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QUASI-SASAKIAN MANIFOLDS

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Introduction. In the study of even-dimensional Riemannian manifolds an almost Hermitian manifold $P(J, G)$ is an almost complex manifold with a Riemannian metric G such that $G(JX, Y) = -G(JX, Y)$ for any vector fields X and Y. If the Nijenhuis tensor field $[J, J]$ of the structure tensor field J vanishes on P, the manifold $P(J, G)$ is called to be Hermitian. A Kähler manifold is a Hermitian manifold with the closed fundamental 2-form Ω, defined by $\Omega(X, Y) = G(JX, Y)$. We know a familiar result that a necessary and sufficient condition for an almost Hermitian manifold $P(J, G)$ to be Kählerian is that $\nabla_X J = 0$ holds for any vector field X with respect to the Riemannian connection ∇ of the metric G(e.g., [7], [15]).

Analogously, we shall consider odd-dimensional Riemannian manifolds. Let $M(f, E, \eta, g)$ be a $(2n+1)$-dimensional almost contact metric manifold, on which a set of tensor fields (f, E, η, g) consisting of a linear transformation field f, a vector field E, a 1-form η and a Riemannian metric g satisfies $f^2 = -I + \eta \otimes E$, $\eta(E) = 1$, $fE = 0$, $\eta(fX) = 0$, $\eta(X) = g(X, E)$, $g(fX, Y) = -g(X, fY)$ for any vector fields X and Y, where I denotes the identity linear transformation field ([8], [10]). An almost complex structure J_0 can be defined on the product $M \times R$ of M and a real line R by $J_0(X, \lambda d/dt) = (fX + \lambda E, -\eta(X)d/dt)$, where λ is a scalar field on $M \times R$. If the structure J_0 is complex analytic, the almost contact metric structure (f, E, η, g) is called to be normal. It is shown that a necessary and sufficient condition for an almost contact metric structure (f, E, η, g) to be normal is that the *torsion* tensor field S, defined by

$$S(X, Y) = [fX, fY] - f[fX, Y] - f[X, fY] + f^2[X, Y] + d\eta(X, Y)E$$

for any X and Y, vanishes on the manifold [9]. A quasi-Sasakian manifold $M(f, E, \eta, g)$ is a normal almost contact metric manifold with the closed fundamental 2-form F, defined by $F(X, Y) = g(fX, Y)$ ([1], [12]).

In this paper, our purpose is to prove the following theorem.

THEOREM. *A necessary and sufficient condition for an almost contact metric manifold $M(f, E, \eta, g)$ to be quasi-Sasakian is that there exists a symmetric linear transformation field A on M such that*

$$(*) \qquad (\nabla_X f)Y = \eta(Y)AX - g(AX, Y)E\,, \qquad fAX = AfX\,,$$

for any vector fields X and Y on M with respect to the Riemannian connection ∇ of the metric g.

A $(2n+1)$-dimensional manifold is said to have a contact structure if it carries a 1-form η with the property $\eta\wedge(d\eta)^n \neq 0$ [5]. It is well known that on a contact manifold there exists an almost contact metric structure (f, E, η, g) with a contact form η defining the contact structure and $d\eta = 2F$ ([8], [9], [10]). A normal contact metric manifold is also called a Sasakian manifold. A normal almost contact metric manifold is cosymplectic if its fundamental 2-form F and contact form η are both closed ([1], [4]). The $(2n+1)$-dimensional Euclidean space R^{2n+1} admits a cosymplectic structure. The sphere S^{2n+1} admits a Sasakian structure [10]. And a product manifold of a Sasakian manifold and a Kähler manifold admits a quasi-Sasakian structure [6].

It follows from the condition $(*)$ that $\nabla_X E = fAX$ holds for any X. In fact, differentiating covariantly $f^2Y = -Y + \eta(Y)E$ with respect to X, we have $(\nabla_X f)fY + f(\nabla_X f)Y = (\nabla_X\eta)(Y)E + \eta(Y)\nabla_X E$. Putting $Y = E$ in this equation, we find the desired relation. Then, since $d\eta(X, Y) = 2F(AX, Y)$, Theorem implies that a necessary and sufficient condition for an almost contact metric structure (f, E, η, g) on a manifold to be cosymplectic (resp. Sasakian) is that $\nabla_X f = 0$ (resp. $(\nabla_X f)Y = \eta(Y)X - g(X, Y)E$) holds ([1], [10]). In this paper, by Theorem we want to consider a quasi-Sasakian manifold being locally a product of a Sasakian manifold and a Kähler manifold, and to see the induced structures on hypersurfaces in Kähler manifolds.

