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TOEPLITZ OPERATORS ON STRONGLY PSEUDOCONVEX DOMAINS IN STEIN SPACES

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0. Introduction. In this paper we study the C^*-algebra generated by the Toeplitz operators defined on strongly pseudoconvex domains in normal Stein spaces. We show that there exist short exact sequences of *-algebras which give elements of Ext. defined by Brown-Douglas-Fillmore ([4]).

Let Ω be a strongly pseudoconvex domain in a normal Stein space M (with or without singularities). Suppose that Ω has a volume form. Let $L^2(\Omega)$ (resp. $L^2(\partial\Omega)$) be the square integrable functions on Ω (resp. on $\partial\Omega$) and let $H^2(\Omega)$ (resp. $H^2(\partial\Omega)$) be the holomorphic square integrable functions on Ω (resp. be the closure of the C^∞-functions on $\partial\Omega$ which are extendible to holomorphic functions in Ω). Let

$$\Pi: L^2(\Omega)) \longrightarrow H^2(\Omega)$$
$$(\text{or } \Pi: L^2(\partial\Omega) \longrightarrow H^2(\partial\Omega))$$

be the orthogonal projection.

For any topological space X, we denote by $C(X)$ the Banach algebra of all complex valued continuous functions on X, endowed with supremum norm.

For $\phi \in C(\bar{\Omega})$ (resp. $\phi \in C(\partial\Omega)$), we define the Toeplitz operator

$$T_\phi[\Omega]: H^2(\Omega) \longrightarrow H^2(\Omega)$$
$$(\text{resp. } T_\phi[\partial\Omega]: H^2(\partial\Omega) \longrightarrow H^2(\partial\Omega))$$

by $T_\phi(f) = \Pi(\phi\cdot f)$.

Let $\mathscr{T}(\Omega)$ (resp. $\mathscr{T}(\partial\Omega)$) denote the C^*-algebra generated by the operators T_ϕ for all ϕ in $C(\bar{\Omega})$ (resp. $C(\partial\Omega)$). Let us define a mapping

$$\xi: C(\bar{\Omega}) \longrightarrow \mathscr{T}(\Omega)$$
$$(\text{resp. } C(\partial\Omega) \longrightarrow \mathscr{T}(\partial\Omega))$$

by $\xi\phi = T_\phi$, then ξ is contractive and *-linear. For any Hilbert space H, we denote by $\mathscr{L}(H)$ the C^*-algebra of all bounded linear operators on H, by $\mathscr{L}\mathscr{C}(H)$ the closed ideal of compact operators on H.

Our main results are as follows.

THEOREM 1. *There exists a *-homomorphism ρ from $\mathscr{T}(\Omega)$ onto $C(\partial\Omega)$ such that*

$$0 \longrightarrow \mathscr{L}\mathscr{C}(H^2(\Omega)) \longrightarrow \mathscr{T}(\Omega) \xrightarrow{\rho} C(\partial\Omega) \longrightarrow 0$$

is exact and $\rho(T_\phi) = \phi|\partial\Omega$ for all $\phi \in C(\partial\Omega)$.

THEOREM 2. *There exists a *-homomorphism ρ from $\mathscr{T}(\partial\Omega)$ onto $C(\partial\Omega)$ with cross section ξ such that*

$$0 \longrightarrow \mathscr{L}\mathscr{C}(H^2(\partial\Omega)) \longrightarrow \mathscr{T}(\partial\Omega) \xrightarrow{\rho} C(\partial\Omega) \longrightarrow 0$$

is exact.

The case when Ω is the unit disc is classical and has been studied by many peoples (see the books of Douglas [8], [9]). When Ω is a strongly pseudoconvex domain in $\boldsymbol{C}^n$, the Theorem 1 has been given by Janas [13], (see the remark of Yabuta [18]). Theorem 2 for multiply connected domains in $\boldsymbol{C}$ is given by Abrahamse [1] and for spheres in $\boldsymbol{C}^n$ is in Coburn [6]. On the other hand, Rossi [15] has proved that each abstract strongly pseudoconvex manifold bounds a Stein space with singularity, but seldom without singularity. Consequently, it will be worth while to extend the result to domains in Stein spaces.

