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LOCALLY INVERTIBLE OPERATORS AND EXISTENCE PROBLEMS IN DIFFERENTIAL SYSTEMS

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1. Introduction. Consider the boundary value problem

$$(*) \qquad x' + A(t)x = F(t, x)$$

$$(**) \qquad Lx = r ,$$

where A is an $n \times n$ matrix, F is an n-vector, L is a bounded linear operator defined on the space of continuous n-vector-valued functions on an interval $[a, b]$, under the sup-norm, and r is a vector in R^n.

The existence of a solution to the problem $((*), (**))$ is shown here under quite general assumptions on the matrix A and the vector F, which include the following two: the homogeneous problem corresponding to $(*)$, $(**)$ has only the zero solution, and the vector $F(t, u)$ is continuously differentiable in the neighborhood of a certain vector $u_0 \in R^n$.

The method followed here is based on the fact that certain (in general nonlinear) operator T associated with the problem $((*), (**))$ is Fréchet differentiable in a neighborhood of a point in its domain. This operator satisfies the conditions of Theorem I, p. 61, in Miranda's monograph [8] (cf. preliminaries, Theorem A). Thus, it is locally invertible (cf. Definition 2.3), and this implies the existence of solutions to the problem $((*), (**))$.

Extensions of the above considerations to second order systems are also considered. In Section 4 it is shown that this method works also for boundary value problems on infinite intervals, or other existence problems.

For work related to the present paper, the reader is referred, for example, to the book of Falb and Jong [2], the papers [10], [11], [12] of Urabe, and the dissertation of McCandless [7].

2. Preliminaries. We start with certain notations and definitions, and (for the sake of completeness) we state the above mentioned theorem in Miranda's monograph [8].

In what follows $R = (-\infty, \infty)$, $R_+ = [0, \infty)$, and $J = [a, b]$, where a, b are two fixed real numbers. The symbol $\|\cdot\|$ will denote the norm in R^n and the corresponding norm for $n \times n$ real matrices $U = [u_{ij}]$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, n$. If E is a subset of R, then by $C[E, R^n]$ we denote the space of all bounded and continuous functions on E with values R^n. $C[E, R^n]$ will always be considered with the norm

$$\|f\|_E = \sup_{t \in E} \|f(t)\| , \tag{2.1}$$

under which it is a Banach space. By C_c we denote the space of all functions $f \in C[R_+, R^n]$ such that $\lim_{t\to\infty} f(t)$ exists and is a finite vector. C_c is a closed subspace of $C[R_+, R^n]$.

DEFINITION 2.1. Let B_1, B_2 be Banach spaces with norms $\|\cdot\|_1$, $\|\cdot\|_2$ respectively. Let S be an open subset of B_1. Let $f: S \to B_2$, $u \in S$ be such that there exists a linear operator $D(u): B_1 \to B_2$ with the property

$$f(u + h) - f(u) = D(u)h + w(u, h) , \tag{2.2}$$

for every $h \in B_1$, where

$$\lim_{\|h\|_1 \to 0} \|w(u, h)\|_2 / \|h\|_1 = 0 . \tag{2.3}$$

Then $D(u)h$ is the "Fréchet differential of f at u with increment h."

It can be shown that if the Fréchet differential of f at $u \in S$ exists, and f is continuous at $u \in S$, then its "Fréchet derivative" $D(u)$ at the point $u \in S$ is a bounded linear operator on B_1 into B_2.

DEFINITION 2.2. Assume that B_1, B_2, S, f are as in Definition 2.1, and that f is continuous and Fréchet differentiable at every point of S. Moreover, assume that for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

(i) If $u_1, u_2 \in S$ with $\|u_1 - u_2\|_1 \leqq \delta(\varepsilon)$, then

$$\|[D(u_1) - D(u_2)]h\|_2 \leqq \varepsilon \|h\|_1 , \quad h \in B_1 ; \tag{2.4}$$

(ii) $\|w(u, h)\|_2 \leqq \varepsilon \|h\|_1$ for every $u \in S$ and $h \in B_1$ with $\|h\| \leqq \delta(\varepsilon)$.

Then f is said to be "C-differentiable on S".

