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RIGIDITY OF HYPERSURFACES WITH CONSTANT MEAN CURVATURE

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Introduction. The concept of rigidity plays an important role in the study of hypersurfaces in a Riemannian manifold. In fact, we always hope to classify hypersurfaces up to isometry of the ambient manifold. In the late of 19th century, Beez obtained the following remarkable theorem: Let M be an isometrically immersed hypersurface of a Euclidean $(n+1)$-space R^{n+1}. If the type number of an isometric immersion is greater than two everywhere and $n \geqq 3$, then M is determined up to isometry of R^{n+1}, i.e., M is rigid (See [1], or [7], Vol. II, p. 42-46). Many attempts have been made to extend this theorem in various ways ([2], [3]). Among them, Eisenhart proved that Beez's result remains true in the case where the ambient space is a space form $\tilde{M}(K)$ (See [4], p. 212). And in 1936, the first simple proof of the above results was given by T. Y. Thomas (See [10], p. 184-188). On the other hand, E. Cartan [2] developed the theory of deformability of hypersurfaces in a Euclidean space. This was used to weaken the assumption on the type number. Recently, by using the above deformability theory, Harle proved that if the ambient space $\tilde{M}(K)$ satisfies $K \neq 0$, $n \geqq 4$, and M has constant scalar curvature and an isometric immersion with the type number greater than one everywhere, then M is rigid (See [6], Theorem 3-3).

The purpose of this paper is to prove the following result: If M is a hypersurface of $\tilde{M}(K)$, $K \neq 0$, $n \geqq 3$, having non-zero constant mean curvature, then M is rigid (Theorem 3.1). This result is a generalization with respect to the assumption on the type number of Eisenhart's result.

The proofs contained in this paper heavily rely on the methods developed by E. Cartan [2], Dolbeault-Lemoine [3] and Harle [6].

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1. Hypersurfaces. All manifolds and maps considered in this paper will be assumed to be of class C^∞. Let $\tilde{M}(K)$ be a simply connected and complete Riemannian $(n+1)$-manifold of constant curvature K. From now on $\tilde{M}(K)$ will be called a space form. Let $f: M \to \tilde{M}(K)$ be an isometric immersion of a Riemannian n-manifold M into $\tilde{M}(K)$. For simplicity, we say that M is a hypersurface immersed in $\tilde{M}(K)$ and, for all local formulas and computations, we may consider f as an imbedding and thus identify $x \in M$ with $f(x) \in \tilde{M}(K)$. The tangent space $T_x(M)$ is identified with a subspace of the tangent space $T_x(\tilde{M}(K))$, and the normal space $T_x^{\perp}$ is the subspace of $T_x(\tilde{M}(K))$ consisting of all $X \in T_x(\tilde{M}(K))$ which are orthogonal to $T_x(M)$ with respect to the Riemannian metric of $\tilde{M}(K)$. For an arbitrary point $x \in M$, we may choose a field of unit normal vector ξ defined in a neighborhood U of x. The second fundamental form α and the corresponding symmetric operator A are defined and related to covariant differentiations $\tilde{\nabla}$ and ∇ in $\tilde{M}(K)$ and M, respectively, by the following formulas:

$$\tilde{\nabla}_X Y = \nabla_X Y + \alpha(X, Y)\,, \quad \alpha(X, Y) = g(AX, Y)\xi\,, \tag{1.1}$$

$$\tilde{\nabla}_X \xi = -AX\,, \tag{1.2}$$

where X and Y are vector fields tangent to M and g a metric induced on M by the immersion f. From now on the operator A will be called *the second fundamental form of f with respect to ξ*. The rank of A at a point x is called the *type number* of f at this point and is commonly denoted by $t(x)$. The Gauss equation is:

$$R(X, Y) = K(X \wedge Y) + AX \wedge AY\,, \quad X, Y \in T_x(M)\,, \tag{1.3}$$

where R denotes the curvature tensor of M and $X \wedge Y$ the skew-symmetric endomorphism of $T_x(M)$ defined by $(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y$. The Codazzi equation is expressed by

$$\nabla_X(AY) - \nabla_Y(AX) = A[X, Y]\,. \tag{1.4}$$

REMARK. In terms of the operator ∇, the curvature tensor of M is expressed as

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z\,, \tag{1.5}$$

where X, Y and Z are vector fields on M.

DEFINITION 1.1. Let f and $\bar{f}$ be isometric immersions of M into $\tilde{M}(K)$. An open subset U of M is said to be *congruent* when there is an isometry ϕ of $\tilde{M}(K)$ such that $\bar{f} = \phi \circ f$ on U.

The following result is basic:

PROPOSITION 1.2. (Ryan). *Let f be an isometric immersion of M as hypersurface in $\tilde{M}(K)$. If the type number of f is greater than one at a point x, then kernel of A_x is given by*

(1.6) $\ker A_x = \{X \in T_x(M) \mid R(X, Y) = K(X \wedge Y) \quad \textit{for all} \quad Y \in T_x(M)\}$.

PROOF. See [8], Proposition 1.1.

