

where we have put

$$h^a_{ijkl}w^l = dh^a_{ijk} - h^a_{ljk}w^l_i - h^a_{ilk}w^l_j - h^a_{ijl}w^l_k + h^b_{ijk}w^a_b . \tag{1.12}$$

Since M is a totally real submanifold of $\bar{M}$, we may choose a local field of orthonormal frames $e_1, \cdots, e_n, Je_1, \cdots, Je_n$ in M such that, restricted to M, the vectors $e_1, \cdots, e_n$ are tangent to M and the normal vectors are given by $e_{1^*} = e_{n+1} = Je_1, \cdots, e_{n^*} = e_{2n} = Je_n$. Then its dual frame field $w^1, \cdots, w^n, w^{1^*}, \cdots, w^{n^*}$ satisfy

$$w^i_j = w^{i^*}_{j^*} , \quad w^{i^*}_j = w^{j^*}_i , \tag{1.13}$$

where here and in the sequel we use the convention that $i^* = n + i$, $j^* = n + j$, etc. From (1.6) and (1.13) we obtain

$$h^{i^*}_{jk} = h^{j^*}_{ik} = h^{k^*}_{ij} . \tag{1.14}$$

If we assume that a Kaehler manifold $\bar{M}$ is of constant holomorphic sectional curvature c, then we have

$$K^A_{BCD} = \frac{1}{4}c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + J_{AC}J_{BD} - J_{AD}J_{BC} + 2J_{AB}J_{CD}) . \tag{1.15}$$

We call such a space a *complex space form*. If a Riemannian manifold M is of constant curvature k, then we have

$$R^i_{jkl} = k(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) . \tag{1.16}$$

We call such a space a *real space form*.

2. Lemmas. In this section we prepare some lemmas on totally real submanifold M of dimension n immersed in a real $2n$-dimensional Kaehler manifold $\bar{M}$. In the following, we put $H_a = (h^a_{ij})$, H_a being a symmetric (n, n)-matrix. If $H_aH_b = H_bH_a$ for all a, b $(=n + 1, \cdots, 2n)$, then the second fundamental form of M is said to be *commutative*, which is equivalent to that $\sum_j h^a_{ij}h^b_{jk} = \sum_j h^a_{jk}h^b_{ij}$ for all a, b, i and k. We say that the normal connection of M is *flat* if $R^a_{bkl} = 0$ for all a, b, k and l.

LEMMA 1 [6]. *Let M be an n-dimensional totally real submanifold of a real $2n$-dimensional Kaehler manifold $\bar{M}$. Then M is flat if and only if the normal connection of M is flat.*

PROOF. From (1.5), (1.8) and (1.13) we have

$$\Omega^{i*}_{j*} = dw^{i*}_{j*} + w^{i*}_{k*} \wedge w^{k*}_{j*} = dw^i_j + w^i_k \wedge w^k_j = \Omega^i_j ,$$

which shows that $R^{i*}_{j*kl} = R^i_{jkl}$ proving our assertion.

LEMMA 2. *Let M be an n-dimensional totally real submanifold of a real $2n$-dimensional Kaehler manifold $\bar{M}$. Then the second fundamental form of M is commutative if and only if we can choose an orthonormal frame for which $h^{i*}_{jk} = 0$ unless $i = j = k$, i.e., $H_i = (h^{i*}_{jk})$ is of the form*

$$H_i \equiv \begin{bmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & \lambda_i & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix} i , \qquad i = 1, \cdots, n .$$

PROOF. Assume that the second fundamental form of M is commutative, that is, $H_a H_b = H_b H_a$, $a, b = n+1, \cdots, 2n$. Then we can choose an orthonormal frame $e_1, \cdots, e_n$ for $T_x(M)$ in such a way that all H_a's are simultaneously diagonal, i.e., $h^a_{ij} = 0$ when $i \neq j$, that is, $h^{k*}_{ij} = 0$ when $i \neq j$. From (1.14) we see that $h^{k*}_{ij} = 0$ unless $i = j = k$. It is easy to see that the converse is also true.