DEFINITION 2.3. Let B_1, B_2, S, f be as in Definition 2.1, and let $u_0 \in S$ with $f(u_0) = v_0 \in B_2$. Then f is said to be "locally invertible" at (u_0, v_0) if there exist numbers $\alpha > 0$, $\beta > 0$ such that for any $v_1 \in B_2$ with $\|v_1 - v_0\|_2 \leqq \beta$, the equation $f(u) = v_1$ has a unique solution with the property $\|u - u_0\|_1 \leqq \alpha$.

Now we are ready for the following

THEOREM A. *Let $f: S \to B_2$ be C-differentiable on S. Moreover, let $D(u_0): B_1 \to B_2$ be one to one and onto for some $u_0 \in S$. Then the function f is locally invertible at $(u_0, f(u_0))$.*

For a proof of this theorem the reader is referred to Miranda [8].

In what follows, we shall denote the Fréchet derivative of f at u_0 by $f'(u_0)$. Now let B_1, B_2, S be as in Definition 2.1 and let $u_0 \in S$. Then if $f\colon S \to B_2$ is Fréchet differentiable at u_0 and $T\colon B_2 \to B_2$ is a bounded linear operator, then the Fréchet derivative of Tf at u_0 exists and equals $Tf'(u_0)$. This follows easily from the definition of the Fréchet differential.

Now assume that the $n \times n$ matrix $A(t)$ in $(*)$ is continuous on J. Then $X(t)$ will denote the fundamental matrix of solutions of

$$x' + A(t)x = 0 \tag{2.5}$$

with $X(0) = I$ (the $n \times n$ identity matrix). Now let L in $(**)$ be a bounded linear operator mapping $C[J, R^n]$ into R^n. Moreover, denote by $\tilde{L}$ the $n \times n$ matrix whose columns are the values of L on the corresponding columns of $X(t)$. Assume that $\tilde{L}$ is nonsingular with inverse $\tilde{L}^{-1}$. Then any solution of $((*), (**))$ (with $F(t, u)$ continuous on $J \times R^n$) satisfies the equation

$$x(t) = X(t)\tilde{L}^{-1}(r - Lp(\cdot\,; x)) + p(t; x)\ , \tag{2.6}$$

$$p(t; x) = \int_0^t X(t)X^{-1}(s)F(s, x(s))ds\ . \tag{2.7}$$

For a proof of this fact the reader is referred to Opial [9]. Let V be an open subset of R^n and assume that $F(t, u)$ is continuous on $J \times R^n$ and continuously differentiable in V with respect to u. Let $F_x(t, u) = [(\partial F_i/\partial x_j)(t, u)]$ denote the Jacobian matrix of F at the point $(t, u) \in J \times U$. Now let $G = \{u \in C[J, R^n]; u(t) \in U, t \in J\}$, and $x_0 \in G$. Then the Fréchet derivative at x_0 of the operator $Q\colon G \to C[J, R^n]$ with $[Qu](t) = F(t, u(t))$, is given by

$$[Q'(x_0)h](t) = F_x(t, x_0(t))h(t)\ , \quad t \in J \tag{2.8}$$

for any $h \in C[J, R^n]$. For a proof of this fact in the case of a real valued function F the reader is referred to Ladas and Lakshmikantham [6, p. 12]. The reader is also referred to the dissertation of McCandless [7], where a good treatment has been given of the Fréchet differentiability in Banach spaces.

Now assume that $F(t, u)$ is defined and continuous on $R_+ \times R^n$ and that it is continuously differentiable in u for any $t \in R_+$. Moreover, assume that the set $\{F(t, u(t)); t \in R_+\}$ is bounded for any $u \in C[R_+, R^n]$. Then the operator Q with $[Qu](t) = F(t, u(t))$ maps the space $C[R_+, R^n]$ into itself. Another assumption that we make on F is the following: for every bounded set K in R^n and every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, K)$ such that for each $u_1, u_2 \in K$ with $\|u_1 - u_2\| \leqq \delta$, $\|F_x(t, u_1) - F_x(t, u_2)\| \leqq \varepsilon$

for every $t \in R_+$. We shall show that the Fréchet derivative of the operator Q defined above is given, at $x_0 \in C[R_+, R^n]$, by

$$[Q'(x_0)h](t) = F_x(t, x_0(t))h(t)\,, \quad t \in R_+ \tag{2.9}$$

for any $h \in C[R_+, R^n]$. In fact, we have, for some functions θ_i, $i = 1, 2, \cdots, n$ with $0 \leqq \theta_i \leqq 1$,