PROPOSITION 1.3. *Let f and $\bar{f}$ be isometric immersions of M as hypersurfaces in $\tilde{M}(K)$. If $t(x)$ for f is $\geqq 2$ for all x, then $t(x) = \bar{t}(x)$, where $\bar{t}(x)$ is the type number of $\bar{f}$ at x.*

PROOF. Denote the right side of (1.6) by $N(x)$. Let A and $\bar{A}$ be the second fundamental forms corresponding to f and $\bar{f}$, respectively. If $\bar{t}(x) \leqq 1$, then $\bar{A}X \wedge \bar{A}Y = 0$ for all X and Y and hence $\dim N(x) = n$ contrary to the assumption $t(x) \geqq 2$. Thus $\bar{t}(x) \geqq 2$ for all x. Hence $\ker \bar{A}_x = N(x) = \ker A_x$. Since A and $\bar{A}$ are symmetric, $\operatorname{Im} A_x = \operatorname{Im} \bar{A}_x = N(x)^{\perp}$. In particular, $t(x) = \bar{t}(x)$.

PROPOSITION 1.4. (Beez [1] and Eisenhart [4]). *Let f and $\bar{f}$ be isometric immersions of an orientable Riemannian $n(\geqq 3)$-manifold M as hypersurfaces in $\tilde{M}(K)$. If the second fundamental forms A and $\bar{A}$ for f and $\bar{f}$, respectively, coincide at each point of M, then there is an isometry ϕ of $\tilde{M}(K)$ such that $\bar{f} = \phi \circ f$.*

PROOF. See for example [9], Theorem 4.

COROLLARY 1.5. *Let M, f and $\bar{f}$ be as in Proposition 1.4. Assume that M is connected. If the second fundamental forms A and $\bar{A}$ coincide at each point of M up to a sign, then there is an isometry ϕ of $\tilde{M}(K)$ such that $\bar{f} = \phi \circ f$.*

PROOF. Let ξ and $\bar{\xi}$ be fields of unit normals globally defined on M for the immersions f and $\bar{f}$, respectively. Let M^+(resp. M^-) be a set of points $x \in M$ at which $A = \bar{A}$ (resp. $A = -\bar{A}$). Then both M^+ and M^- are closed. By assumption, M is a disjoint union of M^+ and M^-. Since M is connected, either $M = M^+$ or $M = M^-$. If $M = M^+$, we are done. If $M = M^-$, we change $\bar{\xi}$ to $-\bar{\xi}$.

PROPOSITION 1.6. (Beez [1] and Eisenhart [4]). *Let f and $\bar{f}$ be isometric immersions of connected M as hypersurfaces in $\tilde{M}(K)$. If $t(x)$ is $\geqq 3$ at each x, then there is an isometry ϕ of $\tilde{M}(K)$ such that $\bar{f} = \phi \circ f$.*

A simple proof is given in [8], Theorem 1.3.

PROPOSITION 1.7. (Harle). *Let M be a connected Riemannian $n(\geqq 3)$-manifold, and let f and $\bar{f}$ be isometric immersions of M into $\tilde{M}(K)$. Assume that M contains no open subset on which f is totally geodesic. If there is a family of open submanifolds $\{U_\alpha\}$ each of wich is congruent and forms a covering of M, then there is an isometry ϕ of $\tilde{M}(K)$ such* *that* $\bar{f} = \phi \circ f$.

PROOF. See [6], Proposition 1-7.

2. Deformability of hypersurfaces. Let M be a Riemannian $n(\geqq 3)$-manifold. From now on we will denote the scalar product by $\langle X, Y\rangle$.

The following fact is basic and will be used without further mentioning.

Let f *and* $\bar{f}$ *be isometric immersions of* M *as hypersurfaces in* $\tilde{M}(K)$. *If* M *contains no open congruent submanifold* (*See Definition* 1.1), *and the type number* $t(x)$ (*resp.* $\bar{t}(x)$) *of the immersion* f (*resp.* $\bar{f}$) *is* $\geqq 2$ *at each* x, *then* $t(x) = \bar{t}(x) = 2$ *at each* x.

In fact, since M contains no open congruent submanifold, in view of Proposition 1.6 the type numbers of f and $\bar{f}$ are at most two at each point. From Proposition 1.3 $t(x) = \bar{t}(x) \geqq 2$ at each x, which shows that the type numbers have to be exactly 2.

The main objective in this section is to prove the following result.

THEOREM 2.1. *Let* M *be a Riemannian* n-*manifold with* $n \geqq 3$ *and let* f *and* $\bar{f}$ *be isometric immersions of* M *in* $\tilde{M}(K)$, $K \neq 0$, *with non-zero constant mean curvature. If the type number* $t(x)$ *of the immersion* f *at each* x *is* $\geqq 2$, *then* M *contains an open congruent submanifold.*

The proof of this theorem will depend on several lemmas.

In order to simplify the statements of these lemmas we prepare the following definition.

DEFINITION 2.2. The complex tangent space $T_x^c(M)$ of a manifold M is the *complexification* of the tangent space $T_x(M)$. A complex vector field is defined by assigning to each point x of M an element of $T_x^c(M)$. Any complex vector field Z can be written uniquely as $Z = Z' + iZ''$ where Z' and Z'' are real vector fields.

LEMMA 2.3. (Gray [5]). *Let* f *be an isometric immersion of* M *in* $\tilde{M}(K)$ *such that its type number is constant and greater than one. Then the nullity distribution* N *of* f *is involutive and its leaves are totally geodesic both in* M *and* $\tilde{M}(K)$.