COROLLARY 1. *Let M be an n-dimensional totally real minimal submanifold of a real $2n$-dimensional Kaehler manifold $\bar{M}$. If the second fundamental form of M is commutative, then M is totally geodesic.*

PROOF. Since M is minimal, we have $\operatorname{Tr} H_i = 0$ and hence $\lambda_i = 0$ for all i, by Lemma 2. This shows that M is totally geodesic.

COROLLARY 2 [6]. *Let M be an n-dimensional totally real and totally umbilical submanifold of a real $2n$-dimensional Kaehler manifold $\bar{M}$. If $n > 1$, then M is totally geodesic.*

PROOF. From the assumption we have $h^{k*}_{ij} = \delta_{ij}(\operatorname{Tr} H_k)/n$. Thus we see that $\sum_j h^{k*}_{ij} h^{l*}_{jm} = \sum_j h^{k*}_{jm} h^{l*}_{ij}$, that is, $H_k H_l = H_l H_k$. Therefore Lemma 2 implies that $h^{k*}_{ij} = 0$ unless $i = j = k$. On the other hand, we have $h^{k*}_{ij} = \lambda_k \delta_{ij}/n$. Setting $i = j \neq k$, we have $\lambda_k = 0$ when $n > 1$ and hence M is totally geodesic.

In the sequel we denote by $\bar{M}^n(c)$ a complex space form of real dimension $2n$ (complex dimension n) with constant holomorphic sectional curvature c. Let M be an n-dimensional totally real submanifold of $\bar{M}^n(c)$. Then the Gauss-equation (1.7) and (1.15) imply that

$$R^i_{jkl} = \frac{1}{4}c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_a (h^a_{ik}h^a_{jl} - h^a_{il}h^a_{jk}) \ . \tag{2.1}$$

If M is totally geodesic, then M is of constant curvature $(1/4)c$. Therefore Corollary 1 and Corollary 2 give the following results proved by Ludden-Okumura-Yano [6].

COROLLARY 3. *Let M be an n-dimensional totally real minimal submanifold of a complex space form $\bar{M}^n(c)$. If the second fundamental form of M is commutative, then M is a real space form of constant curvature* $(1/4)c$.

COROLLARY 4. *Let M be an n-dimensional totally real, totally umbilical submanifold of a complex space form $\bar{M}^n(c)$. If $n > 1$, then M is a real space form of constant curvature* $(1/4)c$.

LEMMA 3 [6]. *Let M be an n-dimensional totally real submanifold of a complex space form $\bar{M}^n(c)$. Then M is a real space form of constant curvature $(1/4)c$ if and only if M has the commutative second fundamental form.*

PROOF. From (1.7), (1.14) and (1.15) we have

$$\begin{aligned} R^i_{jkl} &= K^i_{jkl} + \sum_t (h^{t*}_{ik}h^{t*}_{jl} - h^{t*}_{il}h^{t*}_{jk}) \\ &= \frac{1}{4}c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_t (h^{i*}_{tk}h^{j*}_{tl} - h^{i*}_{tl}h^{j*}_{tk}) \ , \end{aligned}$$

which proves our assertion.

LEMMA 4 [5]. *Let M be an n-dimensional totally real submanifold of a complex space form $\bar{M}^n(c)$. Then we have the following equation:*

$$\begin{aligned} \sum_{t,i,j} h^{t*}_{ij}\Delta h^{t*}_{ij} = {} & \sum_{t,i,j,k} h^{t*}_{ij}h^{t*}_{kkij} + \sum_t \left[\frac{1}{4}(n+1)c \operatorname{Tr} H_t^2 - \frac{1}{2}c(\operatorname{Tr} H_t)^2\right] \\ & + \sum_{t,s} \{\operatorname{Tr}(H_tH_s - H_sH_t)^2 - [\operatorname{Tr}(H_tH_s)]^2 + \operatorname{Tr} H_s \operatorname{Tr}(H_tH_sH_t)\} \ . \end{aligned} \tag{2.2}$$

PROOF. This can be proved by a straightforward computation which uses the Ricci formula and the Codazzi equation $h^{t*}_{ijk} = h^{t*}_{ikj}$.