$$\begin{aligned}\sup_{t \in R_+} \left\| F(t, x_0(t) + h(t)) - F(t, x(t)) - \frac{\partial F}{\partial x}(t, x_0(t))h(t) \right\| \\ \leqq \sup_{t \in R_+} \left\| \left[\frac{\partial F_i}{\partial x_j}(t, x_0(t) + \theta_i h(t)) \right] - \left[\frac{\partial F_i}{\partial x_j}(t, x_0(t)) \right] \right\| \cdot \| h \|_{R_+}\,.\end{aligned} \tag{2.10}$$

Since $\| x_0(t) + \theta_i h(t) \| \leqq \| x_0 \|_{R_+} + \| h \|_{R_+} < \infty$, dividing the above inequality by $\| h \|_{R_+}$, and taking the limit of both members of the resulting inequality as $\| h \|_{R_+} \to 0$, our assertion follows.

Similarly, it can be shown that the above hold for C_c, or $C[R, R^n]$ under analogous assumptions on $F(t, u)$.

3. Problems on finite intervals. In what follows, the notations and definitions of Section 2 will be used without further mention.

THEOREM 3.1. *For the equation*

$$x(t) = f(t) - X(t)\tilde{L}^{-1}Lp(\cdot\,; x) + p(t; x) \tag{3.1}$$

assume the following:

(i) *there exists an open set $U \subseteqq R^n$ such that $F(t, u)$ and its Jacobian $F_x(t, u)$ are defined and continuous on $J \times U$. Moreover, for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $u_2, u_2 \in U$ and $\| u_1 - u_2 \| \leqq \delta(\varepsilon)$ imply $\| F_x(t, u_1) - F_x(t, u_2) \| \leqq \varepsilon$ for every $t \in J$;*

(ii) *let $x_0 \in C[J, R^n]$, with $x_0(t) \in U$ for $t \in J$, be fixed, and let $f_0 \in C[J, R^n]$ be given by*

$$x_0(t) = f_0(t) - X(t)\tilde{L}^{-1}Lp(\cdot\,; x_0) + p(t; x_0)\,; \tag{3.2}$$

(iii) *let the equation*

$$x(t) = f(t) - X(t)\tilde{L}^{-1}Lq(\cdot\,; x_0, x) + q(t; x_0, x) \tag{3.3}$$

have a unique solution $x \in C[J, R^n]$ for every $f \in C[J, R^n]$, where

$$q(t; x_0, x) = \int_0^t X(t)X^{-1}(s)F_x(s, x_0(s))x(s)ds\,. \tag{3.4}$$

Then there exist two constants $\alpha > 0$, $\beta > 0$ such that for every $f \in C[J, R^n]$ with $\| f - f_0 \| \leqq \beta$ there exists a unique solution to the equation

$$x(t) = f(t) - X(t)L^{-1}Lp(\cdot\,; x) + p(t; x) \tag{3.5}$$

with the property $\|x - x_0\| \leqq \alpha$.

PROOF. Let $G = \{x \in C[J, R^n];\ x(t) \in U,\ t \in J\}$, and fix $x_0 \in G$. Now consider the operator $T\colon G \to C[J, R^n]$ with

$$[Tx](t) = x(t) + X(t)\tilde{L}^{-1}Lp(\cdot\,; x) - p(t; x)\,. \tag{3.6}$$

It is easy to see that the operator T is Fréchet differentiable on G, being the composition of an integral operator (which is linear and bounded) and the operator $Q\colon G \to C[J, R^n]$ with $(Qx)(t) = F(t, x(t))$, $t \in J$. Its Fréchet derivative at x_0 and with increment h is given by

$$[T'(x_0)h](t) = h(t) + X(t)\tilde{L}^{-1}Lq(\cdot\,; x_0, h) - q(t; x_0, h)\,. \tag{3.7}$$