PROOF. See, for example [6], Proposition 1-5.

LEMMA 2.4. *Let* M *be an orientable Riemannian* n-*manifold and let* f *and* $\bar{f}$ *be isometric immersions of* M *in* $\tilde{M}(K)$ *with non-zero constant mean curvature and* $t(x) = \bar{t}(x) = 2$ *at each* x. *Assume that there is an orthonormal frame* $\{X, Y, X_3, \cdots, X_n\}$ *defined around each point of* M

in such a way that the vector fields $X_3, \cdots, X_n$ *form a basis for the nullity distribution* N (*which is defined independent of immersions, in view of Proposition* 1.2) *and for the restrictions of* A *and* $\bar{A}$ *to any open orientable submanifold* U *of* M *the equations*

$$\langle AX, X\rangle = \langle \bar{A}X, X\rangle = 0 \tag{2.1}$$

hold at all points of U. *Assume further that* M *contains no open congruent submanifold. Then the distribution* N *is parallel on* M (*See* [6], [7]).

PROOF. In view of Lemma 2.3, it is sufficient to show that the following equations hold around each point of M, i.e.,

$$\begin{aligned} \langle \nabla_X X_i, Y\rangle &= \langle \nabla_Y X_i, X\rangle = 0 \,, \\ \langle \nabla_X X_i, X\rangle &= \langle \nabla_Y X_i, Y\rangle = 0 \,, \qquad i = 3, \cdots, n \,. \end{aligned}$$

In order to show these equations, we first show that the following equations hold around each point of M, i.e.,

$$\langle \nabla_{X_i} X, Y\rangle = 0 \,, \qquad i = 3, \cdots, n \,. \tag{2.2}$$

Assume $\langle \nabla_{X_i} X, Y\rangle$ to be non-zero at a point x of M for some index i. Thus it will be non-zero at all points of an open orientable submanifold U. Then for the restriction of A to U

$$\nabla_{X_i}\langle AY, X\rangle = \langle \nabla_{X_i} AY, X\rangle + \langle AY, \nabla_{X_i} X\rangle \,.$$

Since $\langle AX, X\rangle$ and AX_i are zero at all points of U, the above relation can be written as

$$\nabla_{X_i}\langle AY, X\rangle = \langle [X_i, Y], Y\rangle\langle Y, AX\rangle + \langle AY, \nabla_{X_i} X\rangle \,. \tag{2.3}$$

A similar relation holds for the restriction of $\bar{A}$ to U.

From the Gauss equation it follows that

$$\langle AX, X\rangle\langle AY, Y\rangle - \langle AX, Y\rangle^2 = \langle \bar{A}X, X\rangle\langle \bar{A}Y, Y\rangle - \langle \bar{A}X, Y\rangle^2 \,,$$

which together with (2.1) gives

$$\langle AX, Y\rangle = e\langle \bar{A}X, Y\rangle \,, \qquad e = \pm 1 \,. \tag{2.4}$$

From (2.3) and (2.4), we thus have

$$\langle (A - e\bar{A})Y, \nabla_{X_i} X\rangle = 0$$

or

$$\langle (A - e\bar{A})Y, Y\rangle\langle \nabla_{X_i} X, Y\rangle = 0 \,.$$

Since $\langle \nabla_{X_i} X, Y\rangle$ is assumed to be non-zero, it follows

$$\langle AY, Y\rangle = e\langle \bar{A}Y, Y\rangle . \tag{2.5}$$

Now (2.1), (2.4) and (2.5) show that $A = e\bar{A}$ and therefore U is congruent, which is a contradiction. Thus (2.2) is proved.

From (2.1) it follows that

$$0 = \nabla_{X_i}\langle AX, X\rangle = \langle \nabla_{X_i}AX, X\rangle + \langle AX, \nabla_{X_i}X\rangle .$$

By using (2.1), (2.2) and noting that AX_i vanishes, this relation becomes

$$0 = \langle [X_i, X], AX\rangle = -\langle \nabla_X X_i, AX\rangle = -\langle \nabla_X X_i, Y\rangle\langle AX, Y\rangle .$$

By assumption, N is an $(n-2)$-dimensional distribution, which means that $\langle AX, Y\rangle$ never vanishes. Thus

$$\langle \nabla_X X_i, Y\rangle = 0 , \qquad \text{for all} \quad i \geqq 3 .$$

Next since A is symmetric, $\langle AX, Y\rangle - \langle AY, X\rangle = 0$ around each point of M. By covariant differentiation with respect to X_i, this relation yields

$$\nabla_{X_i}(\langle AX, Y\rangle - \langle AY, X\rangle) = 0 . \tag{2.6}$$

On the other hand, we have

$$\begin{aligned} \nabla_{X_i}\langle AX, Y\rangle &= \langle \nabla_{X_i}AX, Y\rangle + \langle AX, \nabla_{X_i}Y\rangle \\ &= \langle [X_i, X], AY\rangle = -\langle \nabla_X X_i, AY\rangle \\ &= -\langle \nabla_X X_i, X\rangle\langle AX, Y\rangle . \end{aligned} \tag{2.7}$$

Similarly,

$$\nabla_{X_i}\langle AY, X\rangle = -\langle \nabla_Y X_i, Y\rangle\langle AX, Y\rangle . \tag{2.8}$$