1. Domains in Stein spaces. Let M be a complex space. Let $\mathfrak{R}(M)$ denote the set of regular points of M and let $\mathfrak{S}(M) = M - \mathfrak{R}(M)$ denote the set of singular points of M. A Hermitian inner product h_x on each $\Pi_{1,0}(\boldsymbol{C}T_x\mathfrak{R}(M))$, $x \in \mathfrak{R}(M)$, is called a Hermitian metric of M if the following condition is satisfied.

*) There exists a proper resolution

$$f: \tilde{M} \longrightarrow M\,,$$

where $\tilde{M}$ is a nonsingular complex manifold with a Hermitian metric $\tilde{h}$ such that

$$\tilde{h}_y = f^* h_{f(y)}$$

for every $y \in f^{-1}(\mathfrak{R}(M))$.

Then naturally we have the volume form dV on $\mathfrak{R}(M)$, and we can do the integration on M by regarding $\mathfrak{S}(M)$ to be measure zero.

Let Ω be an open variety in M with smooth boundary $\partial\Omega$ such that $\bar{\Omega}$ is compact. Suppose that $\partial\Omega$ is contained in $\mathfrak{R}(M)$ and is defined by the equation $r = 0$ where r is a continuous function, C^∞ on $\mathfrak{R}(M)$, with $r < 0$ inside Ω, $r > 0$ outside $\bar{\Omega}$, and $|dr| = 1$ on $\partial\Omega$. We call Ω a strongly pseudoconvex domain if the Levi form is positive definite at each point of $\partial\Omega$.

Now suppose that Ω is a strongly pseudoconvex domain in a normal Stein space M with a Hermitian metric. The volume form dV naturally induces a volume form dS on $\partial\Omega$.

We define some Hilbert spaces as follows

$L^2(\Omega)$: the space of square integrable functions on Ω

$L^2(\partial\Omega)$: the space of square integrable functions on $\partial\Omega$

$H^2(\Omega)$: the space of square integrable functions on Ω which are holomorphic in Ω

$H^2(\partial\Omega)$: the $L^2(\partial\Omega)$ closure of C^∞-functions on $\partial\Omega$ which are extendible to holomorphic functions on Ω.

We have the proper resolution $\tilde{\Omega}$ of Ω by Hironaka's theorem [12]. Since Ω is normal, the total transform in $\tilde{\Omega}$ of each singular point is connected by the Zariski's main theorem (cf. e.g. [16]). Consequently the holomorphic functions on Ω and the holomorphic functions on $\tilde{\Omega}$ are isomorphic.

It is known that $H^2(\Omega)$ is a closed subspace of $L^2(\Omega)$. Obviously $H^2(\partial\Omega)$ is a closed subspace of $L^2(\partial\Omega)$. Remark that, since Ω has non-constant holomorphic functions, $\partial\Omega$ is connected if $\dim\Omega > 1$ ([10, 5.3.6]). We have the operator $\bar{\partial}_b$ on $L^2(\partial\Omega)$ ([10, Chap. V]). By the extension theorem ([10, 5.3.5]), we know that, if $\dim\Omega > 1$, then $H^2(\partial\Omega)$ is the null space of the operator $\bar{\partial}_b$, and the space $H^2(\partial\Omega)$ is independent of Ω.

The projection $\Pi\colon L^2(\Omega) \to H^2(\Omega)$ (resp. $L^2(\partial\Omega) \to H^2(\partial\Omega)$) is given by the integration with Bergman kernel on $\tilde{\Omega}$ (resp. with the limit of Cauchy-Szegö kernel on $\partial\Omega = \partial\tilde{\Omega}$).

We have the following lemma. For any $\phi \in C(\bar{\Omega})$(resp. $C(\partial\Omega)$), denote by M_ϕ the multiplication by ϕ.

