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ON UNIQUENESS IN CAUCHY'S PROBLEM FOR ELLIPTIC PARTIAL DIFFERENTIAL OPERATORS WITH CHARACTERISTICS OF MULTIPLICITY GREATER THAN TWO

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Introduction. The question of uniqueness in Cauchy's problem for elliptic partial differential operators has been reduced to the proof of certain integral estimates of Carleman type, *viz.*,

$$\sigma \| B(x, D)u(x) \|^2 \leqq C \| A(x, D)u(x) \|^2 \quad \forall u \in C_0^\infty\{x \in \boldsymbol{R}^n : |x| < \delta\}$$

where the norm is a weighted norm depending on the small parameter δ and a large parameter τ. If $\sigma \to \infty$ as $\tau \to \infty$, and C is a constant independent of τ and δ, then such an estimate can be shown (Theorem 4.1) to be incompatible with the assumption that there is a solution $v(x)$ of the differential inequality

$$|A(x, D)v(x)| \leqq C |B(x, D)v(x)|$$

and an $\varepsilon > 0$ such that $v \equiv 0$ for $x_1 \leqq \varepsilon \sum_{j=2}^n x_j^2$ unless there is a full neighborhood of $x = 0$ on which $v \equiv 0$. Examples of such estimates can be found in Hörmander [13, 14, 15], Goorgian [9], Pederson [23, 24, 25], and Watanabe [32], among many others.

It is the purpose of this paper to combine the techniques in Pederson [25] and Watanabe [32] to show that uniqueness in Cauchy's problem is a consequence of a smoothness assumption similar to Pederson's when the characteristics have multiplicity greater than two and not all lower order terms are included (*cf.*, Theorem 2.2). In the case of triple characteristics, we give sufficient conditions on the principal part of the operator which, together with the smoothness of the roots implies uniqueness in Cauchy's problem for equations with arbitrary lower order terms whose coefficients are Lipschitz continuous. This additional assumption is, essentially, that the characteristics are either of constant multiplicity of that the gradients of the roots are linearly independent whenever the roots

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coalesce. When the operator can be represented as a product of operators with simple characteristics

$$A(x, \zeta) = A_1(x, \zeta)A_2(x, \zeta)A_3(x, \zeta)$$

then the smoothness assumptions are always satisfied and the additional assumption amounts to saying that whenever some of the distinct factors of A vanish simultaneously, then their gradients are linearly independent.

1. Notation and Preliminary Lemmas. We shall use the following notation throughout this paper. We use $x=(x_1, \cdots, x_n) \in \boldsymbol{R}^n$ (real Euclidean n-space) to denote the independent variables in the problem and $\xi = (\xi_1, \cdots, \xi_n) = (\xi_1, \xi') \in \boldsymbol{R}^n$ to denote the co-variables. When the co-variables are allowed to have complex values, we denote them by $\zeta = (\zeta_1, \cdots, \zeta_n) = (\zeta_1, \zeta') \in \boldsymbol{C}^n$. We denote $D_k = (1/i)(\partial/\partial x_k)$ and $D^k = (1/i)(\partial/\partial \xi_k)$. We depart from custom and use the classical multi-index notation whereby a multi-index α is a set of integers from 1 to n (or, in some cases, m) $\alpha = (\alpha_1, \cdots, \alpha_k)$ and the number of elements will be called the length of α and written $|\alpha| = k$. The multi-index α^* will be obtained from α by omitting all entries α_j with $\alpha_j = 1$. With this convention,

$$\xi^\alpha = \xi_{\alpha_1}\xi_{\alpha_2} \cdots \xi_{\alpha_{|\alpha|}},\ D^\alpha = \frac{(i)^{-|\alpha|}\partial^{|\alpha|}}{\partial\xi_{\alpha_1}\partial\xi_{\alpha_2}\cdots\partial\xi_{\alpha_{|\alpha|}}}$$

and similarly for D_α. If $A(x, \xi)$ is a polynomial in ξ whose coefficients depend on x, write $A^{(\alpha)}(x, \xi) = D^\alpha A(x, \xi)$, $A_{(\alpha)}(x, \xi) = D_\alpha A(x, \xi)$, and write $A(x, D)$ for the operator obtained from A by replacing ξ^α with D_α. If at a point $A(x, \xi)$ has a factorization

$$A(x, \xi) = \prod_{j=1}^{J} (\xi_1 - \rho_j(x, \xi'))^{r_j}$$

where the r_j are integers summing to m and if α is a multi-index with no entry j repeated more than r_j times then we denote the Lagrange interpolation polynomials by

$${}_\alpha A(x, \xi) = \frac{A(x, \xi)}{\prod_{j=1}^{|\alpha|} (\xi_1 - \rho_{\alpha_j})}\,.$$

We use the symbol $\|\cdot\|$ to denote a weighted L_2 norm

$$\|u\|^2 = \int_{\boldsymbol{R}^n} |u(x)|^2 e^{2\tau\varphi_p(x)}dx$$

where the parameters p, τ, δ and the function φ_p are related by

$$\varphi_p(x) = (x_1 - \delta)^2 + \delta^p \sum_{j=2}^{n} (x_j)^2 \quad \delta > 0,\ \tau \geqq 0,\ p > 0$$

and are not represented explicitly. Throughout the paper we use V to denote an open cone in $\boldsymbol{R}^n$ with vertex at the origin which contains the vector $N_0 = (-1, 0, \cdots, 0)$. (An open cone is a set $V = \{tV_0 : t > 0\}$ where V_0 is an open simply connected set in $\boldsymbol{R}^n$.) For any such open cone we write

$$E(V) = \{\zeta \in \boldsymbol{C}^n : \zeta = \xi + i\tau N,\ N \in V,\ \tau \in \boldsymbol{R},\ \xi \in \boldsymbol{R}^n\}$$
$$E'(V) = \{\zeta' \in \boldsymbol{C}^{n-1} : \zeta = (\zeta_1, \zeta') \in E(V)\} .$$

When we refer to the Fourier Transformation of $u(x)$ we mean the function $\hat{u}(\xi)$ given by

$$\hat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\boldsymbol{R}^n} e^{-i\xi\cdot x} u(x) dx$$

and the inverse transformation

$$u(x) = \frac{1}{(2\pi)^{n/2}} \int_{\boldsymbol{R}^n} e^{i\xi\cdot x} \hat{u}(\xi) d\xi .$$

We use the symbol $[t]$, $t \geqq 0$, to denote the integer ν such that $\nu \leqq t < \nu + 1$. Finally, we use the letter C to denote a positive constant which is not necessarily the same in different expressions, and which may depend on the various given operators and fixed functions but never on the 'variable' function u nor the parameters p, δ, τ, q, etc. which may appear.

We begin by stating, without proof, a result due to Trèves of which we will often make use. It appears in [30], p. 137.

LEMMA 1.1. *Suppose $A(\xi)$ is a polynomial in ξ, of degree m. Then*

$$\tau^{|\alpha|} \delta^{p|\alpha^*|} \| A^{(\alpha)}(D) u \|^2 \leqq 2^m m! \| A(D) u \|^2$$

for $\tau \geqq 0$, $\delta > 0$, $p > 0$, $\forall \alpha$, and $u \in C_0^\infty(\boldsymbol{R}^n)$.

We shall have several occasions to use the partition of unity given by the following lemma.

LEMMA 1.2. *Suppose $k_1, \cdots, k_n$ are n positive constants. Then if $g = (g_1, \cdots, g_n)$ denotes an n-tuple of integers, there is a partition of unity $\theta_g \in C^\infty(\boldsymbol{R}^n)$ and a set of points $x_g \in \boldsymbol{R}^n$ such that*

$$\sum_g \theta_g(x) \equiv 1 \quad x \in \boldsymbol{R}^n ,$$

the support of θ_g is contained in the parallelepiped

$$|x_j - x_{g,j}| \leqq \frac{1}{k_j} \quad j = 1, \cdots, n$$

and for a given m there is a constant C independent of k_j, $j = 1, \cdots, n$, *and g so that*

$$|D_\alpha \theta_g| \leqq Ck^\alpha \quad \forall |\alpha| \leqq m .$$

Furthermore, no more than 2^n *of the functions* θ_g *are non-zero at a given point.*

Proof. Suppose $\theta_0 \in C^\infty(\boldsymbol{R}^n)$ is a non-negative function supported in the cube $I = \{|x_j| \leqq 1/2, j = 1, \cdots, n\}$, and assume that

$$\int_{\boldsymbol{R}^n} \theta_0(x)dx = 1 .$$

The function $\theta(x)$ defined by

$$\theta(x) = \int_I \theta_0(x - y)dy$$

is in C^∞ and is supported in the cube $\{|x_j| \leqq 1, j = 1, \cdots, n\}$. It is evident that

$$\sum_g \theta(x - g) = \int_{\boldsymbol{R}^n} \theta_0(x)dx = 1 ,$$

that there is a common bound, C, for all derivatives of order no greater than m, and that no more than 2^n of the functions $\theta(x - g)$ are non-zero at a point. We now take

$$x_g = (g_1/k_1, g_2/k_2, \cdots, g_n/k_n)$$

and

$$\theta_g(x) = \theta(k_1x_1 - g_1, k_2x_2 - g_2, \cdots, k_nx_n - g_n) .$$

Then the support of θ_g is centered at x_g and is contained in the parallelepiped

$$|x_j - x_{g,j}| \leqq \frac{1}{k_j} \quad j = 1, \cdots, n ,$$

the θ_g form a partition of unity, and the derivatives of order no greater than m are bounded by

$$|D_\alpha \theta_g| \leqq Ck^\alpha \quad |\alpha| \leqq m .$$

This completes the proof of Lemma 1.2.