The relations (2.6), (2.7) and (2.8) give

$$\langle \nabla_X X_i, X\rangle = \langle \nabla_Y X_i, Y\rangle , \qquad i = 3, \cdots, n .$$

From non-zero constancy of mean curvature, we have

$$\langle AY, Y\rangle = \text{constant} \neq 0 .$$

By covariant differentiation with respect to X_i this relation yields

$$\nabla_{X_i}\langle AY, Y\rangle = 0 . \tag{2.9}$$

From (2.2) the left side of (2.9) can be written as

$$\begin{aligned} \nabla_{X_i}\langle AY, Y\rangle &= \langle \nabla_{X_i}AY, Y\rangle + \langle AY, \nabla_{X_i}Y\rangle \\ &= \langle [X_i, Y], AY\rangle = -\langle \nabla_Y X_i, AY\rangle \\ &= -\langle \nabla_Y X_i, X\rangle\langle X, AY\rangle - \langle \nabla_Y X_i, Y\rangle\langle Y, AY\rangle . \end{aligned}$$

Therefore

$$\langle \nabla_Y X_i, X\rangle\langle AX, Y\rangle + \langle \nabla_Y X_i, Y\rangle\langle AY, Y\rangle = 0$$

and of course

$$\langle \nabla_Y X_i, X\rangle\langle \bar{A}X, Y\rangle + \langle \nabla_Y X_i, Y\rangle\langle \bar{A}Y, Y\rangle = 0 .$$

By the argument similar to that in the proof of (2.2), it can be concluded that

$$\langle \nabla_Y X_i, Y\rangle = \langle \nabla_Y X_i, X\rangle = 0 , \qquad i = 3, \cdots, n .$$

Hence Lemma 2.4 is proved.

LEMMA 2.5. *Let M be an orientable Riemannian n-manifold and let f and $\bar{f}$ be isometric immersions of M in $\tilde{M}(K)$ and $t(x) = \bar{t}(x) = 2$ at each x. Assume that there is an orthonormal frame $\{X_1, \cdots, X_n\}$ defined around each point of M such that the vector fields $X_3, \cdots, X_n$ form a basis for the nullity distribution N (See Lemma 2.4). Suppose that there are two linearly independent complex vector fields Z and W, which satisfy the condition that their own scalar products never vanish, belonging to the complexification of the vector space spanned by X_1, X_2 such that for the restrictions of A and $\bar{A}$ to any open orientable submanifold U the equations*

$$\langle AZ, W\rangle = \langle \bar{A}Z, W\rangle = 0 \tag{2.10}$$

hold at all points of U. Finally assume that M contains no open congruent submanifold. Then the following equations hold around each point of M.

$$\langle \nabla_Z X_i, W\rangle = \langle \nabla_W X_i, Z\rangle = 0 , \qquad i = 3, \cdots, n .$$

PROOF. Let x be a point of M, and assume

$$\langle \nabla_Z X_i, W\rangle \neq 0 \qquad \text{for some} \quad i \geqq 3 ,$$

at all points of an open orientable submanifold $U(x)$ containing x.

Since for the restriction of A to $U(x)$ AX_i vanish for all $i \geqq 3$, it follows that

$$\langle X_i, AW\rangle = 0 , \qquad i = 3, \cdots, n .$$

By covariant differentiation with respect to Z, this relation yields

$$\langle \nabla_Z X_i, AW\rangle + \langle X_i, \nabla_Z AW\rangle = 0 . \tag{2.11}$$

In view of (2.10) the first term of the left side of (2.11) can be written as

$$\langle \nabla_Z X_i, AW\rangle = \frac{\langle \nabla_Z X_i, W\rangle}{\langle W, W\rangle}\langle AW, W\rangle , \tag{2.12}$$

while the second term as

$$\begin{aligned}\langle X_i, \nabla_Z A W\rangle &= \langle X_i, \nabla_W AZ + A[Z, W]\rangle \\ &= \langle X_i, \nabla_W AZ\rangle = -\langle \nabla_W X_i, AZ\rangle .\end{aligned}$$

Again by (2.10) this equation becomes

$$\langle X_i, \nabla_Z A W\rangle = -\frac{\langle \nabla_W X_i, Z\rangle}{\langle Z, Z\rangle}\langle AZ, Z\rangle . \tag{2.13}$$

From equations (2.11), (2.12) and (2.13) it follows immediately

$$\frac{\langle \nabla_Z X_i, W\rangle}{\langle W, W\rangle}\langle A W, W\rangle = \frac{\langle \nabla_W X_i, Z\rangle}{\langle Z, Z\rangle}\langle AZ, Z\rangle . \tag{2.14}$$

A similar relation holds for the restriction of $\bar{A}$ to $U(x)$.