LEMMA 5 [2]. *Let M be an n-dimensional totally real submanifold of a real $2n$-dimensional Kaehler manifold $\bar{M}$. Then we have*

$$\sum_{t,s} \operatorname{Tr} H_t^2H_s^2 = \sum_{t,s} (\operatorname{Tr} H_tH_s)^2 \ . \tag{2.3}$$

PROOF. Since $h^{i*}_{jk} = h^{j*}_{ik}$, we have

$$\sum_{t,s} \mathrm{Tr}\, H_t^2 H_s^2 = \sum_{t,s,k,l,m,h} h_{kl}^{t*} h_{lm}^{t*} h_{mh}^{s*} h_{hk}^{s*}$$
$$= \sum_{t,s,k,l,m,h} h_{tl}^{k*} h_{lt}^{m*} h_{sh}^{m*} h_{hs}^{k*} = \sum_{k,m} (\mathrm{Tr}\, H_k H_m)^2 .$$

LEMMA 6. *If M is an n-dimensional totally real submanifold of a complex space form $\bar{M}^n(c)$ and is of constant curvature k, then we have*

$$\left(\frac{1}{4}c - k\right) \sum_t [\mathrm{Tr}\, H_t^2 - (\mathrm{Tr}\, H_t)^2] = \sum_{t,s} [\mathrm{Tr}\, H_t^2 H_s^2 - \mathrm{Tr}\, (H_t H_s)^2] . \tag{2.4}$$

PROOF. From the assumption and (1.7) we have

$$\left(\frac{1}{4}c - k\right)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) = \sum_t (h_{il}^{t*} h_{jk}^{t*} - h_{ik}^{t*} h_{jl}^{t*}) . \tag{2.5}$$

Multiplying by $\sum_s h_{il}^{s*} h_{jk}^{s*}$ the both sides of this equation and summing up with respect to i, j, k and l, we have (2.4) by using (2.3).

LEMMA 7. *If M is an n-dimensional totally real submanifold of a complex space form $\bar{M}^n(c)$ and is of constant curvature k, then we have*

$$(n-1)\left(\frac{1}{4}c - k\right) \sum_t \mathrm{Tr}\, H_t^2 = \sum_{t,s} [\mathrm{Tr}\, H_t^2 H_s^2 - \mathrm{Tr}\, H_s \,\mathrm{Tr}\, (H_t H_s H_t)] . \tag{2.6}$$

PROOF. From (2.5) we obtain

$$(n-1)\left(\frac{1}{4}c - k\right)\delta_{jl} = \sum_{t,i} (h_{il}^{t*} h_{ij}^{t*} - h_{ii}^{t*} h_{il}^{t*}) . \tag{2.7}$$

Multiplying the both sides of (2.7) by $\sum_s h_{jk}^{s*} h_{kl}^{s*}$ and summing up with respect to j, k and l, we have (2.6).

LEMMA 8. *Let M be an n-dimensional totally real submanifold of a complex space form $\bar{M}^n(c)$ with parallel mean curvature vector. If the scalar curvature of M is constant, then the square of the length of the second fundamental form is constant, i.e., $\sum_t \mathrm{Tr}\, H_t^2 \equiv$ constant.*

PROOF. From the Gauss-equation (1.7) we have

$$R = \frac{1}{4}n(n-1)c + \sum_t (\mathrm{Tr}\, H_t)^2 - \sum_t \mathrm{Tr}\, H_t^2 , \tag{2.8}$$

from which we have our assertion since R and $\sum_t (\mathrm{Tr}\, H_t)^2$ are constant by the assumptions.

3. Totally real submanifolds of constant curvature.

PROPOSITION 1. *Let M be an n-dimensional totally real submanifold of a complex space form $\bar{M}^n(c)$ with parallel mean curvature vector. If M is of constant curvature k, then we have*

$$\sum_{t,i,j,k}(h_{ijk}^{t*})^2 = -k\sum_t [(n+1)\,\mathrm{Tr}\,H_t^2 - 2(\mathrm{Tr}\,H_t)^2]\,. \tag{3.1}$$

PROOF. By Lemma 8, $\sum_t \mathrm{Tr}\,H_t^2$ is constant. Thus we have

$$\sum_{t,i,j} h_{ij}^{t*}\Delta h_{ij}^{t*} = \frac{1}{2}\Delta\sum_t \mathrm{Tr}\,H_t^2 - \sum_{t,i,j,k}(h_{ijk}^{t*})^2 = -\sum_{t,i,j,k}(h_{ijk}^{t*})^2\,.$$