The following lemma gives an example of a norm inequality which is implied by the corresponding pointwise inequality for the integrand.

Its proof closely follows the one introduced by Hörmander [14] and does not need to be repeated here. Examples of its proof for various values of p may also be found in Pederson [25], Goorjian [9], and Watanabe [32].

LEMMA 1.3. *Suppose* $Q(\zeta)$ *and* $P_j(\zeta)$, $j = 1, \cdots, J$, $\zeta \in \boldsymbol{C}^n$ *are polynomials with the property that there are non-negative integers* $\nu_0, \nu_1, \cdots, \nu_J$, *non-negative constants* C, τ_0, *and an open cone* $V \subset \boldsymbol{R}^n$ *with* $N_0 = (-1, 0, \cdots, 0) \in V$ *such that*

$$|\tau N|^{\nu_0} |Q(\zeta)|^2 \leqq C \sum_{j=1}^{J} |\tau N|^{\nu_j} |P_j(\zeta)|^2 \tag{1.1}$$

for $\zeta = \xi + i\tau N \in E(V)$ *and* $\tau \geqq \tau_0$. *Then for any choice of* $0 < p$ *there are constants* δ_0 *and* τ_1, *with* $\delta_0 > 0$, $\tau_1 \geqq \tau_0$, *so that*

$$(\tau\delta)^{\nu_0} \| Q(D)u \|^2 \leqq C \sum_{j=1}^{J} (\tau\delta)^{\nu_j} \| P_j(D)u \|^2 \tag{1.2}$$

for $0 < \delta \leqq \delta_0$, $\tau \geqq \tau_1/\delta^p$, $u \in C_0^\infty(|x| < \delta/2)$.

2. Some estimates leading to uniqueness theorems. The results of this paper all deal with a very special kind of operator.

DEFINITION. A homogeneous operator $A(D)$ is said to be *elliptic* when its polynomial $A(\zeta)$ has no zeros for $\zeta \in \boldsymbol{R}^n$. An elliptic operator is said to have *characteristics of multiplicity at most* $r \geqq 1$ if all the polynomials $(\partial^k A(\zeta)/\partial\zeta_1^k)$ $k = 0, 1, \cdots, r$ have no common zero for $\zeta' \in \boldsymbol{R}^{n-1}$ (when considered as polynomials in ζ_1), when $r = 1$ we say the characteristics are *simple*. Since the coefficient of ζ_1^m in $A(\zeta)$ must be bounded away from zero for A to be elliptic, we assume that all elliptic operators of degree m are normalized so that the coefficient of ζ_1^m in $A(\zeta)$ is identically one.

With this notion, we can state the next lemma.

LEMMA 2.1. *Suppose* $A(\zeta)$, $\zeta \in \boldsymbol{C}^n$, *is a homogeneous polynomial of degree* m *with characteristics of multiplicity no greater than* r. *Then there is an open cone* $V \subset \boldsymbol{R}^n$ *so that*

$$|\zeta|^{2m} \leqq C \sum_{k=0}^{r} |\tau N|^{2k} \Big|\frac{\partial^k A}{\partial \zeta_1^k}(\zeta)\Big|^2 \quad \zeta = \xi + i\tau N \in E(V) . \tag{2.1}$$

PROOF. When considered as functions of $(\xi, \tau) \in \boldsymbol{R}^{n+1}$, both sides are homogeneous of degree $2m$, and hence we need only show (2.1) for the set $|\xi|^2 + \tau^2 = 1$. Take $N = N_0$ first. In this case, neither side of (2.1) is ever zero, by the assumption on the characteristics, and hence a constant exists. By continuity, (2.1) will also hold for N in some neighborhood of N_0. Homogeneity then assures that (2.1) holds for N near N_0 and all $(\xi, \tau) \in \boldsymbol{R}^{n+1}$. Since the vector N appears only in the combination (τN), (2.1) must hold for a whole cone $V \subset \boldsymbol{R}^n$. This completes the proof of Lemma 2.1.

The next lemma is a consequence of a very simple calculation.

LEMMA 2.2.

$$|\tau N|^{2m-2|\alpha|}|\zeta^\alpha|^2 \leqq |\zeta|^{2m} \quad \zeta \in E(\boldsymbol{R}^n)\, |\alpha| \leqq m\ .$$

PROOF. We first note that $|\zeta^\alpha|^2 \leqq |\zeta|^{2|\alpha|}$ and then that $|\zeta|^2 = |\xi|^2 + |\tau N|^2 \geqq |\tau N|^2$. The result follows.

The norm estimate corresponding to Lemma 2.2 will prove very useful in the work that follows.

COROLLARY. *There are constants* $p > 0,\ 0 < \delta_0,\ 0 < \tau_0$ *so that*

$$(\tau\delta)^{2m-2|\alpha|}\|D_\alpha u\|^2 \leqq C \sum_{|\beta|=m} \|D_\beta u\|^2$$

$$\forall 0 < \delta \leqq \delta_0,\ \tau \geqq \tau_0/\delta^2,\ u \in C_0^\infty(|x| \leqq \delta/2)\ .$$

We are now in a position to state an important result due to Goorjian, giving weighted norm estimates of the derivatives of u in terms of $A(D)u$. Its proof provides model a for that of Theorem 2.2.

THEOREM 2.1. *Suppose $A(\zeta)$ is s polynomial in $\zeta \in \boldsymbol{C}^n$ of degree m whose elliptic principal part has characteristics of multiplicity no greater than r. Then there are constants* $0 < p < 2,\ 0 < \delta_0 < 1,\ \tau_0 > 0$, *such that*

$$\sum_{|\alpha|\leqq m} \tau^{m-|\alpha|}(\tau\delta^2)^{m-|\alpha|-r}\|D_\alpha u\|^2 \leqq C\|A(D)u\|^2 \tag{2.2}$$

$$\forall 0 < \delta \leqq \delta_0,\ \tau \geqq \tau_0/\delta^2\ ,\ \textit{and}\ u \in C_0^\infty(|x| < \delta/2)\ .$$

PROOF. To simplify the notation somewhat, we first assume that $|\alpha| = m$ and prove the estimate

$$\sum_{|\alpha|=m} (\tau\delta^2)^{-r}\|D_\alpha u\|^2 \leqq C\|A(D)u\|^2$$

$$\forall 0 < \delta \leqq \delta_0,\ \tau \geqq \tau_0/\delta^2\ ,\ \text{and}\ u \in C_0^\infty(|x| < \delta/2)\ .$$

Applying Lemma 1.3 to the inequality (2.1) in Lemma (2.1), we see that for $0 < p < 2$ there are constants $\delta_0 > 0$ and $\tau_0 > 0$ such that

$$\sum_{|\alpha|=m} \|D_\alpha u\|^2 \leqq C \sum_{k=0}^{r} (\tau\delta)^{2k}\|P^{(_k\beta)}(D)u\|^2 \quad \forall u \in C_0^\infty(|x| < \delta/2) \tag{2.3}$$

where $_k\beta = (1, 1, \cdots, 1)$ and $|_k\beta| = k$, and P denotes the principal part of A. If we use Q to denote the terms of A with order less than m, then we can write

$$\sum_{|\alpha|=m} \|D_\alpha u\|^2 \leqq C\sum_{k=0}^{r} (\tau\delta)^{2k} \|A^{(k\beta)}(D)u\|^2 + C\sum_{k=0}^{r} (\tau\delta)^{2k} \|Q^{(k\beta)}(D)u\|^2 . \tag{2.4}$$

Since Q has order less than m, we have from the corollary of Lemma 2.2

$$\|Q^{(k\beta)}(D)u\|^2 \leqq C \sum_{|\alpha|=m-1-k} \|D_\alpha u\|^2 \leqq C(\tau\delta)^{-2-2k} \sum_{|\alpha|=m} \|D_\alpha u\|^2 ,$$

and (2.4) becomes

$$\sum_{|\alpha|=m} \|D_\alpha u\|^2 \leqq C \sum_{k=0}^{r} (\tau\delta)^{2k} \|A^{(k\beta)}(D)u\|^2 + C(\tau\delta)^{-2} \sum_{|\alpha|=m} \|D_\alpha u\|^2 .$$

Now, we are interested only in those values of τ where $(\tau\delta^2) \geqq \tau_0$, and $\tau\delta > \tau\delta^2$ since $\delta \leqq \delta_0 < 1$. Choose τ_0 so large that $C(\tau\delta)^{-2} \leqq C\tau_0^{-2} \leqq 1/2$, so we can write

$$\sum_{|\alpha|=m} \|D_\alpha u\|^2 \leqq C\sum_{k=0}^{r} (\tau\delta)^{2k} \|A^{(k\beta)}(D)u\|^2 . \tag{2.5}$$

Lastly, we apply Lemma 1.1 to the right side of (2.5), and recall that $|{}_k\beta^*| = 0$, to get

$$\sum_{|\alpha|=m} \|D_\alpha u\|^2 \leqq C\sum_{k=0}^{r} (\tau\delta^2)^{k} \|A(D)u\|^2 ,$$

or, by taking $\tau_0 \geqq 1$,

$$(\tau\delta^2)^{-r} \sum_{|\alpha|=m} \|D_\alpha u\|^2 \leqq C\|A(D)u\|^2 .$$

The result now follows from the inequality

$$(\tau\delta)^{2m-|\beta|}\|D_\beta u\|^2 \leqq C \sum_{|\alpha|=m} \|D_\alpha u\|^2 \qquad |\beta| \leqq m \tag{2.6}$$

which is from the corollary to Lemma 2.2.