LEMMA 1. *The operator*

$$(1 - \Pi)M_\phi\colon H^2(\Omega) \to L^2(\Omega))$$
$$(\text{resp.}\colon H^2(\partial\Omega) \to L^2(\partial\Omega))$$

is compact.

PROOF. For smooth $\phi \in C^\infty(\bar{\Omega})$ (or $\phi \in C^\infty(\partial\Omega)$), it is a consequence of the Kohn's solution of $\bar{\partial}$-Neumenn problem or $\bar{\partial}_b$- Neumann problem (if $\dim\Omega > 1$) and has been proved in Venugopalkrishna [17] or in Folland-Kohn [10]. Since any $\phi \in C(\bar{\Omega})$ (or $C(\partial\Omega)$) can be approximated uniformly by smooth ones, the lemma follows in these cases.

Consider the case of $\dim\Omega = 1$. Since M is normal M is an open Riemann surface and $\partial\Omega$ consists of a finite number of non-intersecting smooth Jordan curves. Thus the proof of the lemma is essentially the same as that of Lemma 2.8 in Abrahamse [1]. Indeed, every continuous function on $\partial\Omega$ can be approximated uniformly on $\partial\Omega$ by linear span of meromorphic functions on M with exactly one simple pole in $M\backslash\partial\Omega$. For the proof combine Corollary 2 in Kodama [14] and Satz 12 in Behnke-Stein [3]. Further if $P(z, a)$ is a meromorphic function on M with exactly one simple pole at a point $a \in M\backslash\partial\Omega$, then $(f(z) - f(a))P(z, a) \in H^2(\partial\Omega)$ for every $f \in H^2(\partial\Omega)$. Thus for such a $P(z, a)$ we have

$$(1 - \Pi)M_{P(z,a)}f = f(a)(1 - \Pi)P(z, a) \quad \text{for} \quad f \in H^2(\partial\Omega) .$$

Hence $(1-\Pi)M_P$ is of rank one. Since every $\phi \in C(\partial\Omega)$ can be approximated uniformly by linear span of such P, it follows that $(1 - \Pi)M_\phi$ is compact.

The following is also proved in [17] or [10].

LEMMA 2. *If $\phi \in C(\bar{\Omega})$ satisfies the equation $\phi = 0$ on $\partial\Omega$, then the multiplication by ϕ is a compact operator from $H^2(\Omega)$ to $L^2(\Omega)$.*

2. Proof of the theorems. To prove theorems, we recall the definition of joint spectrum and joint approximate point spectrum. Let B be a commutative Banach algebra with unit. Let $f_1, f_2, \cdots, f_k$ be in B. Then the joint spectrum $\sigma(f_1, f_2, \cdots, f_k)$ is the set of points $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ in C^k such that

$$B(f_1 - \lambda_1) + B(f_2 - \lambda_2) + \cdots + B(f_k - \lambda_k) \neq B .$$

Let us denote by $\mathfrak{M}(B)$ the maximal ideal space of B. Then it is well-known that

$$\sigma(f_1, f_2, \cdots, f_k) = \{(m(f_1), m(f_2), \cdots, m(f_k)); m \in \mathfrak{M}(B)\} .$$

Let $T_1, T_2, \cdots, T_k$ be a finite commuting subset in $\mathscr{L}(H)$. Then $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ in C^k is in the joint approximate point spectrum $\sigma_\pi(T_1, T_2, \cdots, T_k)$ if

$$\mathscr{L}(H)(T_1 - \lambda_1) + \mathscr{L}(H)(T_2 - \lambda_2) + \cdots + \mathscr{L}(H)(T_k - \lambda_k) \neq \mathscr{L}(H) .$$

The joint approximate point spectrum is a compact non-empty subset of C^k. The projection map from C^k to C^l defines a continuous map from $\sigma_\pi(T_1, T_2, \cdots, T_k)$ onto $\sigma_\pi(T_1, T_2, \cdots, T_l)$ for each $1 \leqq l \leqq k$. Thus if $\{T_\alpha; \alpha \in J\}$ is a commuting family of operators in $\mathscr{L}(H)$, then the joint approximate point spectrum $\sigma_\pi(T_\alpha; \alpha \in J)$ is the projective limit $\lim_{\{\alpha_1,\alpha_2,\ldots,\alpha_n\}\subset J} \sigma_\pi(T_{\alpha_1}, T_{\alpha_2}, \cdots, T_{\alpha_n})$ directed for all finite subsets of J.

