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EXISTENCE OF PERIODIC SOLUTIONS OF ONE-DIMENSIONAL DIFFERENTIAL-DELAY EQUATIONS

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1. **Introduction.** This paper is motivated by the data of numerical computing experiments by Professor Y. Ueda and his colleagues. In the study of phase locked loops which are widely used in communication systems also, in order to utilize the frequency range effectively, it has become necessary to consider phase locked loops acting in the high frequency range. In this case, it is necessary to analyze the acting principles of the system with time delays, since we cannot ignore influences to the system of time delays which arise in the parts of the system. In their studies for this purpose, the following difference-differential equation arises:

$$\dot{\theta}(\tau) = \delta - \sin(\theta(\tau - L)) , \quad \tau \geqq 0, \delta \geqq 0, L > 0 . \tag{1.1}$$

Roughly speaking, the variables in (1.1) are related with the model in the following way: τ is the time, $\theta(\tau)$ denotes the phase difference at time τ, δ is the difference between signal frequency and free-running frequency of the voltage-controlled oscillator, and L is the sum of the time delays which arise in the parts of the system. In the case where $0 \leqq \delta \leqq 1$, (1.1) has trivial periodic solutions, namely, the constant functions $\theta(\tau) \equiv \alpha$, $\tau \geqq 0$, where α is a number such that $\sin \alpha = \delta$. In their experiments, they observed the existence of a nontrivial periodic solution for $\delta = 0.3$ and $L = 2$, a periodic solution of the second kind for $\delta = 0.8$ and $L = 2$, and solutions which approach asymptotically to a constant solution. Thus there arise the following problems. Find the relation between δ and L so that (1.1) has periodic solutions, or a constant solution is uniformly asymptotically stable. We shall give sufficient conditions for these problems in Sections 4, 7, and 9.

There are various methods and many results for the existence of periodic solutions of functional differential equations [cf. 1, 2, 3, 4, 5]. We shall show the existence of periodic solutions of a more general system than (1.1) by using a fixed point theorem for the truncated cones of Krasnosel'skii in [2] (see Section 3). Particularly, for the existence of a periodic solution of the second kind of (1.1), we consider also the case where $\delta > 1$ (see Section 8). On the other hand, there are many results on the stability of solutions of functional differential equations [cf. 6, 7]. Using these results, in Section 6, we shall discuss the uniform asymptotic stability of a constant solution, the nonexistence of periodic solutions, and the nonexistence of periodic solutions of the second kind. Moreover, we shall show another example in Section 5.

2. **Notations and assumptions.** For a given $h > 0$, C denotes the space of continuous functions mapping the interval $[-h, 0]$ into R, and for $\phi \in C$, $|\phi| = \sup_{-h\leqq\theta\leqq 0}|\phi(\theta)|$. For any continuous function $x(u)$ defined on $-h \leqq u < A$, $A > 0$, and any fixed t, $0 < t < A$, the symbol x_t will denote the restriction of $x(u)$ to the interval $[t-h, t]$, i.e., x_t is an element of C defined by $x_t(\theta) = x(t+\theta)$, $-h \leqq \theta \leqq 0$.

Consider a nonlinear one-dimensional differential-delay equation

$$\dot{x}(t) = f(x(t-h)) , \qquad t \geqq 0 , \tag{2.1}$$

where $f(x)$ is assumed to satisfy the following conditions.

(H1) For $X_1 > A_1 > 0$, $X_2 > A_2 > 0$, $B_1 > 0$, and $B_2 > 0$, $f(x)$ is defined and continuous for $-X_1 \leqq x \leqq X_2$, $xf(x) < 0$ for $x \neq 0$, $f(-A_1) = B_1$, $f(A_2) = -B_2$ and $-B_2 \leqq f(x) \leqq B_1$ for $-X_1 \leqq x \leqq X_2$.

(H2) $-(B_2/A_2)x \leqq f(x) \leqq p(x)$ for $-A_1 \leqq x \leqq 0$, and $p(x) \leqq f(x) \leqq -(B_2/A_2)x$ for $0 \leqq x \leqq A_2$, where $p(x)$ is a nonincreasing continuous function defined for $-A_1 \leqq x \leqq A_2$ and satisfies $p(-A_1) = B_1$ and $p(-A_2) = -B_2$.

(H3) $f(x) = -Lx + M(x)$ for $L > 0$, where $M(x)$ is the higher order part and satisfies $|M(x) - M(y)| \leqq \mu(\sigma)|x-y|$ for $-X_1 \leqq x$, $y \leqq X_2$, $|x|, |y| \leqq \sigma$ and $\mu(\sigma)$ is continuous and nondecreasing with $\mu(0) = 0$.

3. **Existence of nontrivial periodic solutions.** In this section we shall discuss the existence of nontrivial periodic solutions of (2.1) for $-X_1 < x < X_2$. For any k_1 such that $\max(A_1/X_1, A_2/X_2) < k_1 < 1$, we define the set K by

$$K = \{\phi \in C\colon \phi(-h) = 0,\ \phi(\theta) \text{ is nondecreasing on } [-h, 0],\ \phi(0) \leqq k_1X_2\} . \tag{3.1}$$

LEMMA 3.1. (i) *Let $h \geqq A_2/B_2$ be fixed. For the positive number $v = \min_{A_2\leqq x\leqq k_1X_1}|f(x)|$, let m be the smallest integer such that $m \geqq (X_2 - A_2)/vh$. If $\phi \in K\backslash\{0\}$, then $x(t) = x(t, \phi)$ has its first zero point t_0 such that $0 < t_0 \leqq (m+3)h$, and it is simple.*

(ii) *If $A_2/B_2 \leqq h \leqq (A_2 + k_1X_1)/B_2 + (1/B_2^2)\int_0^{A_2} p(s)ds$ and $\phi \in K\backslash\{0\}$, then $x(t)$ has a minimal value at the time $t_0 + h$ and $x(t_0 + h) \geqq -k_1X_1$.*

(iii) *Let h satisfy the condition in* (ii), *and for the positive number $w = \min_{-k_1X_1\leqq x\leqq -A_1}|f(x)|$, let n be the smallest integer such that $n \geqq (k_1X_1 - A_1)/wh$. If $\phi \in K\backslash\{0\}$, then $x(t)$ has its second zero till the time*

$t_0 + (n + 4)h$.