REMARK. This estimate can be used to prove uniqueness in Cauchy's problem for the inequality

$$|A(D)u| \leqq C \sum_{|\alpha| \leqq m-[r+1/2]} |D_\alpha u|$$

where A may include terms of order less than m. All of the coefficients of A, however, must be constant.

We now show that the assumption of a certain pointwise inequality involving the principal part of an operator is sufficient to prove an estimate analogous to (2.2) but including a large class of operators with variable coefficients. In the next section we give sufficient conditions for this pointwise inequality to be true.

THEOREM 2.2. *Suppose* $A(x, \zeta) = P(x, \zeta) + Q(x, \zeta)$ *where* P *is a homogeneous elliptic polynomial of degree* m *in* $\zeta \in C^n$ *with characteristics* *of multiplicity at most r, r odd, and whose coefficients are C^r, and Q is a polynomial in $\zeta \in \boldsymbol{C}^n$ of degree at most $(m - [(r+1)/2] + 1)$ with Lipschitz continuous coefficients. Assume there is a neighborhood Ω of the origin in $\boldsymbol{R}^n$ and an open cone $V \subset \boldsymbol{R}^n$ containing N_0 such that*

$$|P_{(\alpha)}(x, \zeta)|^2 \leqq C \sum_{0 \leqq |\beta| \leqq |\alpha|} |\tau N|^{2|\beta|} |P^{(\beta)}(x, \zeta)|^2 \tag{2.7}$$

for $1 \leqq |\alpha| \leqq (r-1)$, $x \in \Omega$, $\zeta \in E(V)$. Then there are constants $0 < p < 2$, $0 < \delta_0 < 1$, and $0 \leqq \tau_0$ so that

$$\sum_{|\alpha| \leqq m} (\tau\delta^2)^{m-|\alpha|-r} \tau^{m-|\alpha|} \| D_\alpha u \|^2 \leqq C \| A(x, D)u \|^2 \tag{2.8}$$

for all $0 < \delta \leqq \delta_0$, $\tau \geqq \tau_0/\delta^2$, $u \in C_0^\infty(|x| < \delta/2)$.

REMARK. If P has at most triple characteristics, so that $r = 3$, then Q may have degree at most $(m-1)$. In particular, operators of the form

$$P + Q + ((m-2) \text{ and lower order terms})$$

will have unique solutions of Cauchy's problem as a consequence of Theorem 4.1.

Applying Lemma 1.3 to the inequality (2.7) we find that there are constants $0 < p < 2$, $0 < \delta_0 < 1$, $0 \leqq \tau_0$ so that

$$\| P_{(\alpha)}(x_0, D)u \|^2 \leqq C \sum_{0 \leqq |\beta| \leqq |\alpha|} (\tau\delta)^{2|\beta|} \| P^{(\beta)}(x_0, D)u \|^2 \tag{2.9}$$

for $0 \leqq \delta \leqq \delta_0$, $\tau \geqq \tau_0/\delta^2 \geqq \tau_0/\delta^{2-p}$, $x_0 \in \Omega$, and $u \in C_0^\infty(|x| < \delta/2)$. In order to replace $P^{(\beta)}$ by $A^{(\beta)}$ on the right side, we use the same device as was used to conclude (2.5) from (2.3). We then have

$$\begin{aligned} \| P_{(\alpha)}(x_0, D)u \|^2 \leqq\; & C \sum_{0 \leqq |\beta| \leqq |\alpha|} (\tau\delta)^{2|\beta|} \| A^{(\beta)}(x_0, D)u \|^2 \\ & + C \sum_{0 \leqq |\beta| \leqq |\alpha|} (\tau\delta)^{2|\beta|} \| Q^{(\beta)}(x_0, D)u \|^2 . \end{aligned} \tag{2.10}$$

Since Q is of degree at most $(m - [(r+1)/2] + 1)$ and since its coefficients are bounded in Ω, we have

$$\| P_{(\alpha)}(x_0, D)u \|^2 \leqq C \sum_{0 \leqq |\beta| \leqq |\alpha|} (\tau\delta)^{2|\beta|} \| A^{(\beta)}(x_0, D)u \|^2 + C \sum_{|\beta| = m - [(r+1)/2] + 1} \| D_\beta u \|^2 .$$

Because $\delta^{|\beta|} \leqq \delta^{|\beta^*|}$ and r is odd we have

$$\| P_{(\alpha)}(x_0, D)u \|^2 \leqq C(\tau\delta^{2-p})^{|\alpha|} \| A(x_0, D)u \|^2 + C \sum_{|\beta| = m} (\tau\delta)^{-r+1} \| D_\beta u \|^2 ,$$

where we have assumed $\tau\delta^{2-p} \geqq \tau\delta^2 \geqq \tau_0 \geqq 1$. As a consequence of Theorem 2.1 we have

$$\| P_{(\alpha)}(x_0, D)u \|^2 \leqq C(\tau\delta^{2-p})^{|\alpha|} \| A(x_0, D)u \|^2 + C(\tau\delta)\delta^r \| A(x_0, D)u \|^2$$

and since $|\alpha| \geqq 1$ and $r \geqq 1$,

$$\| P_{(\alpha)}(x_0, D)u \|^2 \leqq C(\tau\delta^{2-p})^{|\alpha|} \| A(x_0, D)u \|^2 . \tag{2.11}$$

It is interesting to note here that Watanabe's [32] proof of our Theorem 2.2 for $P(x, D) = (P_1(x, D))^3$ where P_1 has simple characteristics is essentially similar to the one offered here, differing mainly in that he replaced (2.11) with his Lemma 4, which is a stronger result than (2.11) for that case and whose proof is different.

We now show how the theorem follows from (2.11). We use the partition of unity given in Lemma 1.2 with $k_1 = k_2 = \cdots = k_n = (\tau\delta^q)^{-1/2}$, where $p < q < 2 - p$. Recall that this means that

$$\theta_g(x) = \theta((\tau\delta^q)^{1/2}x - g) ,$$
$$x_g = g/(\tau\delta^q)^{1/2} ,$$

and

$$u_g(x) = \theta_g(x)u(x) .$$

Now choose $u \in C_0^\infty(|x| < \delta/2)$ where $\delta \leqq \delta_0$ and δ_0 will be specified later. Applying Lemma 1.1, Theorem 2.1, and estimate (2.11) to the operator $A(x_g, D)$, whose coefficients are 'frozen' at x_g, and to the function u_g, we find that

$$\begin{aligned} \sum_{|\alpha| \leqq m} (\tau\delta^2)^{m-|\alpha|-r}\tau^{m-|\alpha|} \| D_\alpha u_g \|^2 + \sum_{1 \leqq |\alpha|} \tau^{|\alpha|}\delta^{p|\alpha^*|} \\ \times \| A^{(\alpha)}(x_g, D)u_g \|^2 + \sum_{1 \leqq |\alpha| \leqq r-1} (\tau\delta^{2-p})^{-|\alpha|} \| P_{(\alpha)}(x_g, D)u_g \|^2 \\ \leqq C \| A(x_g, D)u_g \|^2 . \end{aligned} \tag{2.12}$$

Applying Taylor's theorem to the principal part, P, of A gives

$$\begin{aligned} A(x, \xi) - A(x_g, \xi) = Q(x, \xi) - Q(x_g, \xi) + \sum_{1 \leqq |\alpha| \leqq r-1} \frac{(x - x_g)^\alpha}{|\alpha|!} P_{(\alpha)}(x_g, \xi) \\ + \frac{1}{r!} \sum_{|\alpha| = r} (x - x_g)^\alpha P_{(\alpha)}(\bar{x}, \xi) \end{aligned}$$

where $\bar{x}$ is a point between x and x_g. Hence,

$$\begin{aligned} \| A(x, D)u_g - A(x_g, D)u_g \|^2 \leqq C(\tau\delta^q)^{-1} \sum_{|\alpha| = m-[(r+1)/2]+1} \| D_\alpha u_g \|^2 \\ + C \sum_{1 \leqq |\alpha| \leqq r-1} (\tau\delta^q)^{-|\alpha|} \| P_{(\alpha)}(x_g, D)u_g \|^2 \\ + C \sum_{|\alpha| = m} (\tau\delta^q)^{-r} \| D_\alpha u_g \|^2 . \end{aligned} \tag{2.13}$$

Comparing the first sum on the right side of (2.13) with the terms of order $|\alpha| = m - [(r + 1)/2] + 1$ of the first sum on the left of (2.12) gives

$$(\tau\delta^2)^{m-(m-[(r+1)/2]+1)-r}\tau^{m-(m-[(r+1)/2]+1)} + C(\tau\delta^q)^{-1} = (\tau\delta^2)^{-1}(\delta^{-(r-1)} - C\delta^{2-q}) .$$