1. The proof of Theorem. Assume that there exists a symmetric linear transformation field A satisfying $(*)$ on an almost contact metric manifold $M(f, E, \eta, g)$. Since $g((\nabla_X f)Y, Z) = (\nabla_X F)(Y, Z)$ and $dF(X, Y, Z) = (\nabla_X F)(Y, Z) + (\nabla_Y F)(Z, X) + (\nabla_Z F)(X, Y)$ are valid, then the 2-form F is closed. Using the symmetry of the connection ∇, we may reduce the bracket $[fX, fY] = \nabla_{fX} fY - \nabla_{fY} fX = (\nabla_{fX} f)Y - (\nabla_{fY} f)X + f\nabla_{fX} Y - f\nabla_{fY} X = \eta(Y)AfX - \eta(X)AfY - 2g(AfX, Y)E + f\nabla_{fX} Y - f\nabla_{fY} X$. Hence, we have $S = 0$.

Conversely, assume that $S = 0$ and $dF = 0$. The torsion tensor field S reduces to

$$S(X, Y) = (\nabla_{fX}f)Y + (\nabla_X f)fY - (\nabla_{fY}f)X - (\nabla_Y f)fX - \eta(Y)\nabla_X E + \eta(X)\nabla_Y E\,,$$

from which

$$g(S(X, Y), Z) = dF(fX, Y, Z) - (\nabla_Z F)(fX, Y) - \eta(Y)(\nabla_X\eta)(Z) - dF(fY, X, Z) + (\nabla_Z F)(fY, X) + \eta(X)(\nabla_Y\eta)(Z)$$

by virtue of $(\nabla_X F)(Y, Z) = -(\nabla_X F)(Z, Y)$. Hence, it follows that

$$(1.1)\qquad 2(\nabla_Z F)(fX, Y) = \eta(X)(\nabla_Z\eta)(Y) + \eta(Y)(\nabla_Z\eta)(X) - \eta(Y)(\nabla_X\eta)(Z) + \eta(X)(\nabla_Y\eta)(Z)\,.$$

Replacing X, Y and Z by $-fY$, Z and X in (1.1) respectively, we obtain

$$(1.2)\qquad 2(\nabla_X F)(Y, Z) = 2\eta(Y)(\nabla_X F)(E, Z) - \eta(Z)(\nabla_X\eta)(fY) + \eta(Z)(\nabla_{fY}\eta)(X)\,.$$

Putting $X = E$ in (1.1) and using $g(\nabla_Z E, E) = 0$, we find

$$(1.3)\qquad (\nabla_Y\eta)(Z) + (\nabla_Z\eta)(Y) = \eta(Y)(\nabla_E\eta)(Z)\,.$$

On the other hand, we take notice of the vanishing of the torsion tensor field again. Define a tensor field T of type (1.3) on M by $T(X, Y) = (\nabla_{fX}f)Y + (\nabla_X f)fY - \eta(Y)\nabla_X E$ for any X and Y. Then the relations $S(X, Y) = T(X, Y) - T(Y, X)$, $g(T(X, Z), Y) = -g(T(X, Y), Z)$ and $g(S(X, Y), Z) - g(S(X, Z), Y) - g(S(Y, Z), X) = 2g(T(X, Y), Z)$ hold. Therefore, we see that $S = 0$ is valid if and only if $T = 0$. Consequently, we have $\nabla_E E = 0$, $\nabla_E\eta = 0$, $\nabla_E f = 0$, $\nabla_{fX}E = f\nabla_X E$, $(\nabla_{fX}\eta)(Y) + (\nabla_X\eta)(fY) = 0$.

Considering (1.2) and (1.3) together with these relations, we see that E is a Killing vector field, i.e., $L_E g = 0$ where L_E denotes the Lie-derivation with respect to E, and have

$$(\nabla_X f)Y = \eta(Y)\underline{A}X - g(\underline{A}X, Y)E\,,\quad f\underline{A}X = \underline{A}fX,\ g(\underline{A}X, Y) = g(X, \underline{A}Y)$$

for any vector fields X and Y, where we have put $\underline{A}X = -f\nabla_X E$. This completes the proof.