PROOF. (i) First, we consider the case $A_2 < \phi(0) \leqq k_1 X_2$. In this case, $x(t)$ is nonincreasing for $t \geqq 0$ and $\dot{x}(t) \leqq -v$ for $t \geqq h$ as long as $x(t) > A_2$. Hence if $x(h) > A_2$, we have

$$x(t) \leqq x(h) - (t - h)v\,, \qquad t \geqq h\,.$$

Now assume that $x(t)$ does not reach A_2 till the time $(m + 1)h$. Then we obtain

$$x((m + 1)h) \leqq x(h) - mvh \leqq x(h) - X_2 + A_2 \leqq A_2\,.$$

This contradiction shows that $x(t)$ reaches A_2 till the time $(m + 1)h$. Let t_1 be the first time such that $x(t_1) = A_2$. Then $x(t)$ is nonincreasing on $[t_1, t_1 + h]$. It remains only to show that $x(t)$ has a zero point till the time $t_1 + 2h$. If we assume that $x(t)$ has no zero point in $[t_1, t_1 + h]$, then we have $0 < x(t_1 + h) \leqq x(t)$ for $t_1 \leqq t \leqq t_1 + h$. Thus we obtain

$$\begin{aligned} x(t_1 + 2h) &= x(t_1 + h) + \int_{t_1+h}^{t_1+2h} f(x(s - h))ds \\ &\leqq x(t_1 + h) + \int_{t_1+h}^{t_1+2h} \left\{-\frac{B_2}{A_2} x(s - h)\right\} ds \\ &\leqq x(t_1 + h)\left(1 - \frac{B_2}{A_2} h\right) \leqq 0\,, \end{aligned}$$

and hence $x(t)$ has a zero point till the time $(m + 3)h$. In the second case $0 < \phi(0) \leqq A_2$, we can apply the same argument for $t_1 = 0$. It is clear that the zero is simple.

(ii) It is clear that $x(t)$ attains a minimal value at the time $t_0 + h$. Let $\chi \in C$ be a function such that

$$\chi(\theta) = \begin{cases} -B_2\theta\,, & -\dfrac{A_2}{B_2} < \theta \leqq 0\,, \\ A_2\,, & -h \leqq \theta \leqq -\dfrac{A_2}{B_2}\,. \end{cases}$$

Since $p(x)$ is nonincreasing on $[0, A_2]$ and $p(A_2) = -B_2$, we have

$$f(x(t_0 + \theta)) \geqq p(x(t_0 + \theta)) \geqq p(\chi(\theta)) \quad \text{for all } \theta \in [-h, 0]\,,$$

and hence the minimal value is estimated as follows.

$$\begin{aligned} x(t_0 + h) &\geqq \int_{-h}^{0} f(x(t_0 + s))ds \geqq \int_{-h}^{-A_2/B_2} p(\chi(s))ds + \int_{-A_2/B_2}^{0} p(\chi(s))ds \\ &\geqq A_2 - B_2 h + \frac{1}{B_2}\int_0^{A_2} p(s)ds \geqq -k_1 X_1\,. \end{aligned} \tag{3.2}$$

Thus (ii) is true.

(iii) We first show that $x(t)$ reaches $-A_1$ till the time $t_0+(n+2)h$ even if $-k_1X_1 \leqq x(t_0+h) < -A_1$. $x(t)$ is increasing for $t \geqq t_0+h$ as long as $x(t-h)<0$. If we assume that $x(t_0+2h) < -A_1$, then $-k_1X_1 \leqq x(t-h) < -A_1$ and $\dot{x}(t) \geqq w$ for $t \geqq t_0+2h$ as long as $x(t) < -A_1$, and hence we obtain

$$x(t) \geqq x(t_0+2h) + (t-t_0-2h)w \geqq -k_1X_1 + (t-t_0-2h)w$$

as long as $x(t) < -A_1$. Therefore, if $x(t_0+(n+2)h) < -A_1$, we have

$$x(t_0+(n+2)h) \geqq -k_1X_1 + nwh \geqq -A_1 .$$

This contradiction shows that $x(t)$ reaches $-A_1$ till the time $t_0+(n+2)h$ even if $-k_1X_1 \leqq x(t_0+h) < -A_1$. Let $t_2 \in [t_0+h, t_0+(n+2)h]$ be a number such that $-A_1 \leqq x(t_2) < 0$. If $x(t)$ does not reach the t-axis before the time t_2+2h, then $x(t)$ is increasing on $[t_2, t_2+h]$ and we have

$$\begin{aligned} x(t_2+2h) &= x(t_2+h) + \int_{t_2}^{t_2+h} f(x(s))ds \\ &\geqq x(t_2+h) + \int_{t_2}^{t_2+h}\left\{-\frac{B_2}{A_2}x(s)\right\}ds \geqq x(t_2+h)\left(1-\frac{B_2}{A_2}h\right) \geqq 0 . \end{aligned}$$

Since this is a contradiction, $x(t)$ reaches the t-axis till the time $t_0+(n+4)h$ again.

LEMMA 3.2. *Suppose that h satisfies condition in* (ii) *of Lemma* 3.1, *and* $0<k_0<k_1<1$.

(i) *If* $A_2/B_2 < (A_1+k_0X_2)/B_1 - 1/B_1^2\int_{-A_1}^{0} p(s)ds = h_1$, $h \leqq h_1$ *and* $\phi \in K\backslash\{0\}$, *then the maximal value of* $x(t)$ *is not greater than* k_0X_2.