Since $q < 2$, choosing δ_0 so small that

$$C\delta_0^{2-q} \leqq 1/2$$

will allow us to combine these terms. Similarly, comparing the third sum on the right of (2.13) with the terms of order $|\alpha| = m$ in the first sum on the left of (2.12) gives

$$(\tau\delta^2)^{-r} - C(\tau\delta^q)^{-r} = (\tau\delta^2)^{-r}(1 - C\delta^{r(2-q)})$$

and choosing δ_0 so small that

$$C\delta^{r(2-q)} \leqq C\delta_0^{(2-q)} \leqq 1/2$$

allows us to combine these terms. Comparing the middle sum on the right of (2.13) with the third sum on the left of (2.12) gives

$$(\tau\delta^{2-p})^{-|\alpha|} - C(\tau\delta^q)^{-|\alpha|} = (\tau\delta^{2-p})^{-|\alpha|}(1 - C\delta^{|\alpha|(2-q-p)})$$

and choosing δ_0 so small that

$$C\delta^{|\alpha|(2-q-p)} \leqq C\delta_0^{2-q-p} \leqq 1/2$$

allows us to combine these terms. We therefore have

$$\sum_{|\alpha| \leqq m} (\tau\delta^2)^{m-|\alpha|-r}\tau^{m-|\alpha|}\|D_\alpha u_g\|^2 + \sum_{1 \leqq |\alpha|} \tau^{|\alpha|}\delta^{p|\alpha^*|}\|A^{(\alpha)}(x_g, D)u_g\|^2 + \sum_{1 \leqq |\alpha| \leqq r-1} (\tau\delta^{2-p})^{-|\alpha|}\|P_{(\alpha)}(x_g, D)u_g\|^2 \leqq C\|A(x, D)u_g\|^2 . \tag{2.14}$$

We next consider the middle term on the left side of (2.14). For $1 \leqq |\alpha| \leqq r - 1$, Taylor's theorem gives

$$A^{(\alpha)}(x, \xi) - A^{(\alpha)}(x_g, \xi) = \sum_{1 \leqq |\beta| \leqq r-|\alpha|} \frac{(x - x_g)^\beta}{|\beta|!} P_{(\beta)}^{(\alpha)}(x_g, \xi) + \sum_{|\beta| = r-|\alpha|+1} \frac{(x - x_g)^\beta}{|\beta|!} P_{(\beta)}^{(\alpha)}(\bar{x}, \xi) + Q^{(\alpha)}(x, \xi) - Q^{(\alpha)}(x_g, \xi)$$

where $\bar{x}$ is between x and x_g, and hence

$$\tau^{|\alpha|}\delta^{p|\alpha^*|}\|A^{(\alpha)}(x, D)u_g - A^{(\alpha)}(x_g, D)u_g\|^2 \leqq C \sum_{1 \leqq |\beta| \leqq r-|\alpha|} (\tau\delta^q)^{-|\beta|} \times \tau^{|\alpha|}\delta^{p|\alpha^*|}\|P_{(\beta)}^{(\alpha)}(x_g, D)u_g\|^2 + C \sum_{|\beta| = m-|\alpha|} (\tau\delta^q)^{-(r-|\alpha|+1)} \times \tau^{|\alpha|}\delta^{p|\alpha^*|}\|D_\beta u_g\|^2 + C(\tau\delta^q)^{-1}\tau^{|\alpha|}\delta^{p|\alpha^*|} \sum_{|\beta| = m-|\alpha|-[(r+1(/2]+1} \|D_\beta u_g\|^2 .$$

Applying Lemma 1.1 to the first term on the right gives

$$\tau^{|\alpha|}\delta^{p|\alpha^*|}\|A^{(\alpha)}(x, D)u_g - A^{(\alpha)}(x_g, D)u_g\|^2 \leqq C \sum_{1 \leqq |\beta| \leqq r-|\alpha|} (\tau\delta^q)^{-|\beta|} \times \|P_{(\beta)}(x_g, D)u_g\|^2 + C \sum_{|\beta| = m-|\alpha|} \tau^{2|\alpha|-r-1}\delta^{p|\alpha^*|-q(r-|\alpha|+1)} \times \|D_\beta u_g\|^2 + C(\tau\delta^q)^{-1}\tau^{|\alpha|}\delta^{p|\alpha^*|} \sum_{|\beta| = m-[(r+1)/2]+1-|\alpha|} \|D_\beta u_g\|^2 . \tag{2.15}$$

We now use (2.15) for $1 \leqq |\alpha| \leqq r - 1$, and the following consequence of the degree of A,

$$\sum_{|\alpha| \geqq r} \tau^{|\alpha|}\delta^{p|\alpha^*|}\|A^{(\alpha)}(x, D)u_g - A^{(\alpha)}(x_g, D)u_g\|^2 \leqq C \sum_{|\alpha| \geqq r} \tau^{|\alpha|}\delta^{p|\alpha^*|}\|D_{m-|\alpha|}u_g\|^2$$

for $|\alpha| \geqq r$, in combination with (2.14) to give

$$\sum_{|\alpha| \leqq m} (\tau\delta^2)^{m-|\alpha|-r}\tau^{m-|\alpha|}\|D_\alpha u_g\|^2 + \sum_{1 \leqq |\alpha|} \tau^{|\alpha|}\delta^{p|\alpha^*|} \times \|A^{(\alpha)}(x, D)u_g\|^2 \leqq C\|A(x, D)u_g\| \tag{2.16}$$

where we have simply dropped the terms involving $P_{(\alpha)}$ on the left side.

To replace u_g by u we sum on g. Since $u = \sum_g u_g$, then $D_\alpha u = \sum_g D_\alpha u_g$ and $A^{(\alpha)}(x, D)u = \sum A^{(\alpha)}(x, D)u_g$. By Leibnitz' rule, since $u_g = \theta_g u$,

$$A(x, D)u = \sum_\alpha C_\alpha \quad (A^{(\alpha)}(x, D)u)(D_\alpha\theta_g)$$

where the constants C_α depend only on α, and hence summing (2.16) yields

$$\sum_{|\alpha| \leqq m} (\tau\delta^2)^{m-|\alpha|-r}\tau^{m-|\alpha|}\|D_\alpha u\|^2 + \sum_{|\alpha| \geqq 1} \tau^{|\alpha|}\delta^{p|\alpha^*|}\|A^{(\alpha)}(x, D)u\|^2 \leqq C\|A(x, D)u\|^2 + C\sum_{|\alpha| \geqq 1} (\tau\delta^q)^{|\alpha|}\|A^{(\alpha)}(x, D)u\|^2 . \tag{2.17}$$

If $p < q < 2$ and δ_0 is chosen so small that

$$C\delta_0^{q-p} \leqq 1/2$$

then

$$\tau^{|\alpha|}\delta^{p|\alpha^*|} - C\tau^{|\alpha|}\delta^{q|\alpha|} \geqq 1/2\tau^{|\alpha|}\delta^{p|\alpha^*|}$$

and hence (2.17) becomes

$$\sum_{|\alpha| \leqq m} (\tau\delta^2)^{m-|\alpha|-r}\tau^{m-|\alpha|}\|D_\alpha u\|^2 + \sum_{|\alpha| \geqq 1} \tau^{|\alpha|}\delta^{p|\alpha^*|}\|A^{(\alpha)}(x, D)u\|^2 \leqq C\|A(x, D)u\|^2 . \tag{2.18}$$

The proof of Theorem 2.2 is completed by simply dropping the second sum on the left side of (2.20).

REMARK. It is possible to eliminate the inequality (2.7) from Theorem 2.2 if the polynomial $A(x, \zeta)$ is restricted to be homogeneous ($Q = 0$), and one also assumes that the roots of A considered as a polynomial in ζ_1 are locally C^r. This fact is a consequence of Lemma 3.1.

3. Sufficient conditions for theorem 2.2. The reader will observe that the conclusion of Theorem 2.2 is a consequence of an inequality which, per se, does not depend explicitly on any properties of the roots of the principal part of the symbol. In this section we shall give sufficient conditions in order that the inequality (2.7) is satisfied. Our aim is to generalize the result of Watanabe for operators of the form

$$A(x, \zeta) = (P(x, \zeta))^3 + Q(x, \zeta) , \tag{3.1}$$

where P is a homogeneous elliptic polynomial of degree $m/3$ whose characteristics are simple and Q has degree $(m-1)$ and Lipschitz continuous coefficients, by replacing the polynomial $(P)^3$ by more general polynomials with characteristics of multiplicity no greater than three. In the case that $Q \equiv 0$, we have already achieved this objective.

It seems reasonable that it should be possible to generalize Watanabe's result with (3.1) replaced by

$$A(x, \zeta) = P_1(x, \zeta)P_2(x, \zeta)P_3(x, \zeta) + Q(x, \zeta)$$

or

$$A(x, \zeta) = (P_1(x, \zeta))^2 P_2(x, \zeta) + Q(x, \zeta)$$

where P_1, P_2, P_3 are homogeneous elliptic polynomials with simple characteristics whose degrees sum to m and Q is as above. This generalization may not be possible unless additional assumptions are placed on P_1, P_2, P_3, however, and one possible such assumption is that the distinct factors of the principal part have linearly independent gradients in a neighborhood of a multiple root, a condition which is trivially true in the case of a single distinct factor with simple characteristics, *i.e.*, in Watanabe's case. As it turns out, this assumption may be imposed even when the factors are not polynomials, and it is with this in mind that we make the following definition.