On the other hand, the extension of the Gauss equation to complex vector fields gives

$$\langle AZ, Z\rangle\langle A W, W\rangle - \langle AZ, W\rangle^2 = \langle \bar{A}Z, Z\rangle\langle \bar{A} W, W\rangle - \langle \bar{A}Z, W\rangle^2 ,$$

which implies, due to (2.10),

$$\langle AZ, Z\rangle\langle A W, W\rangle = \langle \bar{A}Z, Z\rangle\langle \bar{A} W, W\rangle . \tag{2.15}$$

From (2.14) we obtain

$$\frac{\langle \nabla_Z X_i, W\rangle}{\langle W, W\rangle}\langle AW, W\rangle^2 = \frac{\langle \nabla_W X_i, Z\rangle}{\langle Z, Z\rangle}\langle AZ, Z\rangle\langle AW, W\rangle ,$$

$$\frac{\langle \nabla_Z X_i, W\rangle}{\langle W, W\rangle}\langle \bar{A}W, W\rangle^2 = \frac{\langle \nabla_W X_i, Z\rangle}{\langle Z, Z\rangle}\langle \bar{A}Z, Z\rangle\langle \bar{A}W, W\rangle ,$$

which together with (2.15) yield

$$\frac{\langle \nabla_Z X_i, W\rangle}{\langle W, W\rangle}(\langle AW, W\rangle^2 - \langle \bar{A}W, W\rangle^2) = 0 . \tag{2.16}$$

Since $\langle \nabla_Z X_i, W\rangle$ is assumed to be non-zero, it follows from (2.16) that

$$\langle AW, W\rangle^2 - \langle \bar{A}W, W\rangle^2 = 0$$

at all points of $U(x)$, which means that

$$\langle AW, W\rangle = e\langle \bar{A}W, W\rangle , \qquad e = \pm 1 . \tag{2.17}$$

From (2.15) and (2.17) we obtain

$$\langle AZ, Z\rangle = e\langle \bar{A}Z, Z\rangle . \tag{2.18}$$

Finally, from (2.10), (2.17) and (2.18) it follows that

$$A = e\bar{A} \tag{2.19}$$

and therefore $U(x)$ is congruent. Since M contains no open congruent submanifold, this is a contradiction. Thus $\langle \nabla_Z X_i, W\rangle$ vanishes at x. Since the above proof is symmetric in Z and W it follows that $\langle \nabla_W X_i, Z\rangle$ also vanishes around each point of M. Hence the proof of Lemma 2.5 is complete.

PROOF OF THEOREM 2.1. Assume that M contains no open congruent submanifold. Then in view of Lemma 2.3, in a neighborhood U of each point p of M it is possible to find an orthonormal frame $\{X_1, \cdots, X_n\}$ such that the vectors $X_3, \cdots, X_n$ form a basis of the nullity distribution N.

For the restriction of A to U, we have

$$\langle A[X_1, X_2], X_i\rangle = 0 , \qquad i = 3, \cdots, n .$$

The relation and Codazzi equation (1.4) yield

$$\begin{aligned}(2.20)\qquad &\langle \nabla_{X_1} X_i, X_2\rangle\langle AX_2, X_2\rangle \\ &+ [\langle \nabla_{X_1} X_i, X_1\rangle - \langle \nabla_{X_2} X_i, X_2\rangle]\langle AX_1, X_2\rangle \\ &- \langle \nabla_{X_2} X_i, X_1\rangle\langle AX_1, X_1\rangle = 0\end{aligned}$$

for all $i \geqq 3$ at all points of U. A similar relation holds for the restriction of $\bar{A}$ to U. Consider the following subset P of U: the set of the points q of U such that

$$\begin{aligned}&\langle \nabla_{X_1} X_i, X_2\rangle_q = \langle \nabla_{X_2} X_i, X_1\rangle_q = 0 , \\ &\langle \nabla_{X_1} X_i, X_1\rangle_q = \langle \nabla_{X_2} X_i, X_2\rangle_q , \qquad i = 3, \cdots, n .\end{aligned}$$

In this case the following holds.

(2.21) *The set P has no interior points.*

In fact, consider a point $q \in \operatorname{Int} P$. Locally it is possible to replace X_1, X_2 by unit vector fields X, Y such that

$$(2.22)\qquad \langle X, Y\rangle = 0 , \quad \langle AX, Y\rangle = 0 ,$$

in a neighborhood of q, provided the non-zero eigenvalues of A_q are distinct. A direct computation gives:

$$\begin{aligned}(2.23)\qquad &\langle \nabla_X X_i, Y\rangle = \langle \nabla_Y X_i, X\rangle = 0 , \\ &\langle \nabla_X X_i, X\rangle = \langle \nabla_Y X_i, Y\rangle ,\end{aligned}$$

for all $i \geqq 3$ and at all points of a neighborhood of q.

From non-zero constancy of mean curvature it follows

$$\langle AX, X\rangle + \langle AY, Y\rangle = \text{constant} \neq 0 .$$

By covariant differentiation with respect to X_i of both sides of the above relation, we have

$$\nabla_{X_i}(\langle AX, X\rangle + \langle AY, Y\rangle) = 0\,, \qquad i = 3, \cdots, n\,. \tag{2.24}$$

On the other hand, we have

$$\begin{aligned} \nabla_{X_i}\langle AX, X\rangle &= \langle \nabla_{X_i}AX, X\rangle + \langle AX, \nabla_{X_i}X\rangle \\ &= \langle [X_i, X], AX\rangle = \langle [X_i, X], X\rangle\langle AX, X\rangle \\ &= -\langle \nabla_X X_i, X\rangle\langle AX, X\rangle\,. \end{aligned} \tag{2.25}$$

Similarly,

$$\nabla_{X_i}\langle AY, Y\rangle = -\langle \nabla_Y X_i, Y\rangle\langle AY, Y\rangle\,. \tag{2.26}$$

The relations (2.24), (2.25) and (2.26) give

$$\langle \nabla_X X_i, X\rangle\langle AX, X\rangle + \langle \nabla_Y X_i, Y\rangle\langle AY, Y\rangle = 0\,,$$

which implies, in consequence of (2.23),

$$\langle \nabla_X X_i, X\rangle = \langle \nabla_Y X_i, Y\rangle = 0\,, \qquad \text{for all} \quad i \geqq 3.$$

Thus in this case N is parallel at q.