(ii) *Suppose that* $x(t_0+h) \geqq -\mu > -A_1$ *and* $A_2/B_2 < (\mu+k_0X_2)/a - 1/a^2\int_{-\mu}^{0} p(s)ds = h_2$ *where* $a = \max_{-\mu\leqq x\leqq 0} f(x)$. *If* $A_2/B_2 \leqq h \leqq h_2$, *then the maximal values of* $x(t)$ *are not greater than* k_0X_2.

(iii) *Let* $x(t)$ *attain its first maximal value at the time* $\tau(\phi)>0$. *Then we have* $2h < \tau(\phi) < (m+n+8)h$ *and* $x_{\tau(\phi)} \in K$.

PROOF. (i) Since h satisfies the condition in (ii) of Lemma 3.1, the minimal value of $x(t)$ is estimated by (3.1). Let $t_2 = \inf\{t: t>t_0, x(t)=0\}$. As we have $A_2/B_2 \geqq A_1/B_1$ by assumption (H2), we consider the function

$$\psi_1(\theta) = \begin{cases} B_1\theta , & -\dfrac{A_1}{B_1} < \theta \leqq 0 . \\ -A_1 , & -h \leqq \theta \leqq -\dfrac{A_1}{B_1} , \end{cases}$$

Since $p(x)$ is nonincreasing on $[-A_1, 0]$ and $p(-A_1) = B_1$, we have

$$f(x(t_2+\theta)) \leqq p(x(t_2+\theta)) \leqq p(\psi_1(\theta)) \quad \text{for all } \theta \in [-h, 0] .$$

This yields the following estimate of the maximal value.

$$\begin{aligned} x(t_2+h) &\leqq \int_{-h}^{0} f(x(t_2+s))ds \leqq \int_{-h}^{-A_1/B_1} p(\psi_1(s))ds + \int_{-A_1/B_1}^{0} p(\psi_1(s))ds \\ &\leqq B_1 h - A_1 + \frac{1}{B_1}\int_{-A_1}^{0} p(s)ds \leqq k_0 X_2 . \end{aligned} \tag{3.3}$$

(ii) As we have $A_2/B_2 \geqq \mu/a$ by assumption (H2), we consider the function

$$\psi_2(\theta) = \begin{cases} a\theta, & -\dfrac{\mu}{a} < \theta \leqq 0 , \\ -\mu, & -h \leqq \theta \leqq -\dfrac{\mu}{a} . \end{cases}$$

Since $p_1(x) = \min(p(x), a)$ is nonincreasing on $[-\mu, 0]$ and $p_1(-\mu) = a$ is its maximum value, we obtain

$$f(x(t_2+\theta)) \leqq p_1(x(t_2+\theta)) \leqq p_1(\psi_2(\theta)) \quad \text{for all } \theta \in [-h, 0] ,$$

which implies

$$\begin{aligned} x(t_2+h) &\leqq \int_{-h}^{0} f(x(t_2+s))ds \leqq \int_{-h}^{-\mu/a} p_1(\psi_2(s))ds + \int_{-\mu/a}^{0} p_1(\psi_2(s))ds \\ &\leqq ah - \mu + \frac{1}{a}\int_{-\mu}^{0} p(s)ds \leqq k_0 X_2 . \end{aligned} \tag{3.4}$$

(iii) From Lemma 3.1, we have $2h < \tau(\phi) < (m+n+8)h$. Moreover, $x_{\tau(\phi)}$ is clearly an element of K, and thus the other extremum values are similarly estimated.

DEFINITION 3.1. Let E be a Banach space. A set $K \subset E$ is a cone if

(i) K is closed and convex,

(ii) if ϕ is in K, then $\lambda\phi \in K$, $\lambda \geqq 0$,

(iii) for any $\phi \neq 0$ in E, both ϕ and $-\phi$ cannot belong to K.

A truncated cone is the intersection of a cone with a convex neighborhood of zero. The neighborhood does not need to be closed.

The set K in (3.1) cosidered above is a truncated cone. For $\phi \in K\backslash\{0\}$, define the mapping A by

$$A\phi = x_{\tau(\phi)}(\phi) .$$

Then, under the assumptions in Lemma 3.2, A is a positive mapping relative to K, that is, $A(K) \subset K$. Since we have $2n < \tau(\phi) < (m+n+8)h$, define $\tau(0) = \limsup_{\phi \to 0} \tau(\phi)$. Then $\tau: K \to [0, \infty)$ takes closed bounded sets into bounded sets. Furthermore, since $x(t, \phi)$ is continuous in (t, ϕ), $\tau(\phi)$ is continuous on $K \setminus \{0\}$. On the other hand, A takes bounded sets into bounded sets because $|A\phi| \leqq k_0 X_2$. Moreover, the following lemmas hold.

LEMMA 3.3. *If the conditions in Lemma 3.2 hold and if G is an open bounded neighborhood of zero, then*

$$\inf_{\phi \in \partial G \cap K} |A\phi| > 0 .$$

PROOF. If $\inf_{\phi \in \partial G \cap K} |A\phi| = 0$, then there is a sequence $\{\phi_n\}$ such that $\phi_n \in \partial G \cap K$ and $|A\phi_n| \to 0$ as $n \to \infty$. Taking a subsequence if necessary, we can assume that $\tau(\phi_n) \to \tau_0$ as $n \to \infty$ and $2h \leqq \tau_0 < \infty$. Since $\{x(t, \phi_n)\}$ is uniformly bounded and equicontinuous on $[0, \tau_0]$, we can choose a subsequence of the $\{x(t, \phi_n)\}$ so that $x(t, \phi_n) \to y(t)$ uniformly for $t \in [0, \tau_0]$ as $n \to \infty$. This $y(t)$ must correspond to a solution of (2.1) on $[h, \tau_0]$. It is clear from (2.1) that $y(t) = 0$, $0 \leqq t \leqq \tau_0$. Consequently $\phi_n(0) \to 0$ as $n \to \infty$ and the monotonicity of the ϕ_n implies that $\phi_n \to 0$ as $n \to \infty$. But this is impossible since there is a $\eta > 0$ such that $|\phi_n| \geqq \eta$. This contradiction proves the lemma.