DEFINITION. The homogeneous elliptic differential operator $A(x, D)$ is said to have *non-tangential characteristics of multiplicity* r at a point $(x_0, \zeta_0) \in \boldsymbol{R}^n \times \boldsymbol{C}^n$ if its symbol $A(x, \zeta)$ has the factorization in a neighborhood of (x_0, ζ_0)

$$A(x, \zeta) = \prod_{j=1}^{J} (\zeta_1 - \rho_j(x, \zeta'))^{r_j}$$

with $\rho_j \in C^{r-1}$ $j = 1, \cdots, J$ and where the r_j are positive integers whose sum is m, $K \leqq J$ is an integer such that

$$r_1 + r_2 + \cdots + r_K = r$$

and

$$\rho_1(x_0, \zeta_0') = \rho_2(x_0, \zeta_0') = \cdots = \rho_K(x_0, \zeta_0') \neq \rho_j(x_0, \zeta_0') , \quad j = K+1, \cdots, J$$

and the set of vectors $\{\gamma_1, \cdots, \gamma_K\} \subset \boldsymbol{C}^n$ is linearly independent, where γ_j

is the vector whose first component is $1/i$ and whose remaining components are $-D^k\rho_j(x_0, \zeta_0')$, $k = 2, \cdots, n$. We say that an operator has *non-tangential characteristics of multiplicity at most* r in a set if it has non-tangential characteristics of multiplicity no greater than r at every point in that set.

Observe that we tacitly assumed that the roots of the polynomial are well-defined functions, an assumption that is unnecessary if it is the product of factors with simple roots. In this case, if

$$A(x, \zeta) = (P_1(x, \zeta))^{r_1}(P_2(x, \zeta))^{r_2} \cdots (P_K(x, \zeta))^{r_K}(P_{K+1}(x, \zeta))$$

and if $P_1(x_0, \zeta_0) = \cdots = P_K(x_0, \zeta_0) = 0$ and $P_{K+1}(x_0, \zeta_0) \neq 0$ then A will have non-tangential characteristics if and only if the set of vectors $\{\Gamma_1, \cdots, \Gamma_K\} \subset \boldsymbol{C}^n$ is linearly independent, where Γ_j is the vector whose components are $D^kP_j(x_0, \zeta_0)$, $k = 1, \cdots, n$.

The first lemma containts a pointwise inequality relating the polynomials obtained by differentiating the coefficients of the given polynomial with the Lagrange interpolation polynomials.

LEMMA 3.1. *Let $A(x, \zeta)$ be a homogeneous elliptic polynomial of degree m in $\zeta \in \boldsymbol{C}^n$ for $x \in \boldsymbol{R}^n$. Let $\rho_j(x, \zeta')$ $j = 1, \cdots, J$ be its roots, where the multiplicity of ρ_j is r_j when considered as a polynomial in ζ_1. Suppose there is a neighborhood Ω_0 of the origin in $\boldsymbol{R}^n$ and an open cone $V \subset \boldsymbol{R}^n$ so that*

$$\rho_j \in C^r(\Omega_0 \times E'(V))$$

then for $x \in \Omega_0$, $N \in V$, $\zeta \in E(V)$ and $|\alpha| \leqq \min(r, m)$ we have

$$|A_{(\alpha)}(x, \zeta)|^2 \leqq C \sum_{|\beta| \leqq |\alpha|} |\tau N|^{2|\beta|} |{}_{\beta}A(x, \zeta)|^2 . \tag{3.2}$$

REMARK 1. By this assumption we do not mean to imply that the roots of A are defined globally but merely that for each point $(x, \zeta') \in \Omega_0 \times E'(V)$ there is a neighborhood and m functions $\rho_j(x, \zeta')$ each of which is C^r and satisfies $A(x, \zeta_1, \zeta') = 0$ in that neighborhood.

REMARK 2. In (3.2) the multi-index α has entries running from 1 to n, as have the other multi-indices used until now. The index β, however, has entries running from 1 to J, where the entry j may be repeated at most r_j times, since these are the indices for which ${}_{\beta}A$ is defined. Although these restrictions on β are not explicitly denoted in the various expressions where β appears, it will always be clear from the context that they are required.

PROOF OF LEMMA 3.1. Since

$$A(x, \zeta) = \prod_{j=1}^{J} (\zeta_1 - \rho_j(x, \zeta'))^{r_j} ,$$

then if $|\alpha| = 1$ we have

$$A_{(\alpha)} = \sum_{|\beta|=1} r_{\beta_1}(D_\alpha \rho_{\beta_1}(x, \zeta'))_{\beta_1} A .$$

For $\min(r, m) \geqq |\alpha| \geqq 1$, we write

$$A_{(\alpha)} = \sum_{1 \leqq |\beta| \leqq |\alpha|} R(\alpha, \beta)_\beta A , \tag{3.3}$$

where the $R(\alpha, \beta)$ are all continuous functions in a neighborhood of each point in the set $\Omega_0 \times E'(V)$.

We return now to the consideration of (3.2). Considered as functions of $(\xi, \tau) \in \boldsymbol{R}^{n+1}$ for each $x \in \Omega_0$, each side is a homogeneous function of degree $2m$. We therefore need only show (3.2) for the set $S = \{|\xi|^2 + \tau^2 = 1\}$. For $N = N_0$ and $\tau = 0$ the elliptic polynomial $A(x, \zeta)$ is non-zero and hence, by continuity, there is a number τ_0 with $|A(x, \zeta)|^2$ bounded below by a positive number for $\tau \leqq \tau_0$. Again by continuity, there is a relatively open set, V_0, of unit vectors surrounding N_0 such that $|A(x, \zeta)|^2$ is bounded below by a positive number for $N \in V_0$ and $\tau \leqq \tau_0$. Without loss of generality, we may assume that the cone V is generated by the set V_0. For $x \in \Omega_0$, $\tau \leqq \tau_0$, $N \in V_0$, then, $|A(x, \zeta)|^2$ is bounded below by a positive number. Since $|A_{(\alpha)}(x, \zeta)|^2$ is a continuous function on the closure of these sets, it is bounded above, and hence we have

$$|A_{(\alpha)}(x, \zeta)|^2 \leqq C |A(x, \zeta)|^2 \tag{3.5}$$

for $x \in \Omega_0$, $\tau \leqq \tau_0$, $N \in V_0$, and $(\xi, \tau) \in S$.

Now consider the expression (3.3). As noted, $R(\alpha, \beta)$ is a continuous function on the closure of the set given by $x \in \Omega_0$, $\tau \leqq \tau_0$, $N \in V_0$, $(\xi, \tau) \in S$, and hence is bounded above. Although ρ_j, $j = 1, \cdots, m$ and hence $R(\alpha, \beta)$ are only defined locally, S may be covered by a finite number of neighborhoods on which the functions $R(\alpha, \beta)$ are defined and bounded. We then fix this covering and interpret all subsequent inequalities as holding true at (ξ, τ) for each of the finite number of possible selections of sets $\{\rho_j\}$ corresponding to the members of the covering set in which (ξ, τ) lies. This tells us that

$$|A_{(\alpha)}(x, \zeta)|^2 \leqq C \sum_{1 \leqq |\beta| \leqq |\alpha|} |_\beta A|^2 .$$

For $\tau \geqq \tau_0$ this implies

$$|A_{(\alpha)}(x, \zeta)|^2 \leqq C \sum_{1 \leqq |\beta| \leqq |\alpha|} \left(\frac{\tau}{\tau_0} |N|\right)^{2|\beta|} |_\beta A|^2$$

for $N \in V_0$. Hence

$$|A_{(\alpha)}(x, \zeta)|^2 \leqq C \sum_{1 \leqq |\beta| \leqq |\alpha|} |\tau N|^{2|\beta|} |{}_\beta A|^2 \tag{3.6}$$

for $\tau \geqq \tau_0$ and $N \in V_0$. Adding (3.5) and (3.6) gives (3.2) for $N \in V_0$, $x \in \Omega_0$, and $(\xi, \tau) \in S$. Homogeneity now shows (3.2) for $\zeta \in E(V)$ with $N \in V_0$. The vector N, however, always appears in the combination (τN) and hence (3.2) is true for $\zeta \in E(V)$ without other restriction. This completes the proof of Lemma 3.1.

The following lemma gives sufficient conditions for (2.7) to hold when the characteristics have multiplicity at most three. In this case, the consequence of Theorem 2.2 is sufficient to show uniqueness in Cauchy's problem when arbitrary lower order terms are included so long as they have Lipschitz continuous coefficients.