2. Definition of a structure indicator tensor field $\bar{A}$. We make a substitution of $Y = E$ in (*). The tensor field A described in Theorem is written as $A = \underline{A} + k\eta\otimes E$ for a scalar field k, $k = \eta(AE)$. Since $g(\underline{A}X, \eta(Y)E) = 0$ holds for any X and Y, the subspaces $\underline{A}(T_x(M))$ and

$\eta(T_x(M))E$ are orthogonal in the tangent space $T_x(M)$ to M at each point x. This fact means that the space $\underline{A}(T_x(M))$ is always contained in the value $-f^2(T_x(M))$ at x of the $2n$-dimensional distribution determined by $-f^2$. If there exists a non-zero element X of the intersection of the kernel of $\underline{A}$ and $-f^2(T_x(M))$, the vector fX is non-zero and also belongs to itself. Thus, a relation

$$\text{even} = \operatorname{rank} \underline{A} \leqq \operatorname{rank} A \leqq \operatorname{rank} \underline{A} + 1$$

holds at each point of M.

Let there be given a symmetric linear transformation field A satisfying the condition $(*)$ on an almost contact metric manifold (i.e., a quasi-Sasakian manifold). Define a linear transformation field $\bar{A}$ by $\bar{A} = -f^2A + \eta \otimes E$, which means $\bar{A} = \underline{A} + \eta \otimes E$. We shall call $\bar{A}$ a *structure indicator tensor field* of a quasi-Sasakian structure. Then the relation rank $\bar{A} = 2p + 1 (0 \leqq p \leqq n)$ holds at each point of M. *The following conditions are equivalent.*

$$\text{(a)} \quad \underline{A}^2 = \underline{A}\,, \quad \text{(b)} \quad \bar{A}^2 = \bar{A}\,, \quad \text{(c)} \quad g((\nabla_X \bar{A})Y, E) = 0$$

for any X and Y. In fact, since the covariant derivative $\nabla_X \bar{A}$ of $\bar{A}$ with respect to X reduces to $(\nabla_X \bar{A})Y = g(\bar{A}X, f\bar{A}Y)E + (\nabla_X \eta)(Y)E - f\nabla_X \nabla_Y E + f\nabla_{\nabla_X Y} E + \eta(Y)\nabla_X E$, we have $g((\nabla_X \bar{A})Y, E) = g(f(\bar{A} - \bar{A}^2)X, Y)$. This is used for the verification of "(b)$\Leftrightarrow$(c)".

3. A product M of a Sasakian manifold and a Kähler manifold. In this article we prove (c.f., [1], [12]).

PROPOSITION 1. *Let $\bar{A}$ be a structure indicator tensor field on a manifold $M(f, E, \eta, g)$. If $\bar{A}$ is parallel and has a constant rank $2p + 1 (1 \leqq p \leqq n - 1)$ on M, then the manifold M is locally a product of a Sasakian manifold of dimension $2p + 1$ and a Kähler manifold of dimension $2q$, $n = p + q$.*

PROOF. First we put $\bar{A}^* = I - \bar{A}$. Then the tensor fields $\bar{A}$, $\bar{A}^*$ determine a $(2p + 1)$-dimensional and a $2q$-dimensional distributions $\mathscr{D}(\bar{A})$ and $\mathscr{D}(\bar{A}^*)$, which are complementary. If we put $\psi = \bar{A} - \bar{A}^*$ again, we see that ψ defines an almost product structure, $\psi^2 = I$, satisfying $g(\psi X, \psi Y) = g(X, Y)$ for any X and Y. Since the tensor field ψ is parallel on M, $\mathscr{D}(\bar{A})$ and $\mathscr{D}(\bar{A}^*)$ are both completely integrable ([3], [14], [15]). Denoting the maximal integral manifolds through a point of M corresponding to $\mathscr{D}(\bar{A})$ and $\mathscr{D}(\bar{A}^*)$ by N_1 and N_2 respectively, we find a Sasakian structure on N_1 and a Kähler structure on

N_2 from the given quasi-Sasakian structure. In practice, we denote by f_i, $i = 1, 2$, and g_i the restrictions of f and g to N_i respectively, and use the same symbol ∇ as the induced Riemannian connections with respect to g_i on N_i. Since for any vector fields X and Y belonging to $\mathscr{D}(\bar{A})$ the conditions $(\nabla_X f_1)Y = \eta(Y)X - g(X, Y)E$, $g_1(f_1X, Y) = -g_1(X, f_1Y)$, $E \in \mathscr{D}(\bar{A})$ hold on N_1, and for any vector fields X and Y belonging to $\mathscr{D}(\bar{A}^*)$ the conditions $(\nabla_X f_2)Y = 0$, $g_2(f_2X, Y) = -g_2(X, f_2Y)$ hold on N_2, then the sets of tensor fields (f_1, E, η, g_1) on N_1 and (f_2, g_2) on N_2 define the desired structures. q.e.d.