Now let $f \in C[J, R^n]$ and consider $T'(x_0)h = f$. Then $h(t)$, $t \in C[J, R^n]$ is the unique solution of the linear equation (3.3). Thus, $T'(x_0)$ is one-to-one and onto. Let $\varepsilon > 0$ be given. Then there exists $\delta_1(\varepsilon) > 0$ such that $u_1, u_2 \in U$ with $\|u_1 - u_2\| \leqq \delta_1(\varepsilon)$ implies

$$\|F_x(\cdot, u_1) - F_x(\cdot, u_2)\|_J \leqq \varepsilon/2\mu\,, \tag{3.8}$$

where $\mu = \max\{\mu_1, \mu_2\}$ with

$$\mu_1 = \sup_{t \in J}\left\{\|X(t)\|\,\|\tilde{L}^{-1}\|\,\|L\|\int_0^t \|X(t)X^{-1}(s)\|\,ds\right\}, \tag{3.9}$$

$$\mu_2 = \sup_{t \in J}\left\{\int_0^t \|X(t)X^{-1}(s)\|\,ds\right\}. \tag{3.10}$$

Thus, if $u_1, u_2 \in G$ with $\|u_1 - u_2\|_J \leqq \delta_1(\varepsilon)$,
we obtain

$$\|T(u_1)h - T(u_2)h\|_J \leqq \varepsilon\|h\|_J \tag{3.11}$$

for any $h \in C[J, R^n]$. Moreover, it follows as in (2.17) and (2.18) that if

$$T(x + h) - Tx = T'(x)h + w(x, h) \tag{3.12}$$

for every $x \in G$, $h \in C[J, R^n]$, then there exists $\delta_2(\varepsilon) > 0$ such that

$$\begin{aligned} \sup_{t \in J}\|w(x, h)(t)\| &\leqq \sup_{t \in J}\left\|\left[\frac{\partial F_i}{\partial x_j}(t, x(t) + \theta_i h(t)) - \frac{\partial F_i}{\partial x_j}(t, x(t))\right]\right\| \cdot \|h\|_J \\ &\leqq \varepsilon\|h\|_J \end{aligned} \tag{3.13}$$

for any $h \in C[J, R^n]$ with $\|h\|_J \leqq \delta_2(\varepsilon)$. If we let $\delta(\varepsilon) = \min\{\delta_1(\varepsilon), \delta_2(\varepsilon)\}$ we see that the operator T is C-differentiable on G, and that the rest of the assumptions of Theorem A are satisfied. This completes the proof.

The above theorem has the following important

COROLLARY 3.1. *Let the assumptions of Theorem* 3.1 *be satisfied, and assume that the number* β *in the conclusion of the same theorem is such that* $\|f_0\|_J < \beta$. *Then there exist positive numbers* α, μ *such that for each* $r \in R^n$ *with* $\|r\| \leqq \mu$ *there exists a unique solution of the problem* ((*), (**)) *satisfying* $\|x - x_0\|_J \leqq \alpha$.

PROOF. Let $\varepsilon > 0$ be such that $\|f_0\|_J + \varepsilon < \beta$. Then since

$$\lim_{\|r\|\to 0} \sup_{t \in J} \| X(t)\tilde{L}^{-1}r - f_0(t)\| = \|f_0\|_J , \tag{3.14}$$

there exists $\delta(\varepsilon) > 0$ such that

$$\sup_{t \in J} \| X(t)\tilde{L}^{-1}r - f_0(t)\| \leqq \|f_0\|_J + \varepsilon < \beta \tag{3.15}$$

whenever $\|r\| \leqq \delta(\varepsilon) = \mu$. Thus, for every $r \in R^n$ with $\|r\| \leqq \mu$ the equation (2.13) has a unique solution inside $G_0 = \{x \in C[J, R^n]; \|x - x_0\|_J \leqq \alpha\}$. This completes the proof.

It is evident from the above corollary that to ensure a unique solution of ((*), (**)) for sufficiently small $\|r\|$, it suffices to find an approximate solution $x_0(t)$, satisfying

$$x_0'(t) + A(t)x_0(t) = F(t, x_0(t)) + \lambda(t) , \tag{3.16}$$

and such that $\|f_0\|_J < \beta$, where

$$f_0(t) = -X(t)\tilde{L}^{-1}L\phi(\cdot; \lambda) + \phi(t; \lambda)$$

with

$$\phi(t; \lambda) = \int_0^t X(t)X^{-1}(s)\lambda(s)ds .$$

We also wish to point out that the solutions obtained by Theorem 3.1 and its corollary can be obtained by Newton's method. This follows from the proof of Theorem A, where a contraction mapping is involved.