LEMMA 3.2. *Suppose that Ω is a compact neighborhood of the origin in $\boldsymbol{R}^n$, $V \subset \boldsymbol{R}^n$ is an open cone (containing N_0), and that $A(x, D)$ is a homogeneous elliptic operator with non-tangential characteristics of multiplicity at most three in the set $\Omega \times E(V)$. Then by replacing V by the closure of a smaller cone*

$$|A_{(\alpha)}(x, \zeta)|^2 \leqq C \sum_{0 \leqq |\beta| \leqq |\alpha|} |\tau N|^{2|\beta|} |A^{(\beta)}(x, \zeta)|^2 \tag{3.7}$$

for $x \in \Omega$, $N \in V$, $\zeta \in E(V)$, and $1 \leqq |\alpha| \leqq 2$.

PROOF. In view of homogeneity and the fact that N only appears in the product τN, we need only show (3.7) holds on the compact set $\Omega \times E_0$ where $E_0 = \{\zeta \in E(V): |\xi|^2 + \tau^2 = 1, |N| = 1\}$. We shall accomplish this objective by first observing that the right side of (3.7) can only vanish on the subset $S \subset \Omega \times E_0$ for which A has either double or triple roots. We then consider each point $(x, \zeta) \in \Omega \times E_0$ at which A has double or triple roots and construct a relatively open neighborhood $S_0(x, \zeta) \subset \Omega \times E_0$ of (x, ζ) on which (3.7) holds. Since only a finite number of the sets $S_0(x, \zeta)$ are needed to cover S and the right side of (3.7) is bounded below on the closed set $(\Omega \times E_0)\backslash(\cup S_0(x, \zeta))$, the lemma will be proved.

Before beginning the construction indicated above, we introduce some notation which will simplify the calculations somewhat. By ∇A we will mean the vector in $\boldsymbol{C}^n$ whose components are $A^{(j)}$ $j = 1, \cdots, n$ and whose Euclidean norm is denoted $\|\nabla A\|$. Similarly, if ρ is a function of ζ' we shall use $\nabla_{\zeta'}\rho$ to mean the vector whose components are $D^j\rho$, $j = 2, \cdots, n$.

In light of Lemma 3.1 we already know the inequality

$$|A_{(\alpha)}(x, \zeta)|^2 \leqq C \sum_{|\beta| \leqq |\alpha|} |\tau N|^{2|\beta|} |{}_\beta A(x, \zeta)|^2 \tag{3.8}$$

for $x \in \Omega, N \in V, \zeta \in E(V)$, where the sum is taken over those β's for which the Lagrange interpolation polynomial are defined. (The cone V may have to be made smaller for this application of Lemma 3.1.) Hence the proof of (3.7) is reduced to showing the inequality

$$\sum_{|\beta|=k} |{}_\beta A|^2 \leqq C \sum_{|\beta|=k} |A^{(\beta)}|^2 \quad k = 1, 2 . \tag{3.9}$$

To show (3.9) we must consider five separate cases.

Case 1. Suppose (x_0, ζ_0) is a point in a neighborhood of which we have

$$A(x, \zeta) = (\zeta_1 - \rho_1(x, \zeta'))(\zeta_1 - \rho_2(x, \zeta')) \prod_{j=3}^{J} (\zeta_1 - \rho_j(x, \zeta'))^{r_j} \tag{3.10}$$

where $\zeta_{0,1} = \rho_1(x_0, \zeta'_0) = \rho_2(x_0, \zeta'_0) \neq \rho_j(x_0, \zeta'_0)$, $j = 3, \cdots, J$. In this case we need only show (3.9) for $k = 1$, since if $|\alpha| = 2$ in (3.7) its right side is non-zero at (x_0, ζ_0), and hence in a neighborhood of (x_0, ζ_0).

It is easy to see from (3.10) that

$$\nabla A = \gamma_1\, {}_1A + \gamma_2\, {}_2A + \sum_{j=3}^{J} r_j \gamma_j\, {}_jA , \tag{3.11}$$

where $\gamma_j \in \boldsymbol{C}^n$ represents the vector whose first component is $1/i$ and whose remaining components are those of $-\nabla_{\zeta'}\rho_j$. Hence we have

$$||\nabla A||^2 \geqq C||\gamma_1\, {}_1A + \gamma_2\, {}_2A||^2 - C' \sum_{j=3}^{J} r_j^2 |{}_jA|^2 ||\gamma_j||^2 . \tag{3.12}$$

Since $\rho_j \neq \rho_1$ for $j = 3, \cdots, J$ we see that

$${}_jA = {}_1A \frac{(\zeta_1 - \rho_1)}{(\zeta_1 - \rho_j)} \tag{3.13}$$

and $(\zeta_1 - \rho_1)/(\zeta_1 - \rho_j)$ will be continuous and bounded in a neighborhood $S_0(x_0, \zeta_0)$ of (x_0, ζ_0), and will be $o(1)$ (*i.e.*, will approach zero) as $(x, \zeta) \to (x_0, \zeta_0)$. Hence we have

$$||\nabla A||^2 \geqq C||\gamma_1\, {}_1A + \gamma_2\, {}_2A||^2 - o(1)|{}_1A|^2 \tag{3.14}$$

since the functions γ_j are homogeneous of degree zero and hence bounded. Now, γ_1 and γ_2 are linearly independent and hence the function

$$B(\eta_1, \eta_2) = ||\eta_1\gamma_1 + \eta_2\gamma_2||^2$$

is a positive definite quadratic form for $\eta_1, \eta_2 \in \boldsymbol{C}$. It follows that

$$B(\eta_1, \eta_2) \geqq C(|\eta_1|^2 + |\eta_2|^2) ,$$

and the same inequality for a smaller C holds in a restriction of the neighborhood $S_0(x_0, \zeta_0)$, and hence that

$$\|\nabla A\|^2 \geqq C(|_1A|^2 + |_2A|^2) - o(1)|_1A|^2 . \tag{3.15}$$

By restricting $S_0(x_0, \zeta_0)$ further so that

$$o(1) \leqq 1/2\,C ,$$

we have

$$\|\nabla A\|^2 \geqq C(|_1A|^2 + |_2A|^2) .$$

The equality (3.13) also implies that

$$\sum_{|\beta|=1} |_\beta A|^2 \leqq (1 + C)|_1A|^2 + |_2A|^2$$

and (3.9) and hence (3.7) follows for this case.

Case 2. Suppose (x_0, ζ_0) is a point in a neighborhood of which we have

$$A(x, \zeta) = (\zeta_1 - \rho_1(x, \zeta'))^2 \prod_{j=2}^{J} (\zeta_1 - \rho_j)^{r_j} , \tag{3.16}$$

where $\zeta_{0,1} = \rho_1(x_0, \zeta_0') \neq \rho_j(x_0, \zeta')$ $j = 2, \cdots, J$. We again need only show (3.9) for $k = 1$.

From (3.16) we have

$$\nabla A = 2\gamma_1\, {}_1A + \sum_{j=2}^{J} r_j \gamma_j\, {}_jA . \tag{3.17}$$

The equality (3.13) applies in this case as well as Case 1, and we have

$$|_jA|^2 \leqq o(1)|_1A|^2 , \quad j = 2, \cdots, J . \tag{3.18}$$

As a consequence of (3.17), we see that in a neighborhood $S_0(x_0, \zeta_0)$

$$\|\nabla A\|^2 \geqq C\|\gamma_1\|^2|_1A|^2 - o(1)|_1A|^2 . \tag{3.19}$$

Because of its definition, $\|\gamma_1\| \geqq 1$, and by restricting $S_0(x_0, \zeta_0)$ (3.9), and hence (3.7), follows, as in Case 1.

The subsequent three cases are made more difficult because of the necessity of dealing with second derivatives and the attendant notational difficulties. For convenience we shall identify the set of linear maps from C^n to C^n with the set of complex $n \times n$ matrices (with respect to the standard basis) and with the tensor product space $C^n \otimes C^n$. These spaces will be normed by the square root of the sum of the squares of the matrix entries. If $v, w \in C^n$ are vectors whose components are v_j, w_j, $j = 1, \cdots, n$, then $v \otimes w \in C^n \otimes C^n$ has components $v_j w_k$ $j, k = 1, \cdots, n$. We denote by $\nabla^2 A$ the matrix whose entries are $A^{(\beta)}$ $|\beta| = 2$ and hence $\|\nabla^2 A\|^2 = \sum_{|\beta|=2} |A^{(\beta)}|^2$. In addition, $\nabla\gamma_j$ will be the matrix whose entries are $D^\beta(\zeta_1 - \rho_j(x, \zeta'))$, $|\beta| = 2$.

Since the second derivative of the roots will be homogeneous of degree (-1) in ζ', we must be assured that A can have no roots at

$\zeta' = 0$. Since A is elliptic, however, when $\zeta' = 0$, $A = \zeta_1^m$ and

$$|A|^2 = |\zeta_1|^{2m} = (\xi^2 + (\tau N_1)^2)^m .$$

If necessary we restrict the cone V to be so small that if $N \in V$ and $|N| = 1$ then $N_1 \geqq 1/2$. Hence we have

$$|A|^2 \geqq \frac{1}{4^m}(\xi^2 + \tau^2)^m = 1/4^m .$$

By continuity, there is an ε so that if $|\zeta'| < \varepsilon$, $|A|^2 \geqq 8^{-m} > 0$, so we may assume $|\zeta'| \geqq \varepsilon$.