An operator T in $\mathscr{L}(H)$ is called hyponormal if $TT^* \leqq T^*T$. Bunce [5] has proved the following;

THEOREM (Bunce). *If $\{T_\alpha\}$ is a commuting family of hyponormal* *operators in* $\mathscr{L}(H)$, $\mathscr{T}$ *is the* C^**-algebra generated by* $\{T_\alpha\}$, *and* $\mathscr{C}(\mathscr{T})$ *is the commutator ideal for* $\mathscr{T}$, *then there exists a *-homomorphism* η *from* $\mathscr{T}$ *onto* $C(\sigma_\pi(T_\alpha;\, \alpha \in J))$ *such that the sequence*

$$0 \longrightarrow \mathscr{C}(\mathscr{T}) \xrightarrow{\ i\ } \mathscr{T} \xrightarrow{\ \eta\ } C(\sigma_\pi(T_\alpha;\, \alpha \in J)) \longrightarrow 0$$

is exact, where i *is the inclusion.*

The homomorphism η *satisfies*

$$\eta(T_\alpha)(\lambda) = P_\alpha(\lambda)\ ,$$

where $\lambda \in \sigma_\pi(T_\alpha;\, \alpha \in J)$ *and* $P_\alpha\colon \sigma_\pi(T_\alpha;\, \alpha \in J) \to C$ *denotes the projection to the* α*-th component.*

Let $\pi\colon \mathscr{T} \to \mathscr{T}/\mathscr{C}(\mathscr{T})$ denote the natural projection. Then $\mathscr{T}/\mathscr{C}(\mathscr{T})$ is a commutative Banach algebra.

COROLLARY. *If* $T_1, T_2, \cdots, T_k$ *are in* $\{T_\alpha\}$, *then we have*

$$\sigma_\pi(T_1, T_2, \cdots, T_k) = \sigma(\pi(T_1), \pi(T_2), \cdots, \pi(T_k))\ .$$

PROOF. We have

$$\begin{aligned}\sigma_\pi(T_1, T_2, \cdots, T_k) &= \{(P_1(\lambda), P_2(\lambda), \cdots, P_k(\lambda));\ \lambda \in \sigma_\pi(T_\alpha;\, \alpha \in J)\}\\ &= \{(\eta(T_1)(\lambda), \eta(T_2)(\lambda), \cdots, \eta(T_k)(\lambda));\ \lambda \in \sigma_\pi(T_\alpha;\, \alpha \in J)\}\\ &= \{(\xi(\eta(T_1)), \xi(\eta(T_2)), \cdots, \xi(\eta(T_k)));\ \xi \in \mathfrak{M}(C(\sigma_\pi(T_\alpha;\, \alpha \in J)))\}\\ &= \{(\zeta(\pi(T_1)), \zeta(\pi(T_2)), \cdots, \zeta(\pi(T_k)));\ \zeta \in \mathfrak{M}(\mathscr{T}/\mathscr{C}(\mathscr{T}))\}\\ &= \sigma(\pi(T_1), \pi(T_2), \cdots, \pi(T_k))\ ,\end{aligned}$$

which completes the proof.