Let us now consider the following boundary value problem:

$$x'' = F(t, x) \tag{3.17}$$

$$x(a) = x(b) = 0 . \tag{3.18}$$

Here F is assumed to be defined and continuous on $J \times R^n$. Then finding a solution of the above problem amounts to finding a solution $x(t)$, $t \in J$ of the integral equation

$$x(t) = \frac{1}{a-b}\Big[(b-t)\int_a^t (s-a)F(s, x(s))ds + (t-a)\int_t^b (b-s)F(s, x(s))ds\Big] . \tag{3.19}$$

We state the following theorem, whose proof can be carried out as in Theorem 3.1 and its corollary.

THEOREM 3.2. *Assume that Hypothesis* (i) *of Theorem* 3.1 *is satisfied. Furthermore, fix* $x_0 \in C[J, R^n]$ *with* $x_0(t) \in U$ *for every* $t \in J$, *and let* $f_0 \in C[J, R^n]$ *be given by*

$$x_0(t) = f_0(t) + \frac{1}{a-b}\Big[(b-t)\int_a^t (s-a)F(s, x_0(s))ds + (t-a)\int_t^b (b-s)F(s, x_0(s))ds\ . \tag{3.20}$$

Let the equation

$$x(t) = f(t) + \frac{1}{a-b}\Big[(b-t)\int_a^t (s-a)F_x(s, x_0(s))x(s)ds + (t-a)\int_t^b (b-s)F_x(s, x_0(s))x(s)ds\Big] \tag{3.21}$$

have a unique solution $x \in C[J, R^n]$ *for every* $f \in C[J, R^n]$. *Then there exist two positive numbers* α, β *such that for every* $f \in C[J, R^n]$ *with* $||f - f_0||_J \leqq \beta$, *there exists a unique solution* $x \in C[J, R^n]$ *of the equation*

$$x(t) = f(t) + \frac{1}{a-b}\Big[(b-t)\int_a^t (s-a)F(s, x(s))ds + (t-a)\int_t^b (b-s)F(s, x(s))ds\Big] \tag{3.22}$$

with the property $||x - x_0||_J \leqq \alpha$. *Moreover, if* $||f_0||_J \leqq \beta$, *then the problem* ((3.17), (3.18)) *has a unique solution* $x(t)$ *satisfying* $||x - x_0||_J \leqq \alpha$.

4. **Problems on infinite intervals.** It is evident that Theorem 3.1 and Corollary 3.1 can be extended to problems of the form ((*), (**)). We formulate below such a result and omit its proof, which follows as before. Moreover, we formulate a theorem concerning the existence of bounded solutions of the equation (*) on the whole axis R.

THEOREM 4.1. *For the problem* ((*), (**)) *assume the following*:

(i) $\lim_{t\to\infty} X(t) = X(\infty)$ *exists and is a finite matrix. Moreover,*

$$\sup_{t \in R_+} \int_0^t ||X(t)X^{-1}(s)||\,ds < +\infty\ ;$$

(ii) $T: C_c \to R^n$ *is a bounded linear operator such that* $\tilde{L}^{-1}$ *exists;*

(iii) $F(t, u): R_+ \times R^n \to R^n$ *is continuous, and continuously differentiable in* u. *Moreover, for every* $\varepsilon > 0$ *there exists* $\delta(\varepsilon) > 0$ *such that for every* $u_1, u_2 \in R^n$ *with* $||u_1 - u_2|| \leqq \delta(\varepsilon)$,

$$\|F_x(t, u_1) - F_x(t, u_2)\| \leqq \varepsilon\,, \quad t \in R_+\,. \tag{4.1}$$

(iv) *for every* $u \in C[R_+, R^n]$, *the set* $\{F(t, u(t));\, t \in R_+\}$ *is bounded; moreover,*

$$\int_0^\infty \|X^{-1}(s)F(s, u(s))\|\,ds < +\infty\;;$$

(v) *for every* $f \in C_c$ *the equation* (3.3) *has a unique solution* $x \in C_c$, *where* q *is given by* (3.4) *and* $x_0 \in C_c$ *is fixed and such that*

$$\int_0^\infty \|X^{-1}(s)F_x(s, u(s))\|\,ds < +\infty$$

for every u *in some neighborhood of* x_0.