Consequently (2.2) becomes

$$\begin{aligned}\sum_{t,i,j,k}(h_{ijk}^{t*})^2 &= -\sum_t\left[\frac{1}{4}(n+1)c\,\mathrm{Tr}\,H_t^2 - \frac{1}{2}c(\mathrm{Tr}\,H_t)^2\right] \\ &\quad - \sum_{t,s}\{\mathrm{Tr}\,(H_tH_s - H_sH_t)^2 - [\mathrm{Tr}\,(H_tH_s)]^2 + \mathrm{Tr}\,H_s\,\mathrm{Tr}\,(H_tH_sH_t)\}\,.\end{aligned} \tag{3.2}$$

Substituting (2.4) and (2.6) into (3.2) and using (2.3), we have our equation (3.1).

PROPOSITION 2. *Let M be an n-dimensional totally real submanifold of a complex space form $\bar{M}^n(c)$ $(n > 1)$ with parallel mean curvature vector and of constant curvature k. If $(1/4)c \geqq k$, then either $k \leqq 0$ or M is totally geodesic ($(1/4)c = k$).*

PROOF. From (2.8) we obtain

$$\left(\frac{1}{4}c - k\right)n(n-1) = \sum_t [\mathrm{Tr}\,H_t^2 - (\mathrm{Tr}\,H_t)^2]\,.$$

By the assumption we see that

$$\sum_t \mathrm{Tr}\,H_t^2 \geqq \sum_t (\mathrm{Tr}\,H_t)^2\,. \tag{3.3}$$

If $k > 0$, (3.1) implies that

$$\begin{aligned}0 &= \sum_t [(n+1)\,\mathrm{Tr}\,H_t^2 - 2(\mathrm{Tr}\,H_t)^2] \\ &= \sum_t \{(n-1)\,\mathrm{Tr}\,H_t^2 + 2[\mathrm{Tr}\,H_t^2 - (\mathrm{Tr}\,H_t)^2]\}\,,\end{aligned}$$

which shows that $\sum_t \mathrm{Tr}\,H_t^2 = 0$ and hence M is totally geodesic. Except for this possibility, we have $k \leqq 0$.

PROPOSITION 3. *Let M be an n-dimensional totally real submanifold of a complex space form $\bar{M}^n(c)$ $(n > 1)$ with parallel second fundamental form and of constant curvature k. If $(1/4)c \geqq k$, then either M is totally geodesic ($(1/4)c = k$) or flat ($k = 0$).*

REMARK. If M is minimal in $\bar{M}^n(c)$ and of constant curvature k, then $(1/4)c \geqq k$. Thus Proposition 2 and Proposition 3 are generalizations of theorems of Chen-Ogiue [2] for totally real minimal submanifolds.

There is an example that has parallel second fundamental form but is not minimal (see Theorem 1 and Theorem 2 in §4).

PROPOSITION 4. *Let M be an n-dimensional totally real submanifold of a complex space form $\bar{M}^n(c)$ with parallel mean curvature vector. If the second fundamental form of M is commutative, then we have*

$$\sum_{t,i,j,k}(h_{ijk}^{t*})^2 = -\frac{1}{4}c(n-1)\sum_t \operatorname{Tr} H_t^2 \, . \tag{3.4}$$

PROOF. Using Lemma 2 and Lemma 3, we can transform (3.1) into (3.4).

PROPOSITION 5. *Let M be an n-dimensional totally real submanifold of a complex space form $\bar{M}^n(c)$ $(n > 1)$ with parallel mean curvature vector and with commutative second fundamental form. Then either M is totally geodesic or $c \leqq 0$.*

PROPOSITION 6. *Let M be an n-dimensional totally real submanifold of a complex space form $\bar{M}^n(c)$ $(n > 1)$ with parallel second fundamental form. If the second fundamental form of M is commutative, then M is either totally geodesic or flat.*

PROOF. By the assumption and Lemma 3, M is of constant curvature $(1/4)c$. On the other hand, by (3.4), M is either totally geodesic or $c = 0$ in which case M is flat.