4. **Hypersurfaces in Kähler manifolds.** Let $P(J, G)$ be a Kähler manifold of dimension $2n+2$, and let N be a $(2n+1)$-dimensional manifold imbedded in P with imbedding map $i: N \to P$. With identification in mind we express the hypersurface $i(N)$ by N. Take a unit normal vactor field ζ over the hypersurface N. Then we have the relations $Ji_*X = i_*fX + \eta(X)\zeta$, $J\zeta = -i_*E$ for any X on N, where f is a linear transformation field, η is a 1-form, E is a vector field, and i_* denotes the differential of i. We denote the induced metric of G by g, $G(i_*X, i_*Y) = g(X, Y)$. The equations of Gauss and Weingarten are $\nabla_{i_*X} i_*Y = i_*\nabla_X Y + h(X, Y)\zeta$ $(h(X, Y) = h(Y, X))$ and $\nabla_{i_*X}\zeta = -i_*HX$ $(g(HX, Y) = h(X, Y))$, where h and H are the second fundamental tensor fields (of type $(0, 2)$ and $(1, 1)$ respectively) on N with respect to ζ. Then, since $(\nabla_X f)Y = \eta(Y)HX - g(HX, Y)E$, a set (f, E, η, g) of the induced tensor fields defines an almost contact metric structure satisfying $dF = 0$. *A necessary and sufficient condition for the induced structure* (f, E, η, g) *on* N *in a Kähler manifold* $P(J, G)$ *to be quasi-Sasakian is that* H *commutes with* f. In fact, if H commutes with f, since $S(X, Y) = \eta(X)(fH - Hf)Y - \eta(Y)(fH - Hf)X$, then $S = 0$. Conversely, if we have $S = 0$, then, by $dF = 0$, $\nabla_X E = fHX$ and E is a Killing vector field. Hence, we obtain $Hf = fH$. When we specialize the structure (f, E, η, g) in two cases, we have that the set (f, E, η, g) defines a cosymplectic structure if and only if $H = \lambda\eta \otimes E$ for any scalar field λ, and that the set (f, E, η, g) defines a Sasakian structure if and only if $H = I + \lambda\eta \otimes E$ for any scalar field λ.

Consider hypersurfaces in a Kähler manifold $P_0(J, G)$ of constant holomorphic sectional curvature $\tilde{c}$. It is shown that the complex projective space $P^n(C)$, the complex n-space C^n and the open unit ball D^n in C^n are simply-connected complete Kähler manifolds of constant holomorphic sectional curvature according as to be positive, zero and negative ([7], [11]). Denoting the curvature tensor fields on P_0 and on N by $\tilde{R}$ and R respectively, we have

$$\tilde{R}(X, Y, Z, W) = \frac{\tilde{c}}{4}\{G(X, Z)G(Y, W) - G(X, W)G(Y, Z) + \Omega(X, Z)\Omega(Y, W) - \Omega(X, W)\Omega(Y, Z) + 2\Omega(X, Y)\Omega(Z, W)\}$$

for any vector fields X, Y, Z and W on P_0. It follows that for any vector fields X, Y, Z and W on N

$$\tilde{R}(i_*X, i_*Y, i_*Z, i_*W) = \frac{c}{4}\{g(X, Z)g(Y, W) - g(X, W)g(Y, Z) + F(X, Z)F(Y, W) - F(X, W)F(Y, Z) + 2F(X, Y)F(Z, W)\},$$

where c denotes the restriction of $\tilde{c}$ to N. By the equations of Gauss and Codazzi we obtain

$$R(X, Y, Z, W) = h(X, Z)h(Y, W) - h(X, W)h(Y, Z) + \frac{c}{4}\{g(X, Z)g(Y, W) - g(X, W)g(Y, Z) + F(X, Z)F(Y, W) - F(X, W)F(Y, Z) + 2F(X, Y)F(Z, W)\},$$

$$(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) = \frac{c}{4}\{\eta(X)F(Y, Z) - \eta(Y)F(X, Z) - 2\eta(Z)F(X, Y)\}.$$

Let $\bar{A}$ be a structure indicator tensor field on a hypersurface N in P_0. Since the tensor field H is written as $H = \bar{A} + a\eta \otimes E$ for a scalar field a on N, a totally umbilic ($H = I$) hypersurface N imbedded in P_0 imposes the induced structure (f, E, η, g) to be Sasakian and N to be of constant curvature 1, and a totally geodesic hypersurface N imbedded in P_0 imposes the structure (f, E, η, g) to be cosymplectic and N to be flat.