Next assume that the non-zero eigenvalues of A_q are equal. If they are equal in a neighborhood of q, it is possible to find vector fields X, Y satisfying (2.22) and therefore to show that N is parallel at q.

Finally assume that the non-zero eigenvalues of A_q are equal at q, but each neighborhood of q contains a point at which they are distinct. A simple continuity argument shows that in this case N is also parallel at q. Therefore it turns out that N is parallel. The parallelism of N and the fact that its leaves are totally geodesic imply that

$$\langle R(X, X_i)X_i, X\rangle = 0\ ,$$

where X is a unit vector field orthogonal to X_i. On the other hand, we have by $AX_i = 0$

$$\langle R(X, X_i)X_i, X\rangle = K \neq 0\,. \tag{2.27}$$

This is a contradiction. Hence (2.21) is proved.

Next it will be shown that:

(2.28) *$U - P$ has no interior points .*

To show this, consider a point $q \in \mathrm{Int}\,(U - P)$; this means that for some index $i_0 \geqq 3$ the numbers

$$\langle \nabla_{X_1}X_{i_0}, X_2\rangle_q,\ \langle \nabla_{X_1}X_{i_0}, X_1\rangle_q - \langle \nabla_{X_2}X_{i_0}, X_2\rangle_q,\ \langle \nabla_{X_2}X_{i_0}, X_1\rangle_q \tag{2.29}$$

are not simultaneously zero. For any $i \geqq 3$, let Δ^i denote the function on U:

$$\Delta^i = [\langle \nabla_{X_1}X_i, X_1\rangle - \langle \nabla_{X_2}X_i, X_2\rangle]^2 + 4\langle \nabla_{X_1}X_i, X_2\rangle\langle \nabla_{X_2}X_i, X_1\rangle\,.$$

The following two subcases should be discussed:

(a) $\langle \nabla_{X_1} X_{i_0}, X_2 \rangle_q \neq 0$ (or $\langle \nabla_{X_2} X_{i_0}, X_1 \rangle_q \neq 0$) $\begin{cases} \Delta^{i_0}(q) \neq 0 \\ \Delta^{i_0}(q) = 0 \end{cases}$ $\begin{cases} \Delta^{i_0} = 0 \text{ in a neighborhood of } q . \\ \text{Any neighborhood of } q \text{ has a point at which } \Delta^{i_0} \neq 0 . \end{cases}$

(b) $\begin{matrix} \langle \nabla_{X_1} X_{i_0}, X_2 \rangle_q = 0 \\ \langle \nabla_{X_2} X_{i_0}, X_1 \rangle_q = 0 \end{matrix}$ $\begin{cases} \Delta^{i_0}(q) = 0 \\ \Delta^{i_0}(q) \neq 0 \end{cases}$ $\begin{cases} \text{The functions } \langle \nabla_{X_1} X_{i_0}, X_2 \rangle, \langle \nabla_{X_2} X_{i_0}, X_1 \rangle \text{ both vanish on a neighborhood of } q . \\ \text{Any neighborhood of } q \text{ contains a point at which either one of } \langle \nabla_{X_1} X_{i_0}, X_2 \rangle, \langle \nabla_{X_2} X_{i_0}, X_1 \rangle \text{ is non-zero at this point} . \end{cases}$

(a) Assume $\langle \nabla_{X_1} X_{i_0}, X_2 \rangle$, Δ^{i_0} to be non-zero at all points of a neighborhood V_q of q.

In view of the assumption made above, the quadratic equation

$$\langle \nabla_{X_1} X_{i_0}, X_2 \rangle t^2 - [\langle \nabla_{X_1} X_{i_0}, X_1 \rangle - \langle \nabla_{X_2} X_{i_0}, X_2 \rangle] t - \langle \nabla_{X_2} X_{i_0}, X_1 \rangle = 0 ,$$

defines two complex valued C^∞-functions α, β such that

$$\alpha\beta = - \frac{\langle \nabla_{X_2} X_{i_0}, X_1 \rangle}{\langle \nabla_{X_1} X_{i_0}, X_2 \rangle} , \quad \alpha + \beta = \frac{\langle \nabla_{X_1} X_{i_0}, X_1 \rangle - \langle \nabla_{X_2} X_{i_0}, X_2 \rangle}{\langle \nabla_{X_1} X_{i_0}, X_2 \rangle} . \tag{2.30}$$

Consider the complex vector fields Z, W on V_q defined by

$$Z = \alpha X_1 + X_2 , \quad W = \beta X_1 + X_2 , \tag{2.31}$$

which are linearly independent at each point since α and β take different values at each point. Then for the restriction of A to V_q

$$\begin{aligned} \langle AZ, W \rangle &= \alpha\beta \langle AX_1, X_1 \rangle + (\alpha + \beta) \langle AX_1, X_2 \rangle + \langle AX_2, X_2 \rangle \\ &= \frac{1}{\langle \nabla_{X_1} X_{i_0}, X_2 \rangle} [\langle \nabla_{X_1} X_{i_0}, X_2 \rangle \langle AX_2, X_2 \rangle + (\langle \nabla_{X_1} X_{i_0}, X_1 \rangle \\ &\quad - \langle \nabla_{X_2} X_{i_0}, X_2 \rangle) \langle AX_1, X_2 \rangle - \langle \nabla_{X_2} X_{i_0}, X_1 \rangle \langle AX_1, X_1 \rangle] , \end{aligned} \tag{2.32}$$

which is zero due to (2.20). A similar relation holds for the restriction of $\bar{A}$ to V_q.