Now we define the subspace A in $C(\bar{\Omega})$ (resp. $C(\partial\Omega)$) by the sup. norm closure of the continuous functions on $\bar{\Omega}$ (resp. on $\partial\Omega$) each of which can be extended to a holomorphic function in a neighborhood of $\bar{\Omega}$. Let $\Gamma(A)$ denote the Shilov boundary of A. Since Ω is a strongly pseudoconvex domain in a Stein space, we obtain (cf. [11, IX, C7])

$$\Gamma(A) = \partial\Omega\ .$$

Let $\mathscr{T}(A, \Omega)$ (resp. $\mathscr{T}(A, \partial\Omega)$) be the C^*-algebra on $H^2(\Omega)$ (resp. $H^2(\partial\Omega)$) generated by T_ϕ for all $\phi \in A$.

LEMMA 3. *We have*

$$\mathscr{T}(A, \Omega) = \mathscr{T}(\Omega)$$
$$(\textit{resp. } \mathscr{T}(A, \partial\Omega) = \mathscr{T}(\partial\Omega))\ .$$

PROOF. Since A separates points in $\bar{\Omega}$ (resp. $\partial\Omega$), the set $\{\phi\bar{\psi};\ \phi, \psi \in A\}$ is linearly dense in $C(\bar{\Omega})$ (resp. $C(\partial\Omega)$) by the Stone-Weierstrass theorem.

On the other hand, we have $\|T_\phi\| \leqq \|f\|_\infty$ for all $f \in C(\bar{\Omega})$ (resp. $f \in C(\partial\Omega)$). Thus $\mathscr{T}(\Omega)$(resp. $\mathscr{T}(\partial\Omega)$) coincides with the C^*-algebra generated by $\{T_\phi[\Omega];\ \phi \in A\}$(resp. $\{T_\phi[\partial\Omega];\ \phi \in A\}$).

For a C^*-subalgebra $\mathscr{T}$ of $\mathscr{L}(H)$, we denote by $\mathscr{C}(\mathscr{T})$ the commutator ideal of $\mathscr{T}$.

LEMMA 4.

$$\mathscr{C}(\mathscr{T}) = \mathscr{L}\mathscr{C}\,,$$

where $\mathscr{T}$ *denotes the* C^**-algebra* $\mathscr{T}(\Omega)$*(resp.* $\mathscr{T}(\partial\Omega)$*) and* $\mathscr{L}\mathscr{C}$ *denotes* $\mathscr{L}\mathscr{C}(H^2(\Omega))$*(resp.* $\mathscr{L}\mathscr{C}(H^2(\partial\Omega))$*).*

PROOF. $\mathscr{T}$ is irreducible. Assume otherwise, there exists a reducing subspace for $\mathscr{T}$. Then there exists a non-trivial orthogonal projection $Q(\neq 0, 1)$ such that $QT_\phi = T_\phi Q$ for all $\phi \in C(\bar{\Omega})$.(*resp.* $C(\partial\Omega)$). Put $g = Q1 \in H^2(\Omega)$(*resp.* $H^2(\partial(\Omega))$. Then we have, for all $\phi, \psi \in A$,

$$\begin{aligned}(g\phi, \psi) &= (Q\phi, \psi) \\ &= (Q^2\phi, \psi) = (Q\phi, Q\psi) = (g\phi, g\psi) = (|g|^2\phi, \psi)\end{aligned}$$

and we have

$$\int_\Omega (g - |g|^2)\phi\bar{\psi}\,dV = 0$$

$$(\text{resp. } \int_{\partial\Omega} (g - |g|^2)\phi\bar{\psi}\,dS = 0)\,.$$

Since A separates points in $\bar{\Omega}$ (resp. $\partial\Omega$), by the Stone-Weierstrass theorem, the set $\{\phi\bar{\psi};\ \phi, \psi \in A\}$ is linearly dense in $C(\bar{\Omega})$(resp. $C(\partial\Omega)$). Hence we have

$$(*) \qquad g = |g|^2 \quad \text{a.e.}\,.$$

Thus we know that g is real valued function in $H^2(\Omega)$ (resp. $H^2(\partial\Omega)$). Since g must be constant, and by $(*)$, either $g = 0$ or $g = 1$, which contradicts the assumption $Q \neq 0, 1$. Next we show that $\mathscr{C}(\mathscr{T}) \neq \{0\}$. Assume otherwise. Then for all $\phi \in A$,