Then if f_0 *is given by* (3.2), *there exist two constants* $\alpha > 0$, $\beta > 0$ *such that for every* $f \in C[J, R^n]$ *with* $\|f - f_0\|_{R_+} \leqq \beta$, *there exists a unique solution* $x \in C_c$ *to the equation* (3.5) *with* $\|x - x_0\|_{R_+} \leqq \alpha$. *Moreover, if* $\|f_0\|_{R_+} < \beta$, *then the problem* ((*), (**)) *has a solution for* $\|r\|$ *sufficiently small.*

Now assume that P_1, P_2 are $n \times n$ projections, $P_1 + P_2 = I$. The following theorem ensures the existence of a solution to the integral equation

$$\begin{aligned} x(t) = &\int_{-\infty}^t X(t)P_1X^{-1}(s)F(s, x(s))ds \\ &- \int_t^\infty X(t)P_2X^{-1}(s)F(s, x(s))ds\,. \end{aligned} \tag{4.2}$$

In connection with (4.2) we shall consider the equations

$$\begin{aligned} x_0(t) = f_0(t) + &\int_{-\infty}^t X(t)P_1X^{-1}(s)F(s, x_0(s))ds \\ &- \int_t^\infty X(t)P_2X^{-1}(s)F(s, x_0(s))ds\,, \end{aligned} \tag{4.3}$$

$$\begin{aligned} x(t) = f(t) + &\int_{-\infty}^t X(t)P_1X^{-1}(s)F_x(s, x_0(s))x(s)ds \\ &- \int_t^\infty X(t)P_2X^{-1}(s)F_x(s, x_0(s))x(s)ds\,. \end{aligned} \tag{4.4}$$

THEOREM 4.2. *For the system* (*) *assume the following*:

(i) $\displaystyle\sup_{t \in R_+}\left[\int_{-\infty}^t \|X(t)P_1X^{-1}(s)\|\,ds + \int_t^\infty \|X(t)P_2X^{-1}(s)\|\,ds\right] < +\infty\;;$

(ii) $F(t, u)$ *satisfies* (iii) *of Theorem* 4.1 *with* R_+ *replaced by* R;

(iii) *for every* $u \in C[R, R^n]$, *the set* $\{F(t, u(t));\, t \in R\}$ *is bounded;*

$$\text{(iv)} \qquad \sup_{t \in R_+} \left[\int_{-\infty}^{t} \| X(t) P_1 X^{-1}(s) F_x(s, u(s)) \| ds + \int_{t}^{\infty} \| X(t) P_2 X^{-1}(s) F_x(s, u(s)) \| ds \right] < +\infty ,$$

for every u *in some neighborhood of* $x_0 \in C[R, R_+]$, *and the equation* (4.4) *has a unique solution in* $C[R, R^n]$ *for every* $f \in C[R, R^n]$.

Then there exist two constants $\alpha, \beta > 0$ *such that, if* $f_0 \in C[R, R^n]$ *is given by* (4.3), *then for every* $f \in C[R, R^n]$ *with* $\| f - f_0 \|_R \leqq \beta$, *there exists a unique solution to the equation*

$$x(t) = f(t) + \int_{-\infty}^{t} X(t) P_1 X^{-1}(s) F(s, x(s)) ds - \int_{t}^{\infty} X(t) P_2 X^{-1}(s) F(s, x(s)) ds \tag{4.5}$$

satisfying $\| x - x_0 \|_R \leqq \alpha$. *Moreover, if* $\| f_0 \| \leqq \beta$, *the system* $(*)$ *has a unique solution* $x \in C[R, R^n]$ *such that* $\| x - x_0 \|_R \leqq \alpha$. *This is given by* (4.5), *where* $f(t) \equiv 0, \quad t \in R$.

5. Remarks. The assumed uniqueness of the solutions of the integral equations (3.3), (3.21), can be ensured by considering sufficiently small intervals J, in which case we obtain contraction mappings. Similarly, small enough integrals in (v) of Theorem 4.1 and (iv) of Theorem 4.2, ensure the uniqueness of solutions of (3.3), (4.4) respectively. Corollary (3.1) overlaps in the case $Tx = x(0) - x(\omega) = 0$ (ω-periodicity) with a result of Urabe [10]. For initial value problems connected to Fréchet derivatives, the reader is referred to Alekseev [1]. For boundary value problems on infinite intervals related papers are those of Kartsatos [3-5].

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