Making a comparision with this fact, we have the following result.

PROPOSITION 2. *On an induced quasi-Sasakian hypersurface $N(f, E, \eta, g)$ in a Kähler manifold $P_0(J, G)$ of constant holomorphic sectional curvature $\tilde{c}$ if the structure indicator tensor field $\bar{A}$ is parallel, then the second fundamental tensor field H must have the form $H = \bar{A} - (c/4)\eta \otimes E$ or $H = \lambda\eta \otimes E$ for any scalar field λ, hence the curvature tensor field R is determined by the structure tensor fields.*

PROOF. Differentiating covariantly $H = \bar{A} + a\eta \otimes E$ with respect to X, we have $(\nabla_X h)(Y, Z) = \eta(Y)\eta(Z)Xa + a\{\eta(Y)(\nabla_X\eta)(Z) + \eta(Z)(\nabla_X\eta)(Y)\}$. It follows that

$$\frac{c}{4}\{\eta(X)F(Y, Z) - \eta(Y)F(X, Z) - 2\eta(Z)F(X, Y)\}$$
$$= \eta(Y)\eta(Z)Xa - \eta(X)\eta(Z)Ya + a[\eta(Y)(\nabla_X\eta)(Z) - \eta(X)(\nabla_Y\eta)(Z)$$
$$+ \eta(Z)\{(\nabla_X\eta)(Y) - (\nabla_Y\eta)(X)\}] .$$

Putting $Y = Z = E$ in this equation, we find $Xa = \eta(X)Ea$, which is used to see that the terms $\eta(Y)\eta(Z)Xa - \eta(X)\eta(Z)Ya$ vanish. Putting $Z = E$ again in the above, hence, we have $af\bar{A} + (c/4)f = 0$. Taking account of $\bar{A}^2 = \bar{A}$, we obtain $(a + c/4)f\bar{A} = 0$. q.e.d.

COROLLARY 3. *On a hypersurface N in a Kähler manifold $P_0(J, G)$ of constant holomorphic sectional curvature $\tilde{c}$ the set (f, E, η, g) of the induced tensor fields defines a Sasakian structure if and only if H has the form $H = I - (c/4)\eta \otimes E$.*

REFERENCES

[1] D. E. BLAIR, The theory of quasi-Sasakian structures, J. Differential Geometry, 1 (1967), 331-345.
[2] S. S. CHERN, Einstein hypersurfaces in a Kählerian manifold of constant holomorphic curvature, J. Differential Geometry, 1 (1967), 21-31.
[3] C. CHEVALLEY, Theory of Lie Groups, Princeton Univ. Press, 1946.
[4] S. I. GOLDBERG and K. YANO, Noninvariant hypersurfaces of almost contact manifolds, J. Math. Soc. Japan, 22 (1970), 25-34.
[5] J. W. GRAY, Some global properties of contact structures, Ann. of Math., 69 (1959), 421-450.
[6] S. KANEMAKI, Products of f-manifolds, TRU Math., 10 (1974), 11-17.
[7] S. KOBAYASHI AND K. NOMIZU, Foundations of Differential Geometry, II, Wiley (Interscience), New York, 1969.
[8] S. SASAKI, On differentiable manifolds with certain structures which are closely related to almost contact structure I, Tôhoku Math. J., 12 (1960), 459-476.
[9] S. SASAKI AND Y. HATAKEYAMA, On differentiable manifolds with certain structures which are closely related to almost contact structure II, ibid, 13 (1961), 281-294.
[10] S. SASAKI AND Y. HATAKEYAMA, On differentiable manifolds with contact metric structures, J. Math. Soc. Japan, 14 (1962), 249-271.
[11] B. SMYTH, Differential geometry of complex hypersurfaces, Ann. of Math., 85 (1967), 246-266.
[12] S. TANNO, Quasi-Sasakian structures of rank $2p+1$, J. Differential Geometry, 5 (1971), 317-324.
[13] Y. TASHIRO, On contact structure of hypersurfaces in complex manifolds I, Tôhoku Math. J., 15 (1963), 62-78.
[14] A. G. WALKER, Connexions for parallel distributions in the large, Quart. J. Math. (Oxford) (2) 6 (1955), 301-308; II, 9 (1958), 221-231.
[15] K. YANO, Differential Geometry on Complex and Almost Complex Spaces, Pergamon Press, New York, 1965.

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