$$(T_\phi T_{\bar{\phi}}1, 1) = (T_{\bar{\phi}} T_\phi 1, 1)\,.$$

Hence

$$(T_{\bar{\phi}}1,\ T_{\bar{\phi}}1) = (T_\phi 1,\ T_\phi 1) = (\phi, \phi) = (\bar{\phi}, \bar{\phi})\,,$$

and we have

$$\|\Pi\bar{\phi}\| = \|\bar{\phi}\|\,.$$

Then it follows that $\bar{\phi}$ belongs to $H^2(\Omega)$ (resp. $H^2(\partial\Omega)$), a contradiction. Thirdly we see $\mathscr{C}(\mathscr{T}) \subset \mathscr{L}\mathscr{C}$. We have, for any $\phi, \psi \in C(\bar{\Omega})$(resp. $C(\partial\Omega)$),

$$T_\phi T_\psi - T_{\phi\psi} = \Pi M_\phi \Pi M_\psi - \Pi M_\phi M_\psi$$
$$= \Pi M_\phi(\Pi - 1)M_\psi .$$

By Lemma 1, we obtain that $T_\phi T_\psi - T_{\phi\psi} \in \mathscr{L}\mathscr{C}$. Thus we have $T_\phi T_\psi - T_\psi T_\phi \in \mathscr{L}\mathscr{C}$ and the inclusion $\{0\} \neq \mathscr{C}(\mathscr{T}) \subset \mathscr{L}\mathscr{C}$ follows. Now the irreducibility of $\mathscr{T}$ shows that $\mathscr{L}\mathscr{C} \subset \mathscr{C}(\mathscr{T})$ ([Dixmier 7, 2.11.3, 4.1.10]), and we have $\mathscr{C}(\mathscr{T}) = \mathscr{L}\mathscr{C}$.

LEMMA 5. *If a finite number of functions* $\phi_1, \cdots, \phi_n$ *are in* A, *then*

$$\sigma_\pi(T_{\phi_1}, T_{\phi_2}, \cdots, T_{\phi_n}) = \{(\phi_1(x), \phi_2(x), \cdots, \phi_n(x)); x \in \partial\Omega\} ,$$

where $T_{\phi_j} = T_{\phi_j}(\Omega)$(resp. $T_{\phi_j} = T_{\phi_j}(\partial\Omega)$).

PROOF. By the corollary to the Bunce's theorem, we have

$$\sigma_\pi(T_{\phi_1}, T_{\phi_2}, \cdots, T_{\phi_n}) = \sigma(\pi(T_{\phi_1}), \pi(T_{\phi_2}), \cdots, \pi(T_{\phi_n})) .$$

Now let $\lambda \notin \{(\phi_1(x), \phi_2(x), \cdots, \phi_n(x)); x \in \partial\Omega\}$. Then there exists $\psi_1, \psi_2, \cdots, \psi_n \in C(\bar{\Omega})$(resp. $C(\partial(\Omega))$ such that the function $\psi \in C(\bar{\Omega})$(resp. $C(\partial\Omega)$) defined by

$$\phi(x) = \psi_1(x)(\phi_1(x) - \lambda_1) + \cdots + \psi_n(x)(\phi_n(x) - \lambda_n)$$

satisfies the relation

$$\phi(x) = 1 \quad \text{for} \quad x \in \partial\Omega .$$

We define the function $\phi - 1$ by $(\phi - 1)(x) = \phi(x) - 1$. Then

$$T_{\psi_1}(T_{\phi_1} - \lambda_1) + \cdots + T_{\psi_n}(T_{\phi_n} - \lambda_n) = I + T_{\phi-1}$$

and

$$\pi(T_{\psi_1})(\pi(T_{\phi_1}) - \lambda_1) + \cdots + \pi(T_{\psi_n})(\pi(T_{\phi_n}) - \lambda_n) = I + \pi(T_{\phi-1}) .$$