LEMMA 3.4. *If $h > \pi/2L$, there is a zero $\lambda = \rho + i\sigma$ of*

$$\lambda e^{h\lambda} = -L \tag{3.5}$$

with $\rho > 0$, $0 < \sigma h < \pi$.

For the proof, refer to Lemma 29.4 in [8].

The linear part of (2.1) is

$$\dot{x}(t) = -Lx(t-h) , \qquad t \geqq 0 , \tag{3.6}$$

and (3.5) is the characteristic equation of (3.6). Let $(\lambda_0, \bar{\lambda}_0)$ be the characteristic roots of (3.5) whose existence was guaranteed by Lemma 3.4. We decompose C by $(\lambda_0, \bar{\lambda}_0)$ as $C = U \oplus S$, $\dim U = 2$, and denote by Π_U the projection operator onto U.

LEMMA 3.5. *If $h < \pi/2L$, then for any ε, $0 < \varepsilon \leqq k_1 X_2$, we have*

$$\inf_{\phi \in \partial B(\varepsilon) \cap K} |\Pi_U \phi| > 0 ,$$

where $B(\varepsilon) = \{\phi \in C: |\phi| < \varepsilon\}$.

PROOF. Let $\phi(\theta) = e^{\lambda_0 \theta}/(1 + h\lambda_0)$, $-h \leqq \theta \leqq 0$, $\psi(s) = e^{-\lambda_0 s}$, $0 \leqq s \leqq h$, $\Phi = (\phi, \bar{\phi})$, and $\Psi = \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$. The adjoint equation for (3.6) is

$$\dot{y}(t) = Ly(t+h)$$

and the bilinear form is given by

$$(\psi, \phi) = \psi(0)\phi(0) - L\int_{-h}^{0} \psi(s+h)\phi(s)ds \ .$$

We have $(\Psi, \Phi) =$ identity, and $\Pi_U = \Phi(\Psi, \xi)$ for any $\xi \in C$. If there is a sequence $\{\phi_n\}$ in $\partial B(\varepsilon) \cap K$ such that $\Pi_U \phi_n \to 0$ as $n \to \infty$, then necessarily $|(\Psi, \phi_n)| \to 0$ as $n \to \infty$. This implies that $(\psi, \phi_n) \to 0$ as $n \to \infty$. Since $\phi_n(0) = \varepsilon$, if we let R_n and I_n be the real and imaginary parts of (ψ, ϕ_n), respectively, then

$$R_n = \varepsilon - L\int_{-h}^{0} \phi_n(s)e^{-\rho_0(s+h)} \cos \sigma_0(s+h)ds \ ,$$

$$I_n = L\int_{-h}^{0} \phi_n(s)e^{-\rho_0(s+h)} \sin \sigma_0(s+h)ds = L\int_{0}^{h} \phi_n(s-h)e^{-\rho_0 s} \sin \sigma_0 s ds \ .$$

Since $0 < \sigma_0 h < \pi$ and $I_n \to 0$ as $n \to \infty$, it follows that $\phi_n(\theta) \to 0$ as $n \to \infty$ for $-h \leqq \theta < 0$. Thus $R_n \to \varepsilon$ as $n \to \infty$. This contradiction yields the desired inequality.

We are now ready to show the existence of a nontrivial periodic solution of (2.1) for $-X_1 < x < X_2$ by using the following theorem, which is found in [8].

Suppose K is a cone (or a truncated cone) such that for any $\phi \in K$, there is a time $\tau(\phi) > 0$ such that $x_{\tau(\phi)}(\phi) \in K$. If we let $A\phi = x_{\tau(\phi)}(\phi)$, $\phi \in K$, then $A: K \to K$ is a positive operator.

THEOREM 3.1. *Suppose A is the same as defined above, A is continuous, $\tau(\phi) \geqq h$, $\phi \in K$, τ and A take closed bounded sets into bounded sets and the following conditions are satisfied*:

(I) *For any open bounded set $G \subset C$, $0 \in G$,*

$$\inf_{\phi \in \partial G \cap K} |A\phi| > 0 \ .$$

(II) *If F is the set of positive eigenvectors of A, there is an $M > 0$ such that $\phi \in F$, $|\phi| = M$, $A\phi = \mu\phi$ imply $\mu < 1$.*

(III) *For any $\varepsilon > 0$,*

$$\inf_{\phi \in \partial B(\varepsilon) \cap K} |\Pi_U \phi| > 0 \ .$$

Uuder these conditions, there exists a nontrivial periodic solution of (2.1) *with period greater than h. In* (II), *$\phi \neq 0$ is called a positive eigenvector if $A\phi = \mu\phi$ for a positive operator A.*

Among the assumptions of Theorem 3.1, the continuity of A is given by the continuity of solutions for the initial conditions. Also we have shown that τ and A take closed bounded sets into bounded sets. Furthermore, under the conditions of Lemma 3.2, (I) holds by Lemma 3.3, (II) holds for $M > k_0 X_2$ by Lemma 3.2, and (III) also holds by Lemma 3.5. Hence we have the following theorem.