On the other hand, neither $\langle Z, Z \rangle$ nor $\langle W, W \rangle$ vanishes. In fact, assume $\langle Z, Z \rangle$ (resp. $\langle W, W \rangle$) to be zero at a point of V_q. Since α and β are conjugate to each other, it follows from (2.31) that

$$\alpha\beta = 1 \quad \text{and} \quad \alpha + \beta = 0$$

hold at x. In view of (2.20) and (2.32), it follows

$$\langle AX_1, X_1\rangle_x + \langle AX_2, X_2\rangle_x = 0 ,$$

which contradicts non-zero constancy of mean curvature.

Thus Z and W satisfy the conditions of Lemma 2.5, i.e.,

$$\langle \nabla_Z X_i, W\rangle = \langle \nabla_W X_i, Z\rangle = 0 , \qquad i = 3, \cdots, n . \tag{2.33}$$

From (2.31) and (2.33) it follows that

$$\alpha\beta\langle \nabla_{X_1} X_i, X_1\rangle + \alpha\langle \nabla_{X_1} X_i, X_2\rangle + \beta\langle \nabla_{X_2} X_i, X_1\rangle + \langle \nabla_{X_2} X_i, X_2\rangle = 0 ,$$
$$\alpha\beta\langle \nabla_{X_1} X_i, X_1\rangle + \beta\langle \nabla_{X_1} X_i, X_2\rangle + \alpha\langle \nabla_{X_2} X_i, X_1\rangle + \langle \nabla_{X_2} X_i, X_2\rangle = 0 ,$$

which yield

$$(\alpha - \beta)(\langle \nabla_{X_1} X_i, X_2\rangle - \langle \nabla_{X_2} X_i, X_1\rangle) = 0 .$$

Since α and β take different values at each point, this equation becomes

$$\langle \nabla_{X_1} X_i, X_2\rangle - \langle \nabla_{X_2} X_i, X_1\rangle = 0 , \qquad i = 3, \cdots, n ,$$

which, together with (2.30), imply $\alpha\beta = -1$.

From $\alpha\beta = -1$ and non-zero constancy of mean curvature, it follows

$$\frac{\langle AZ, Z\rangle}{\langle Z, Z\rangle} + \frac{\langle AW, W\rangle}{\langle W, W\rangle} = \text{constant} \neq 0 .$$

By covariant differentiation with respect to X_i of both sides of the above relation, we have

$$\nabla_{X_i}\left(\frac{\langle AZ, Z\rangle}{\langle Z, Z\rangle} + \frac{\langle AW, W\rangle}{\langle W, W\rangle}\right) = 0 . \tag{2.34}$$

On the other hand, we have

$$\begin{aligned} \nabla_{X_i}\langle AZ, Z\rangle &= \langle \nabla_{X_i} AZ, Z\rangle + \langle AZ, \nabla_{X_i} Z\rangle \\ &= \langle [X_i, Z], AZ\rangle + \langle AZ, \nabla_{X_i} Z\rangle \\ &= 2\langle \nabla_{X_i} Z, AZ\rangle - \langle \nabla_Z X_i, AZ\rangle \\ &= \frac{2\langle \nabla_{X_i} Z, Z\rangle\langle AZ, Z\rangle - \langle \nabla_Z X_i, Z\rangle\langle AZ, Z\rangle}{\langle Z, Z\rangle} , \end{aligned}$$

which implies

$$\nabla_{X_i}\left(\frac{\langle AZ, Z\rangle}{\langle Z, Z\rangle}\right) = -\frac{\langle \nabla_Z X_i, Z\rangle\langle AZ, Z\rangle}{\langle Z, Z\rangle^2} . \tag{2.35}$$

Similarly,

$$\nabla_{X_i}\Big(\frac{\langle AW, W\rangle}{\langle W, W\rangle}\Big) = -\frac{\langle \nabla_W X_i, W\rangle\langle AW, W\rangle}{\langle W, W\rangle^2}\,. \tag{2.36}$$

The relations (2.34), (2.35) and (2.36) give

$$\frac{\langle \nabla_Z X_i, Z\rangle\langle AZ, Z\rangle}{\langle Z, Z\rangle^2} + \frac{\langle \nabla_W X_i, W\rangle\langle AW, W\rangle}{\langle W, W\rangle^2} = 0$$

and of course

$$\frac{\langle \nabla_Z X_i, Z\rangle\langle \bar{A}Z, Z\rangle}{\langle Z, Z\rangle^2} + \frac{\langle \nabla_W X_i, W\rangle\langle \bar{A}W, W\rangle}{\langle W, W\rangle^2} = 0\,.$$

By the argument similar to that in the proof of Lemma 2.5, it can be concluded that

$$\langle \nabla_Z X_i, Z\rangle = \langle \nabla_W X_i, W\rangle = 0\,, \qquad i = 3, \cdots, n\,,$$

which, together with (2.33), imply that N is parallel in the neighborhood V_q.