Since $(\phi - 1)(x) = 0$ on $\partial\Omega$, $\pi(T_{\phi-1}) = 0$ by Lemmas 2, 3 and 4. Thus we get

$$\lambda \notin \sigma(\pi(T_{\phi_1}), \cdots, \pi(T_{\phi_n})) = \sigma_\pi(T_{\phi_1}, \cdots, T_{\phi_n}) ,$$

and hence

$$\sigma_\pi(T_{\phi_1}, \cdots, T_{\phi_n}) \subset \{(\phi_1(x), \cdots, \phi_n(x)); x \in \partial\Omega\} .$$

Now we show the inverse implication. First we see that $\|\phi\|_\infty = \|T_\phi\|$ for all $\phi \in A$. Indeed, if $\phi \in A$, we have

$$\left(\int |\phi|^j dV\right)^{1/j} = \left(\int |(T_\phi 1)^j| dV\right)^{1/j} = \|(T_\phi 1)^j\|_2^{1/j} \|1\|_2^{1/j} \leqq \|T_\phi\| \, \|1\|_2^{2/j}$$
$$(j = 1, 2, \cdots) .$$

Letting $j \to \infty$, we have

$$||\phi||_\infty = ||\phi||_{L^\infty(dV)} \leqq ||T_\phi|| .$$

Since $||T_\phi|| \leqq ||\phi||_\infty$, we get $||\phi||_\infty = ||T_\phi||$. Now let $\mathscr{A}$ denote the set $\mathscr{A} = \{T_\phi;\ \phi \in A\}$. Then $\mathscr{A}$ is a commutative Banach algebra with identity. We define a map $\tau: A \to \mathscr{A}$ by $\tau(\phi) = T_\phi$ for $\phi \in A$. The map τ is an isometrical isomorphism. The map τ induces a map $\Gamma(\tau)$ between the Shilov boundaries $\tau_*: \Gamma(A) \to \Gamma(\mathscr{A})$ by $\tau_* x(T_\phi) = x(\tau^{-1}(T_\phi)) = x(\phi) = \phi(x)$ for $x \in \Gamma(A)$, $\phi \in A$. Then τ_* is a homeomorphism. By a result of Żelazko ([19, in the proof of theorem, p. 240]), for every $\zeta \in \Gamma(A)$, we have

$$(\zeta(T_{\phi_1}), \cdots, \zeta(T_{\phi_n})) \in \sigma_\pi(T_{\phi_1}, \cdots, T_{\phi_n}) .$$

Consequently we induce that for each $x \in \Gamma(A)$,

$$(\phi_1(x), \cdots, \phi_n(x)) = (\tau_* x(T_{\phi_1}), \cdots, \tau_* x(T_{\phi_n})) \in \sigma_\pi(T_{\phi_1}, \cdots, T_{\phi_n}) .$$

Since $\Gamma(A) = \partial\Omega$, we obtain

$$\sigma_\pi(T_{\phi_1}, \cdots, T_{\phi_n}) \supset \{(\phi_1(x), \cdots, \phi_n(x));\ x \in \partial\Omega\} ,$$

which completes the proof.

PROOF OF THE THEOREMS. Let x be a point in $\partial\Omega$. For any $\phi \in A$, the number $\phi(x)$ is the ϕ-th component of an element in $\sigma_\pi(T_\phi;\ \phi \in A)$ by Lemma 5. Define a mapping β from $\partial\Omega$ to $\sigma_\pi(T_\phi;\ \phi \in A)$ by

$$\beta(x) = \{\phi(x);\ \phi \in A\} \in \boldsymbol{C}^A.$$

Since Ω is Stein, A separates points in $\partial\Omega$. Hence β is injective. It is easy to see that β is continuous and by Lemma 5, it is surjective. Since $\partial\Omega$ is compact, β is a homeomorphism. Thus the mapping $\beta^*: C(\sigma_\pi(T_\phi: \phi \in A)) \to C(\partial\Omega)$ defined by $\beta^*(f) = f \circ \beta$ is an isometrical *-isomorphism. Thus Theorems 1 and 2 are consequences of Lemmas 3 and 4 applied to the Bunce's theorem.