THEOREM 3.2. *Under the conditions of Lemma 3.2, there exists a nontrivial periodic solution of (2.1) for $-X_1 < x < X_2$, and its period is greater than $2h$ and less than $(m+n+8)h$.*

4. **The equation $\dot{u}(t) = \delta - \sin(u(t-h))$.** Consider the difference-differential equation

$$\dot{u}(t) = \delta - \sin(u(t-h)), \qquad t \geqq 0, \quad 0 \leqq \delta < 1. \tag{4.1}$$

Let $\alpha = \sin^{-1}\delta$. Then (4.1) has a constant solution $u(t) \equiv \alpha$. Substituting $x(t) = u(t) - \alpha$, $t \geqq -h$ into (4.1), we have an equivalent equation

$$\dot{x}(t) = \delta - \sin(x(t-h) + \alpha), \qquad t \geqq 0, \quad 0 \leqq \delta < 1, \tag{4.2}$$

which has the zero solution $x(t) \equiv 0$. (4.2) is a special case of (2.1) where $f(x) = \delta - \sin(x+\alpha)$ and $f(x)$ satisfies (H1) for $X_1 = \pi + 2\alpha$, $X_2 = \pi - 2\alpha$, $A_1 = \pi/2 + \alpha$, $A_2 = \pi/2 - \alpha$, $B_1 = 1 + \delta$, $B_2 = 1 - \delta$. Let

$$p(x) = \begin{cases} 1 + \delta, & -\dfrac{\pi}{2} - \alpha \leqq x < -\alpha - 1, \\ -x + \delta - \alpha, & -\alpha - 1 \leqq x < -\alpha, \\ -\dfrac{\delta}{\alpha}x, & -\alpha \leqq x < 0, \\ -\sqrt{1-\delta^2}\,x, & 0 \leqq x < \dfrac{1-\delta}{\sqrt{1-\delta^2}}, \\ \delta - 1, & \dfrac{1-\delta}{\sqrt{1-\delta^2}} \leqq x \leqq \dfrac{\pi}{2} - \alpha. \end{cases}$$

Then $f(x)$ satisfies (H2) for this $p(x)$. Also $f(x)$ satisfies (H3) for $L = \sqrt{1-\delta^2}$. For k_1, $1/2 < k_1 < 1$, we define the following truncated cone K by

$$K = \{\phi \in C\colon \phi(-h) = 0,\ \phi(\theta) \text{ is nondecreasing on } [-h, 0],\ \phi(0) \leqq (\pi - 2\alpha)k_1\}.$$

LEMMA 4.1. *Let*

$$F(\alpha) = \begin{cases} \dfrac{11(\pi - 2\alpha)}{8a} - \dfrac{(3\pi - 14\alpha)^2 + 16(3\pi - 10\alpha)\sin\alpha}{128a^2}, & 0 \leqq \alpha < \dfrac{3\pi}{14} \\ \dfrac{19(\pi - 2\alpha)}{16a}, & \dfrac{3\pi}{14} \leqq \alpha < \dfrac{\pi}{2}, \end{cases}$$

where $a=\sin\alpha+\cos(7\alpha/4+\pi/8)$, *and let* $G(\alpha)=\min(3(\pi-2\alpha)/8(1-\sin\alpha)+1/(2\cos\alpha), F(\alpha))$ *and* $H(\alpha)=\max((\pi-2\alpha)/(1+\sin\alpha)+(\pi+2\alpha-2\cos\alpha)/2(1+\sin\alpha)^2, G(\alpha))$. *If* $(\pi-2\alpha)/2(1-\delta)\leqq h<H(\alpha)$, *then there exist* k_0, k_1 *such that* $1/2<k_0<k_1<1$, *and for all* $\phi\in K\backslash\{0\}$, *the minimal values of* $x(t)$ *are greater than* $-(\pi+2\alpha)k_1$, *and the maximal values of* $x(t)$ *are less than* $(\pi-2\alpha)k_0$. *Furthermore, we have* $2h<\tau(\phi)<(m+n+8)h$ *and* $x_{\tau(\phi)}\in K$.

PROOF. First of all, we prove the following inequality

$$\pi\leqq\frac{(\pi-2\alpha)\sqrt{1-\delta^2}}{1-\delta}<4\,,\quad 0\leqq\alpha<\frac{\pi}{2}\,, \tag{4.3}$$

where the equality holds only if $\alpha=0$. Put $k(\alpha)=(\pi-2\alpha)\cos\alpha/(1-\sin\alpha)$, $0\leqq\alpha<\pi/2$. Then we have $k'(\alpha)=(\pi-2\alpha-2\cos\alpha)/(1-\sin\alpha)^2>0$, $k(0)=\pi$, $\lim_{\alpha\to\pi/2-0}k(\alpha)=4$, and consequently (4.3).

Next, we prove $(\pi-2\alpha)/2(1-\delta)<H(\alpha)$. Let $g(\alpha)=3(\pi-2\alpha)/8(1-\sin\alpha)+1/(2\cos\alpha)$ and $h(\alpha)=(\pi-2\alpha)/(1+\sin\alpha)+(\pi+2\alpha-2\cos\alpha)/2(1+\sin\alpha)^2$, $0\leqq\alpha<\pi/2$. Since $g(\alpha)$ is increasing and $h(\alpha)$ is decreasing and since $g(\pi/8)<h(\pi/8)$, we have $G(\alpha)\leqq g(\alpha)<h(\alpha)\leqq H(\alpha)$ for $0\leqq\alpha\leqq\pi/8$. On the other hand, we obtain $(\pi-2\alpha)/2(1-\delta)<H(\alpha)$ for $0\leqq\alpha\leqq\pi/8$ since we have $g(\alpha)>(\pi-2\alpha)/2(1-\delta)$ by (4.3). It remains only to show the following two inequalities

$$\frac{1}{2(1-\delta)}+\frac{(3\pi-14\alpha)^2+16(3\pi-10\alpha)\sin\alpha}{128a^2(\pi-2\alpha)}<\frac{11}{8a}\,,\quad\frac{\pi}{8}<\alpha<\frac{3\pi}{14}\,, \tag{4.4}$$

$$8a+19\sin\alpha<19\,,\qquad\qquad \frac{3\pi}{14}\leqq\alpha<\frac{\pi}{2}\,. \tag{4.5}$$

Let $q(\alpha)=\sin\alpha+\cos((7\alpha/4)+(\pi/8))=a$ and $r(\alpha)=(\sin\alpha)/(\pi-2\alpha)$, $0<\alpha<(\pi/2)$. Since $q'(\pi/8)<0$ and $q''(\alpha)=-\sin\alpha-(49/16)\cos((7\alpha/4)+(\pi/8))<0$ for $(\pi/8)<\alpha<(3\pi/14)$, $q(\alpha)$ is decreasing for $(\pi/8)<\alpha<(3\pi/14)$. Thus we have $a\geqq q(\pi/6)\geqq 0.758$ for $(\pi/8)<\alpha\leqq(\pi/6)$, and consequently $128a^2\geqq 73.544$. Moreover, we obtain the following inequality.