The next case to be analyzed is that of

$$\langle \nabla_{X_1} X_{i_0}, X_2\rangle \neq 0$$

in a neighborhood V_q of q and Δ^{i_0} vanishing at all points of V_q. In this case the functions α, β coincide at each point, and the vector field

$$X = \frac{\alpha X_1 + X_2}{\|\alpha X_1 + X_2\|}$$

satisfies the conditions of Lemma 2.4 and therefore N is parallel in the neighborhood V_q.

The parallelism of N in the last subcase of (a) is proved by using the reasoning of the proof of the first subcase and a simple continuity argument.

(b) The first subcase cannot occur, for otherwise all functions listed in (2.29) would vanish at q. Hence to study the next case it may be assumed that in a neighborhood V_q of q, the functions $\langle \nabla_{X_1} X_{i_0}, X_2\rangle$, $\langle \nabla_{X_2} X_{i_0}, X_1\rangle$ vanish, while Δ^{i_0} is never zero. Using again (2.20) we obtain

$$[\langle \nabla_{X_1} X_{i_0}, X_1\rangle - \langle \nabla_{X_2} X_{i_0}, X_2\rangle]\langle AX_1, X_2\rangle = 0$$

and of course

$$[\langle \nabla_{X_1} X_{i_0}, X_1\rangle - \langle \nabla_{X_2} X_{i_0}, X_2\rangle]\langle \bar{A}X_1, X_2\rangle = 0\,,$$

which show that the vector fields X_1, X_2 satisfy the condition of Lemma 2.5, and by the same argument with the first subcase of (a), the parallelism of N is established in V_q.

Finally, the last case of (b) can be related to the first case of (a), and as before, a continuity argument proves the parallelism of N at q, in this case.

Thus N is parallel at $q \in \mathrm{Int}\,(U - P)$, which contradicts (2.27). Hence (2.28) is proved.

The conclusions (2.21) and (2.28) are obviously incompatible, and hence M contains an open congruent submanifold.

3. **Rigidity of hypersurfaces.** The main purpose in this section is to prove the following theorem.

THEOREM 3.1. *Let M be a Riemannian n-manifold with $n \geqq 3$ and let f and $\bar{f}$ be isometric immersions of M in $\tilde{M}(K)$, $K \neq 0$ with non-zero constant mean curvature. Then there is an isometry ϕ of $\tilde{M}(K)$ such that $\bar{f} = \phi \circ f$.*

The proof of Theorem 3.1 will depend on the following proposition and lemmas.

PROPOSITION 3.2. *Let M, $\tilde{M}(K)$, f and $\bar{f}$ be as in Theorem 3.1. Assume further that the type number of the immersion f at each x is $\geqq 2$. Then there is an isometry ϕ of $\tilde{M}(K)$ such that $\bar{f} = \phi \circ f$.*

PROOF. In view of the assumption made above, Proposition 1.3 implies that A and $\bar{A}$ have rank $\geqq 2$ everywhere.

Let U be the subset of M consisting of these points which are contained in some open congruent neighborhood (this neighborhood may depend on the point). It follows from Theorem 2.1 that $M - U$ has no interior points, i.e., that U is dense in M. Since U is covered by open congruent submanifolds, each connected component of U is congruent. Let x be a point of M, and V an orientable neighborhood of x. It will be shown that there is a function $e(y)$ defined on V, assuming only the values $+1$ or -1, and such that

$$\bar{A}_y = e(y) A_y \qquad \text{for all} \quad y \in V . \tag{3.1}$$

In fact, if $y \in U$, this follows from the congruence of each component. On the other hand, if $y \notin U$, it can be approximated by points at which (3.1) holds, and by continuity (3.1) holds at y. Again the continuity of A and $\bar{A}$ gives the continuity of e. Since V is assumed to be connected, e must be constant, and V is congruent. This argument shows that M can be covered by congruent neighborhoods, and thus from Proposition 1.7 the proof of Proposition 3.2 is complete.

LEMMA 3.3. (Ryan). *Let M be a hypersurface in $\tilde{M}(K)$ whose principal* *curvatures are constant. If exactly two principal curvatures* $\lambda \neq \mu$ *are distinct, then* $\lambda\mu + K = 0$.

PROOF. See [8], Theorem 2.5.

LEMMA 3.4. *Let* M, $\tilde{M}(K)$, f *and* $\bar{f}$ *be as in Theorem* 3.1. *Then* M *contains an open congruent submanifold.*

PROOF. Assume that M contains no open congruent submanifold. In view of Proposition 3.2, the type numbers of f and $\bar{f}$ are at most one at all points, which shows that the type numbers of f and $\bar{f}$ have to be exactly 1. Thus M satisfies the conditions of Lemma 3.3, i.e.,

$$K = 0\ ,$$

which contradicts $K \neq 0$. Hence Lemma 3.4 is proved.

PROOF OF THEOREM 3.1. We repeat the same argument of Proposition 3.2 using Lemma 3.4.

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