Case 3. Suppose (x_0, ζ_0) is a point in a neighborhood of which we have

$$A(x, \zeta) = (\zeta_1 - \rho_1)(\zeta_1 - \rho_2)(\zeta_1 - \rho_3)\prod_{j=4}^{J}(\zeta_1 - \rho_j)^{r_j} \tag{3.20}$$

where $\zeta_{0,1} = \rho_1(x_0, \zeta_0') = \rho_2(x_0, \zeta_0') = \rho_3(x_0, \zeta_0') \neq \rho_j(x_0, \zeta_0')$ $j = 3, \cdots, J$. We need to show (4.3) for both $k = 1$ and $k = 2$. When $k = 1$, the proof is essentially similar to Case 1, and will not be repeated here.

For $k = 2$, we must calculate $\nabla^2 A$. Differentiating (3.20) twice gives

$$\begin{aligned}\nabla^2 A = {} & (\nabla\gamma_1)\,{}_1A + (\nabla\gamma_2)\,{}_2A + (\nabla\gamma_3)\,{}_3A + \sum_{j=4}^{J} r_j(\nabla\gamma_j)\,{}_jA \\ & + \gamma_1 \otimes \sum_{j=2}^{J} r_j(\gamma_j)\,{}_{j1}A + \gamma_2 \otimes \Big(\sum_{j\neq 2} r_j(\gamma_j)\,{}_{j2}A\Big) \\ & + \gamma_3 \otimes \Big(\sum_{j\neq 3} r_j(\gamma_j)\,{}_{j3}A\Big) + \sum_{\substack{j_1=4\\ j_2=4}}^{J} r_{j_1}r_{j_2}\gamma_{j_1} \otimes \gamma_{j_2}\,{}_{j_1j_2}A \\ & + \sum_{j=4}^{J} r_j(r_j - 1)\gamma_j \otimes \gamma_j\,{}_{jj}A\end{aligned}$$

where we take ${}_{jj}A = 0$ if $r_j = 1$. Hence we have, recalling that ${}_{jk}A = {}_{kj}A$,

$$\begin{aligned}\nabla^2 A = {} & \sum_{j=1}^{J} r_j(\nabla\gamma_j)_jA + (\gamma_1 \otimes \gamma_2 + \gamma_2 \otimes \gamma_1)\,{}_{12}A \\ & + (\gamma_1 \otimes \gamma_3 + \gamma_3 \otimes \gamma_1)\,{}_{13}A + (\gamma_2 \otimes \gamma_3 + \gamma_3 \otimes \gamma_2)\,{}_{23}A \\ & + \sum_{j=4}^{J} r_j\,{}_{j1}A(\gamma_j \otimes \gamma_1 + \gamma_1 \otimes \gamma_j) \\ & + \sum_{j=4}^{J} r_j\,{}_{j2}A(\gamma_j \otimes \gamma_2 + \gamma_2 \otimes \gamma_j) + \sum_{j=4}^{J} r_j\,{}_{j3}A(\gamma_j \otimes \gamma_3 + \gamma_3 \otimes \gamma_j) \\ & + \sum_{\substack{j_1=5\\ 4\leqq j_2<j_1}}^{J} r_{j_1}r_{j_2} \times {}_{j_1j_2}A(\gamma_{j_1} \otimes \gamma_{j_2} + \gamma_{j_2} \otimes \gamma_{j_1}) \\ & + \sum_{j=4}^{J} r_j(r_j - 1)\gamma_j \otimes \gamma_j\,{}_{jj}A .\end{aligned} \tag{3.21}$$

We make use of the following identities.

$$(3.22)\qquad {}_{j}A = {}_{12}A\frac{(\zeta_1 - \rho_1)(\zeta_1 - \rho_2)}{(\zeta_1 - \rho_j)} \quad j = 1, \cdots, J$$

$$ {}_{j1}A = {}_{21}A\frac{(\zeta_1 - \rho_2)}{(\zeta_1 - \rho_j)} \quad j = 4, \cdots, J$$

$$ {}_{j2}A = {}_{21}A\frac{(\zeta_1 - \rho_1)}{(\zeta_1 - \rho_j)} \quad j = 4, \cdots, J$$

$$ {}_{j3}A = {}_{31}A\frac{(\zeta_1 - \rho_1)}{(\zeta_1 - \rho_j)} \quad j = 4, \cdots, J$$

$$ {}_{j_1 j_2}A = {}_{12}A\frac{(\zeta_1 - \rho_1)(\zeta_1 - \rho_2)}{(\zeta_1 - \rho_{j_1})(\zeta_1 - \rho_{j_2})} \quad j_1 = 4, \cdots, J \quad 4 \leqq j_2 \leqq j_1$$

where ${}_{jj}A$ is taken to be zero if $r_j = 1$. Since the indicated quotients in (3.22) are all continuous, bounded, and $o(1)$ in some neighborhood $S_0(x_0, \zeta_0)$, and since γ_j and $\nabla\gamma_j$ are bounded for $|\zeta'| \geqq \varepsilon$, we have

$$(3.23)\qquad \|\nabla^2 A\|^2 \geqq C\|(\gamma_1 \otimes \gamma_2 + \gamma_2 \otimes \gamma_1)\,{}_{12}A + (\gamma_1 \otimes \gamma_3 + \gamma_3 \otimes \gamma_1)\,{}_{13}A + (\gamma_2 \otimes \gamma_3 + \gamma_3 \otimes \gamma_2)\,{}_{23}A\|^2 - o(1)(|{}_{12}A|^2 + |{}_{13}A|^2)\,.$$

It is not hard to see that if $\{\gamma_1, \gamma_2, \gamma_3\}$ is linearly independent, then so is the set

$$\{\gamma_i \otimes \gamma_j : i, j = 1, 2, 3\} \subset \boldsymbol{C}^n \otimes \boldsymbol{C}^n$$

linearly independent. Hence there is a constant C and a further restriction of $S_0(x_0, \zeta_0)$ so that

$$\|\nabla^2 A\|^2 \geqq C(|{}_{12}A|^2 + |{}_{13}A|^2 + |{}_{23}A|^2)\,.$$

Using (3.22) once more gives

$$\sum_{|\beta|=2} |{}_{\beta}A|^2 \leqq C(|{}_{12}A|^2 + |{}_{13}A|^2 + |{}_{23}A|^2)\,,$$

and (3.9) for $k = 2$, and hence (2.7) follows.

Case 4. Suppose (x_0, ζ_0) is a point in a neighborhood of which we have

$$(3.24)\qquad A(x, \zeta) = (\zeta_1 - \rho_1)^3 \prod_{j=2}^{J} (\zeta_1 - \rho_j)^{r_j}\,,$$

where $\zeta_{0,1} = \rho_1(x_0, \zeta_0') \neq \rho_j(x_0, \zeta_0')$ $j = 2, \cdots, J$. We must show (3.9) for $k = 1$ and $k = 2$.

The proof of (3.9) for $k = 1$ follows just as in Case 2.

For $k = 2$, we differentiate (3.24) twice to get that

$$(3.25)\quad \nabla^2 A = 6\gamma_1 \otimes \gamma_1\, {}_{11}A + \sum_{j=1}^{J} r_j \nabla\gamma_j\, {}_jA + \sum_{j=2}^{J} r_j(\gamma_j \otimes \gamma_1 + \gamma_1 \otimes \gamma_j)\, {}_{ij}A$$

$$+ \sum_{\substack{j_1=3 \\ 2\leqq j_2<j_1}}^{J} r_{j_1} r_{j_2}(\gamma_{j_1} \otimes \gamma_{j_2} + \gamma_{j_2} \otimes \gamma_{j_1})_{j_1 j_2}A$$

$$+ \sum_{j=2}^{J} r_j(r_j - 1)\gamma_j \otimes \gamma_j\, {}_{jj}A\ .$$

Now, we have the following equalities

$$(3.26)\qquad {}_jA = {}_{11}A \frac{(\zeta_1 - \rho_1)^2}{(\zeta_1 - \rho_j)} \qquad j = 1, \cdots, J$$

$${}_{1j}A = {}_{11}A \frac{(\zeta_1 - \rho_1)}{(\zeta_1 - \rho_j)} \qquad j = 2, \cdots, J$$

$${}_{j_1 j_2}A = {}_{11}A \frac{(\zeta_1 - \rho_1)^2}{(\zeta_1 - \rho_{j_1})(\zeta_1 - \rho_{j_2})} \qquad j_1, j_2 = 2, \cdots, J$$

where all the indicated ratios are continuous, bounded, and $o(1)$ in some neighborhood $S_0(x_0, \zeta_0)$. In addition, since $|\zeta'| \geqq \varepsilon$, $\gamma_{j_1} \otimes \gamma_{j_2}$ and $\nabla\gamma_j$ are all bounded so we have as a consequence of (3.25)

$$(3.27)\qquad \|\nabla^2 A\|^2 \geqq C|{}_{11}A|^2 - o(1)|{}_{11}A|^2\ ,$$

and for a possibly smaller neighborhood $S_0(x_0, \zeta_0)$

$$\|\nabla^2 A\|^2 \geqq C|{}_{11}A|^2\ .$$

Since it is also true that

$$\sum_{|\beta|=2} |{}_\beta A|^2 \leqq C|{}_{11}A|^2$$

in $S_0(x_0, \zeta_0)$, we have (3.9), and hence (3.7), for $k = 2$.