Remark that the theorems hold if we extend to the matrix case (see [9, 2.3]).

Finally in this section we remark that one can prove the theorems 1 and 2 using Theorem 1.4 in [20] instead of the Bunce's theorem and Zelazko's theorem. In fact, after noting the isometry between A and $\mathscr{A}$ one has by that theorem the following: There exists a closed set X in $\bar{\Omega}$ (resp. $\partial\Omega$) containing $\Gamma(A) = \partial\Omega$ and a *-homomorphism ρ from $\mathscr{T}$ onto $C(X)$ such that the short sequence

$$0 \longrightarrow \mathscr{C}(\mathscr{T}) \xrightarrow{\ i\ } \mathscr{T} \xrightarrow{\ \rho\ } C(X) \longrightarrow 0$$

is exact and $\rho(T_\phi) = \phi | X$ for all $\phi \in C(\bar{\Omega})$(resp. $C(\partial(\Omega))$. Now combining this with Lemmas 2 and 4 one gets the theorems.

3. **Remarks.** Brown-Douglas-Fillmore [4] or Atiyah has shown that, for a compact metrizable space X, the set of isomorphism classes of short exact sequences of *-algebra

$$0 \longrightarrow \mathscr{L}\mathscr{C}(H) \otimes M_n \longrightarrow \mathfrak{A} \otimes M_n \longrightarrow C(X) \otimes M_n \longrightarrow 0 ,$$

where $\mathfrak{A}$ is a subalgebra of the bounded linear operators $\mathscr{L}(H)$ of a Hilbert space H and M_n is the set of $(n \times n)$-matrices, is a group and is isomorphic to the group $K_1(X)$. Consequently Theorems 1 and 2 give elements $\mathscr{T}$ in $K_1(\partial\Omega)$.

On the other hand, Atiyah [2] has defined a class of operators on a compact Hausdorff space X called elliptic operators on X, denoted by $\mathrm{Ell}\,(X)$. Then he defined a natural map $\mathrm{Ell}\,(X) \to K_0(X)$. Let us extend naturally the Toeplitz operator T_ϕ, for $\phi \in C_{M_n}(\bar{\Omega})$(resp. $C_{M_n}(\partial\Omega)$)(C_{M_n} is the Banach algebra of M_n-valued continuous functions) as an operator

$$\tilde{T}_\phi\colon L^2_{C^n}(\Omega) \longrightarrow L^2_{C^n}(\Omega)(\text{resp. } L^2_{C^n}(\partial\Omega) \longrightarrow L^2_{C^n}(\partial\Omega))$$

by $T_\phi\Pi + (1 - \Pi)$. Then it is easy to see that $\tilde{T}_\phi$ belongs to $\mathrm{Ell}\,(\partial\Omega)$ if $\phi(x) \neq 0$ for any $x \in \partial\Omega$. Thus we naturally obtain elements $\{T_\phi\}$ in $K_0(\partial\Omega)$.

The homotopy classes of ϕ in $C(X) \otimes M_n$ define elements $\{\phi\}$ in $K^1(\partial\Omega)$. We have a natural bilinear mapping

$$\cap\colon K_1(\partial\Omega) \otimes K^1(\partial\Omega) \longrightarrow K_0(\partial\Omega)$$

by

$$\mathscr{T} \cap \{\phi\} = \{T_\phi\} .$$

Note that the operator T_ϕ is not a pseudo-differential operator in the usual sense if $\dim \Omega > 1$. To know the class $\mathscr{T}$ in $K_1(\partial\Omega)$ will be an interesting problem. The Brieskorn varieties give strongly pseudoconvex domains in a Stein spaces. The calculation for such manifolds is also not known.

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