$$\frac{(3\pi-14\alpha)^2}{\pi-2\alpha}\leqq(3\pi-14\alpha)\Big(7-\frac{4\pi}{\pi-2\alpha}\Big)<\frac{25\pi}{12}<6.545\ \text{ for }\ \frac{\pi}{8}<\alpha\leqq\frac{\pi}{6}\,.$$

In addition, we have the estimate

$$\frac{16(3\pi-10\alpha)\sin\alpha}{\pi-2\alpha}\leqq16\Big(5-\frac{2\pi}{\pi-2\alpha}\Big)<23.673\,,$$

since $r'(\alpha)=((\pi-2\alpha)\cos\alpha+2\sin\alpha)/(\pi-2\alpha)^2>0$. By these estimates, the second term in the left-hand side of (4.4) is less than 0.411. From this and $1/2(1-\delta)\leqq1$, we can conclude that the left-hand side of (4.4) is less than 1.411. For the right-hand side of (4.4), we obtain $(11/8a) > 1.608$ since $a < p(\pi/8) < 0.855$. Thus (4.4) holds for $(\pi/8) < \alpha \leqq (\pi/6)$. Similarly we can prove (4.4) for the cases $(\pi/6) < \alpha \leqq (\pi/5)$ and $(\pi/5) < \alpha \leqq (3\pi/14)$.

Now we prove (4.5). Let $s(\alpha) = 27 \sin\alpha + 8\cos((7\alpha/4) + (\pi/8))$ and $S(\alpha) = s((\pi/2) - \alpha)$, $0 < \alpha < (\pi/2)$. It is sufficient to show

$$S(\alpha) < 19\,, \quad 0 < \alpha \leqq \frac{2\pi}{7}\,. \tag{4.6}$$

Since we have $S(0) = 19$, it is sufficient to prove $S'(\alpha) < 0$ for $0 < \alpha \leqq (2\pi/7)$. $S'(0) = 0$, $S''(0) < 0$, and $S'''(\alpha) < 0$ for $0 < \alpha \leqq (2\pi/7)$ imply $S''(\alpha) < 0$ and $S'(\alpha) < 0$ for $0 < \alpha \leqq (2\pi/7)$. From this, we obtain (4.6) and consequently (4.5).

Now, let $(\pi - 2\alpha)/2(1-\delta) \leqq h < (\pi + 2\alpha)/(1-\delta)$. Then, by (3.2), we have

$$x(t_0 + h) \geqq (\delta - 1)h + \frac{1-\delta}{2\sqrt{1-\delta^2}} > -\pi - 2\alpha\,. \tag{4.7}$$

This yields that if we choose k_1, $(1/2) < k_1 < 1$, sufficiently near 1, then minimal values of $x(t)$ are greater than $-(\pi + 2\alpha)k_1$ uniformly for $\phi \in K\backslash\{0\}$. For the estimate of maximal values, by (3.3),

$$x(t_2 + h) \leqq (1+\delta)h - \frac{\pi + 2\alpha - 2\cos\alpha}{2(1+\sin\alpha)}\,. \tag{4.8}$$

Thus for $(\pi-2\alpha)/2(1-\delta) \leqq h < (\pi-2\alpha)/(1+\sin\alpha) + (\pi+2\alpha-2\cos\alpha)/2(1+\sin\alpha)^2$, $(1+\delta)h - (\pi + 2\alpha - 2\cos\alpha)/2(1+\sin\alpha)$ is less than $\pi - 2\alpha$, and hence if we choose k_0, $(1/2) < k_0 < 1$, sufficiently near 1, then maximal values are less than $(\pi - 2\alpha)k_0$ uniformly for $\phi \in K\backslash\{0\}$ by (4.8). Clearly, since the above argument has no meaning for large α, we need an estimate for arbitrary α. For $(\pi-2\alpha)/2(1-\delta) \leqq h < 3(\pi-2\alpha)/8(1-\delta) + 1/(2\cos\alpha)$, minimal values of $x(t)$ are greater than $-(3(\pi-2\alpha)/8)$ by (4.7). If we choose $\mu = 3(\pi - 2\alpha)/8$ in Lemma 3.2, then we have $\max_{-\mu \leqq x \leqq 0} f(x) = \delta - \sin(-\mu + \alpha) = a$. We can easily prove $3(\pi-2\alpha)/8a < (\pi-2\alpha)/2(1-\delta)$ by the similar method to the one for (4.6). From this and (3.4), if $0 \leqq \alpha < (3\pi/14)$, then $\mu > \alpha$ and maximal values are estimated by

$$x(t_2 + h) \leqq ah - \frac{3(\pi - 2\alpha)}{8} + \frac{(3\pi - 14\alpha)^2 + 16(3\pi - 10\alpha)\sin\alpha}{128a}\,. \tag{4.9}$$

Thus for

$$\frac{\pi - 2\alpha}{2(1-\delta)} \leqq h < \frac{11(\pi - 2\alpha)}{8a} - \frac{(3\pi - 14\alpha)^2 + 16(3\pi - 10\alpha)\sin\alpha}{128a^2}\,,$$

the right-hand side of (4.9) is less than $\pi - 2\alpha$, and therefore, if we choose k_0, $(1/2) < k_0 < 1$, sufficiently near 1, maximal values are less than $(\pi - 2\alpha)k_0$ uniformly for $\phi \in K\backslash\{0\}$. Similarly, maximal values for $(3\pi/14) \leqq \alpha < (\pi/2)$ are estimated by

$$x(t_2 + h) \leqq ah - \mu + \int_{-\mu/a}^{0} p(as)ds \leqq ah - \frac{3(\pi - 2\alpha)}{16} . \tag{4.10}$$

Hence for $(\pi - 2\alpha)/2(1 - \delta) \leqq h < (19(\pi - 2\alpha)/16a)$, if we choose k_0, $(1/2) < k_0 < 1$, sufficiently near 1, maximal values are less than $(\pi - 2\alpha)k_0$ uniformly for $\phi \in K\backslash\{0\}$. Since we can change k_1 into a greater one if necessary, we can assume $k_0 < k_1 < 1$. Also $x_{\tau(\phi)} \in K$ and we have $2h < \tau(\phi) < (m + n + 8)h$ by Lemma 3.2.