Case 5. Suppose (x_0, ζ_0) is a point in a neighborhood of which we have

$$(3.28)\qquad A(x, \zeta) = (\zeta_1 - \rho_1)^2(\zeta_1 - \rho_2) \prod_{j=3}^{J} (\zeta_1 - \rho_j)^{r_j}$$

where $\zeta_{0,1} = \rho_1(x_0, \zeta_0') = \rho_2(x_0, \zeta_0') \neq \rho_j(x_0, \zeta_0')$ $\quad j = 3, \cdots, J$.

The proof of this case is not essentially different from those of the previous cases.

The assumption of non-tangential characteristics is necessary for the inequality (2.7) to hold, as may be seen from the following example.

COUNTEREXAMPLE. The operator $A(x, D) = ((D_1)^2 + (D_2)^2 + (D_3)^2)^3 + x_1^3(D_3)^6$ has characteristics which are C^2, triple at $x_1 = 0$, but are not non-tangential there. The polynomial $A(x, \zeta)$ does not satisfy the inequality

$$(3.29)\qquad |D_1 A|^2 \leqq C(|A|^2 + |\tau N_0|^2 \sum_{|\beta|=1} |A^{(\beta)}(x, \zeta)|^2)$$

for $\zeta = \xi + i\tau N_0$, and $\forall x_1$ in a small neighborhood of 0. This operator is elliptic for $|x_1| \leqq k < 1$.

PROOF. Consider the symbol

$$A = (\zeta_1^2 + \zeta_2^2 + \zeta_3^2)^3 + x_1^3 \zeta_3^6$$

for $|x_1| < 1/2$. The roots of this polynomial in ζ_1 are easily seen to be smooth when ζ_2 and ζ_3 have small imaginary parts. At points given by $N = N_0 = (-1, 0, \cdots, 0)$, $\zeta_2 = \xi_2 = 0$, $\zeta_3 = \xi_3 \neq 0$, $\xi_1 = 0$, and $\zeta_1 = i\tau$ where

$$\tau = \xi_3 \sqrt{1 + x_1^2}\,,$$

the left side of (3.29) becomes $O(x_1^4)$ as $x_1 \to 0$ while the right side of (3.29) is $O(x_1^6)$ as $x_1 \to 0$.

REMARK. It is unknown to this author whether a fifth order term can be added to the operator A given above in such a way that uniqueness in Cauchy's problem is violated.

In the case that the principal part, P, of the operator A has quintuple characteristics, Theorem 2.2 is not strong enough to conclude uniqueness in Cauchy's problem if terms of degree $(m - 1)$ are included, even if they have very smooth coefficients. If the assumption of non-tangential characteristics is made, however, the conclusion of Theorem 2.2 still yields new information, *viz.*, that terms of degree $(m - 2)$ with Lipschitz continuous coefficients may be added to P while retaining uniqueness in Cauchy's problem. The proof of a lemma analogous to Lemma 3.2 involves great notational difficulty but no theoretical difficulty and is not presented here.

4. A theorem on unique continuation.

The next theorem illustrates how uniqueness in Cauchy's problem is a consequence of weighted L_2 estimates such as those in Theorems 2.1 and 2.2.

THEOREM 4.1. *Suppose that $A(x, D)$ is a differential operator of degree m for which the following estimate is known. There are constants $1 < p$, $\delta_0 > 0$, $\tau_0 \geqq 0$ and an integer $r > 0$ such that*

$$(4.1) \qquad \sum_{|\alpha| \leqq m - [(r+1)/2]} (\tau\delta^2)^{m-|\alpha|-r} \tau^{m-|\alpha|} \|D_\alpha u\|^2 \leqq C \|A(x, D)u\|^2$$

$$\forall 0 < \delta \leqq \delta_0,\ \tau \geqq \tau_0/\delta^2,\ u \in C_0^\infty(|x| < \delta/2)\,.$$

Suppose that $v(x)$ is a solution of the differential inequality

$$(4.2) \qquad |A(x, D)v| \leqq C \sum_{|\alpha| \leqq m - [(r+1)/2]} |D_\alpha v|$$

and which vanishes for $x_1 \leqq \varepsilon(x_2^2 + \cdots + x_n^2)$, $\varepsilon > 0$, when x is in a neighborhood Ω_0 of the origin. Then there is a neighborhood of the origin on which $v \equiv 0$.

PROOF. Choose a function $\chi \in C_0^\infty(|x| < \delta/2)$ such that $\chi \equiv 1$ for $|x_1| \leqq \delta/4$. Then set $\chi(x)v(x) = u(x)$. Without loss of generality, we may assume $u \in C_0^\infty(|x| < \delta/2)$ since such functions are dense in L_2. By (4.1) we have

$$\sum_{|\alpha| \leqq m-[(r+1)/2]} (\tau\delta^2)^{m-|\alpha|-r}\tau^{m-|\alpha|} \int_{|x|<\delta/2} |D_\alpha u|^2 e^{2\tau\varphi_p} dx \leqq C \int_{|x|<\delta/2} |A(x, D)u|^2 e^{2\tau\varphi_p} dx . \tag{4.3}$$

We now shrink the region of integration on the left of (4.3) and apply (4.2) to the right (with $u \equiv v$ for $|x| \leqq \delta/4$) to get

$$\begin{aligned} &\sum_{|\alpha| \leqq m-[(r+1)/2]} (\tau\delta^2)^{m-|\alpha|-r}\tau^{m-|\alpha|} \int_{\substack{|x|\leqq\delta/2 \\ x_1 \leqq \delta/4}} |D_\alpha u|^2 e^{2\tau\varphi_p} dx \\ &\quad\leqq C \sum_{|\alpha| \leqq m-[(r+1)/2]} \int_{\substack{|x|\leqq\delta/2 \\ x_1 \leqq \delta/4}} |D_\alpha u|^2 e^{2\tau\varphi_p} dx \\ &\quad\quad + C \int_{\substack{|x|\leqq\delta/2 \\ x_1 \geqq \delta/4}} |A(x, D)u|^2 e^{2\tau\varphi_p} dx . \end{aligned}$$

Hence we have

$$\sum_{|\alpha| \leqq m-[(r+1)/2]} ((\tau\delta^2)^{m-|\alpha|-r}\tau^{m-|\alpha|} - C) \int_{x_1 \leqq \delta/4} |D_\alpha u|^2 e^{2\tau\varphi_p} dx \leqq C \int_{x_1 \geqq \delta/4} |A(x, D)u|^2 e^{2\tau\varphi_p} dx . \tag{4.4}$$

Now for r an odd integer and $|\alpha| \leqq m - [(r+1)/2]$

$$(\tau\delta^2)^{m-|\alpha|-r}\tau^{m-|\alpha|} \geqq \tau\delta^{-2(r-[(r+1)/2])} \geqq \tau\delta^{-2} ,$$

and for γ an even integer and $|\alpha| \leqq m - [(r+1)/2]$

$$(\tau\delta^2)^{m-|\alpha|-r}\tau^{m-|\alpha|} \geqq \delta^{-2(r-[(r+1)/2])} \geqq \delta^{-2} .$$

In either case we may choose δ small enough and $\tau \geqq 1$ so that

$$(\tau\delta^2)^{m-|\alpha|-r}\tau^{m-|\alpha|} - C \geqq 1 .$$

With these choices, we drop all but the $\alpha = 0$ term in (4.4) and restrict the region of integration still further to get

$$\int_{x_1 \leqq \delta/16} |D_\alpha u|^2 e^{2\tau\varphi_p} dx \leqq C \int_{x_1 \geqq \delta/4} |A(x, D)u|^2 e^{2\tau\varphi_p} dx . \tag{4.5}$$

We now investigate the behavior of φ_p in the two regions of integration in (5.5). For $x_1 \leqq \delta/16$ we have

$$\varphi_p(x) = (x_1 - \delta)^2 + \delta^p \sum_{j=2}^n (x_j)^2 \geqq (x_1 - \delta)^2 \geqq \frac{225}{256}\delta^2 .$$

Now, we choose δ small enough that $\delta^{p-1} \leqq \varepsilon$, and leave it fixed afterwards. For

$$\delta^p \sum_{j \geqq 2} (x_j)^2 \leqq \varepsilon\delta \sum_{j \geqq 2} (x_j)^2 \leqq \delta x_1$$

we have

$$\varphi_p(x) = (x_1 - \delta)^2 + \delta^p \sum_{j=2}^{n} (x_j)^2 \leqq x_1^2 - \delta x_1 + \delta^2 ,$$

and if $\delta/4 \leqq x_1 \leqq \delta/2$, by monotonicity of the right side, we have

$$\varphi_p(x) \leqq \frac{\delta^2}{16} - \frac{\delta^2}{4} + \delta^2 = \frac{13}{16}\delta^2 = \frac{208}{256}\delta^2 .$$

Hence (4.5) becomes

$$e^{(225\delta^2/128)\tau} \int_{x_1 \leqq \delta/16} |D_\alpha u|^2 dx \leqq C e^{(208\delta^2/128)\tau} \int_{x_1 \geqq \delta/4} |A(x, D)u|^2 dx \tag{4.6}$$

This is clearly impossible for all large τ unless u, and hence v, is zero for $x_1 \leqq \delta/16$. This completes the proof of Theorem 4.1.

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