PROPOSITION 7. *If M is an n-dimensional flat totally real submanifold of a real $2n$-dimensional flat Kaehler manifold $\bar{M}$ and if the mean curvature vector is parallel, then the second fundamental form is parallel.*

PROOF. By Lemma 3, M has the commutative second fundamental form. Consequently we have our assertion by equation (3.4).

4. **Flat totally real submanifolds.** A simply connected complete Kaehler manifold of constant holomorphic sectional curvature c and of real dimension $2n$ can be identified with the complex projective space $P_n(C)$, the open unit ball D_n in C^n or C^n according as $c > 0$, $c < 0$ or $c = 0$.

Now we give an example of a totally real submanifold in C^n. Let J be the complex structure of C^n given by

$$J \equiv \begin{bmatrix} \begin{array}{cc|} 0 & -1 \\ 1 & 0 \\ \hline \end{array} & & & \\ & \begin{array}{|cc|} 0 & -1 \\ 1 & 0 \\ \hline \end{array} & & \\ & & \ddots & \\ & & & \begin{array}{|cc} \hline 0 & -1 \\ 1 & 0 \end{array} \end{bmatrix} .$$

Let $S^1(r_i) = \{z_i \in C\colon |z_i|^2 = r_i^2\}$, $i = 1, \cdots, n$. We consider $S^1(r_1) \times S^1(r_2) \times \cdots \times S^1(r_n)$ in C^n, which is flat. The position vector X of $S^1(r_1) \times S^1(r_2) \times \cdots \times S^1(r_n)$ in C^n has components given by

$$X \equiv \begin{bmatrix} r_1 \cos u^1 \\ r_1 \sin u^1 \\ r_2 \cos u^2 \\ r_2 \sin u^2 \\ \vdots \\ r_n \cos u^n \\ r_n \sin u^n \end{bmatrix} .$$

Putting $X_i = \partial_i X = \partial X/\partial u^i$, we have

$$X_1 \equiv r_1 \begin{bmatrix} -\sin u^1 \\ \cos u^1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad X_2 \equiv r_2 \begin{bmatrix} 0 \\ 0 \\ -\sin u^2 \\ \cos u^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, X_n \equiv r_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -\sin u^n \\ \cos u^n \end{bmatrix} .$$

On the other hand, we can take unit normal vectors as

$$N_1 \equiv -\begin{bmatrix} \cos u^1 \\ \sin u^1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad N_2 \equiv -\begin{bmatrix} 0 \\ 0 \\ \cos u^2 \\ \sin u^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, N_n \equiv -\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \cos u^n \\ \sin u^n \end{bmatrix}.$$

Then we obtain

$$JX_1 = r_1 N_1, \quad JX_2 = r_2 N_2, \cdots, JX_n = r_n N_n .$$

Consequently $S^1(r_1) \times S^1(r_2) \times \cdots \times S^1(r_n)$ is a flat totally real submanifold in C^n and it has parallel mean curvature vector and flat normal connection (see [7: p. 111] and [3]). In view of Lemma 3 and Proposition 7, this example has parallel and commutative second fundamental form.

THEOREM 1. *Let M be a compact n-dimensional ($n > 1$) totally real submanifold of C^n with parallel mean curvature vector. If the second fundamental form of M is commutative, then*

$$M \equiv S^1(r_1) \times S^1(r_2) \times \cdots \times S^1(r_n) .$$

PROOF. By the assumption and Lemma 3 we see that M is flat. Therefore Proposition 7 shows that the second fundamental form of M is parallel. Since M is flat, the normal connection of M also is flat by Lemma 1. Consequently Lemma 2 and Theorem 3.2 of Yano-Ishihara [7] imply our statement.

THEOREM 2. *Let M be a compact n-dimensional ($n > 1$) totally real submanifold of a simply connected complete complex space form $\bar{M}^n(c)$ with parallel second fundamental form. If the second fundamental form of M is commutative and if M is not totally geodesic, then*

$$M \equiv S^1(r_1) \times S^1(r_2) \times \cdots \times S^1(r_n) \quad in \quad C^n .$$

PROOF. By the assumption and Proposition 6, we have $c = 0$ and the ambient space $\bar{M}$ is C^n. Thus Theorem 2 follows from Theorem 1.

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