REMARK. Since $f(x)$ satisfies (H2) by setting $p(x) = f(x)$, we can change $F(\alpha)$ in (4.3) into a greater one, namely,

$$F_1(\alpha) = \frac{11(\pi - 2\alpha)}{8a} - \frac{3(\pi - 2\alpha)\sin\alpha + 8\cos\alpha - 8\cos\left(\dfrac{14\pi - 3\alpha}{8}\right)}{8a^2} . \tag{4.11}$$

We have the following theorem by Theorem 3.1 and Lemma 4.1.

THEOREM 4.1. *Let $(\pi - 2\alpha)/2(1 - \delta) \leqq h < H(\alpha)$ be fixed, where $(\pi/2) < h < H(0)$ for $\alpha = 0$. Then (4.1) has a nontrivial periodic solution for $-\pi < u < \pi$. Its period is greater than $2h$ and less than $(m + n + 8)h$.*

REMARK. (i) In particular, for $\delta = 0.3$ and $h = 2$, Ueda and his colleagues have observed the existence of a nontrivial periodic solution. We can conclude from Theorem 4.1 that there exists a nontrivial periodic solution for $\delta = 0.3$, $1.81 \leqq h \leqq 2.45$, and $h = 2$, $|\delta| \leqq 0.445$.

(ii) It is easy to see that (4.1) has a nontrivial periodic solution for $-\pi < u < \pi$ if $\pi/(2\cos\alpha) < h < H_1(\alpha)$, where $H_1(\alpha)$ corresponds to $F_1(\alpha)$ in (4.4).

5. Another application. Consider the difference-differential equation

$$\dot{u}(t) = \delta - g(u(t - h)) , \qquad t \geqq 0 , \quad 0 \leqq \delta < 1 , \tag{5.1}$$

where $g(x) = (1/2)(|x + 1| - |x - 1|)$. Since $g(\delta) = \delta$ for $0 \leqq \delta < 1$, (5.1) has a constant solution $u(t) \equiv \delta$. Substituting $x(t) = u(t) - \delta$, $t \geqq -h$ into (5.1), we have an equivalent equation

$$\dot{x}(t) = \delta - g(x(t - h) + \delta) , \qquad t \geqq 0 , \quad 0 \leqq \delta < 1 , \tag{5.2}$$

which has the zero solution $x(t) \equiv 0$. For a fixed $h > (\pi/2)$, let $X_1 = X_2 < (2h - 1)(1 + \delta)/2$, $A_1 = B_1 = 1 - \delta$. Then $f(x) = \delta - g(x + \delta)$ satisfies

(H1) and (H2) by setting $p(x) = f(x)$. As the linear part of $f(x)$ is $-x$, $f(x)$ satisfies (H3) also for $L = 1$. We consider the truncated cone such that

$$K = \{\phi \in C: \phi(-h) = 0,\ \phi(\theta) \text{ is nondecreasing on } [-h, 0],\ \phi(0) \leqq X_1\} .$$

We have the following theorem from Theorem 3.1.

THEOREM 5.1. *If* $h > (\pi/2)$, *then* (5.1) *has a nontrivial periodic solution.*

It is not difficult to prove this theorem. Since $h > (\pi/2) > (A_2/B_2)$, we can easily prove by the similar method to the one for Lemma 3.1 that $x_{\tau(\phi)}(\phi) \in K$ for a $\tau(\phi) > 0$, where $\{\tau(\phi)\}$ is uniformly bounded for $\phi \in K \backslash \{0\}$. Also, the maximal value $x(\tau(\phi))$ is estimated by

$$x(\tau(\phi)) \leqq \int_{-h}^{-1} (1 + \delta) ds + \int_{-1}^{0} f((1 + \delta)s) ds \leqq \frac{(2h - 1)(1 + \delta)}{2} < X_1 .$$

Moreover, since $h > (\pi/2L) = (\pi/2)$, Lemmas 3.3, 3.4, and 3.5 hold. Thus all assumptions of Theorem 3.1 are satisfied, and hence we have Theorem 5.1.

6. Stability of a constant solution. In this section, using a theorem of Yorke [6], we consider the nonexistence of nontrivial periodic solutions of (2.1) for $-X_1 < x < X_2$ and the uniform asymptotic stability of the zero solution of (2.1) when h is smaller than that in Theorem 3.2. Furthermore, we consider the nonexistence of periodic solutions of the second kind when $f(x)$ can be extended as a continuous periodic function with period $X_1 + X_2$, where $x(t)$ is called a periodic solution of the second kind if there exist $X \neq 0$ and $T > 0$ such that $x(t + T) = x(t) + X$ for $t \geqq 0$.

Consider a nonlinear one-dimensional differential-delay equation

$$\dot{x}(t) = F(t, x_t) , \qquad t \geqq 0 . \tag{6.1}$$

Let $C_\beta = \{\phi \in C: |\phi| < \beta\}$ and let $F: [0, \infty) \times C_\beta \to R$ be continuous.

DEFINITION 6.1. We say 0 is uniformly stable for (6.1) if for any $\eta > 0$ there exists a $\rho = \rho(\eta)$ in $(0, \eta]$ such that for any $t_0 \geqq 0$ and $\phi \in C_\beta$ and any solution $x(t) = x(t, t_0, \phi)$ we have for all $t > t_0$ in the domain of $x(t)$

$$|\phi| < \rho \text{ implies } |x(t, t_0, \phi)| < \eta .$$

DEFINITION 6.2. Let $\gamma > 0$. We say 0 is uniformly asymptotically stable with attraction radius γ for (6.1) if

(i) 0 is uniformly stable,

(ii) for each $t_0 \geqq 0$, each solution $x(t, t_0, \phi)$ with $|\phi| \leqq \gamma$ exists for all $t \geqq t_0$,

(iii) there exists $T = T(\gamma_1)$ for each $\gamma_1 \in (0, \gamma)$ such that for each $t_0 \geqq 0$ and each solution $x(t)$ of (6.1) with $|\phi| \leqq \gamma_1$, $|x(t_0 + s)| \leqq (\gamma_1/2)$ for all $s \geqq T(\gamma_1)$.

For $\phi \in C_\beta$, define $M(\phi) = \max\{0, \sup_{-h \leq \theta \leq 0} \phi(\theta)\}$. The following theorem can be found in [6].

THEOREM 6.1 (Yorke). *Let $\beta > 0$ and $h > 0$. Let $F: [0, \infty) \times C_\beta \to R$ be continuous. Assume for some $c \geqq 0$*

$$-cM(\phi) \leqq F(t, \phi) \leqq cM(-\phi) \quad \text{for all} \quad \phi \in C_\beta . \tag{6.2}$$

(i) *Assume $ch \leqq (3/2)$. Then $x(t) \equiv 0$ is a solution and is uniformly stable.*

(ii) *Assume $0 < ch < (3/2)$ and*

$$\begin{cases} \text{for all sequences } t_n \to \infty \text{ and } \phi_n \in C_\beta \text{ converges to a constant nonzero} \\ \text{function in } C_\beta,\ F(t_n, \phi_n) \text{ does not converge to } 0. \end{cases} \tag{6.3}$$

Then 0 is uniformly asymptotically stable, and if $t_0 \geqq 0$ and $|\phi| \leqq (2\beta/5)$, then $x(t) \to 0$ as $t \to \infty$.

REMARK. (i) and (ii) can be made more specific as follows [6].

$$\begin{cases} \text{If } t_0 \geqq 0 \text{ and } |\phi| < (2\beta/5), \text{ then the solution } x(t) \text{ is defined and} \\ \text{satisfies } |x(t)| \leqq (5/2)|\phi| \text{ for all } t \geqq t_0. \end{cases} \tag{6.4}$$

$$\begin{cases} V(t) \overset{\text{def}}{=} \sup\limits_{t \leqq s \leqq t+3h} |x(s)| \text{ is a monotonic nonincreasing function for} \\ t \geqq t_0, \text{ and if } 0 < ch < (3/2), \text{ then } V(t) \to 0. \end{cases} \tag{6.5}$$

Let $h < (A_2/B_2)$, $\xi_0 = \min(X_1, X_2)$, $\xi_1 = \max\left(-X_1, \int_{-h}^{0} p(-B_2 s)ds\right)$, and let $\xi_2 = \min\left(X_2, (A_2B_1/B_2) - A_1 + (1/B_1)\int_{-A_1}^{0} p(s)ds\right)$. Let $c_1, c_2 \geqq (B_2/A_2)$ satisfy

$$\frac{f(x)}{x} \geqq -c_1 \quad \text{for all } x \neq 0 \text{ in } [\xi_1, \xi_2] \tag{6.6}$$

and

$$\frac{f(x)}{x} \geqq -c_2 \quad \text{for all } x \neq 0 \text{ in } [-\xi_0, \xi_0] . \tag{6.7}$$

Then we have the following results as a corollary of Theorem 6.1.

COROLLARY 6.1. (i) *If $h < 3/(2c_1)$, then* (2.1) *has no nontrivial* *periodic solution for* $-X_1 < x < X_2$.

(ii) *If* $h < (3/2c_2)$, *then for any* γ, $0 < \gamma < (2\xi_0/5)$, 0 *is uniformly asymptotically stable with attraction radius* γ *for* (2.1).

(iii) *Suppose that* $f(x)$ *can be extended as a continuous periodic function with period* $X_1 + X_2$. *If* $h < \min(3/(2c_2), 4\xi_0/(5B_1), 4\xi_0/(5B_2))$, *then* (2.1) *has no periodic solution of the second kind.*

PROOF. (i) Let $h < (3/2c_1)$. Let $q(t)$ be a nontrivial periodic solution of (2.1) for $-X_1 < x < X_2$, and let ζ_1, ζ_2 be numbers such that $\xi_1 < \zeta_1$, $\zeta_2 < \xi_2$, $\zeta_1 \leqq q(t) \leqq \zeta_2$ for $t \geqq 0$. If we take a positive number η, $0 < \eta < \min(\zeta_1 - \xi_1, \xi_2 - \zeta_2)$, then from (6.6), we have $(f(x)/x) \geqq -c_1$ for all $x \neq 0$ in $[\zeta_1 - \eta, \zeta_2 + \eta]$. If we define

$$f_1(x) = \begin{cases} f(\zeta_1 - \eta), & x < \zeta_1 - \eta, \\ f(x), & \zeta_1 - \eta \leqq x \leqq \zeta_2 + \eta, \\ f(\zeta_2 + \eta), & x > \zeta_2 + \eta, \end{cases}$$

then clearly $xf_1(x) < 0$ for $x \neq 0$, $f_1(0) = 0$, and hence $f_1(x)$ satisfies (6.3) for $\beta = (5/2)\max(\eta - \zeta_1, \zeta_2 + \eta)$, and for $c = c_1$, we have

$$\frac{f_1(x)}{x} \geqq -c, \qquad x \neq 0. \tag{6.8}$$

Moreover, $q(t)$ is a solution of the equation

$$\dot{x}(t) = f_1(x(t-h)), \qquad t \geqq 0. \tag{6.9}$$

Since we have $f_1(\phi) = f_1(\phi(-h))$, it is clear that (6.2) and (6.8) are equivalent. From this and Theorem 6.1, 0 is uniformly asymptotically stable for (6.9). On the other hand, since $|q_0| < (2\beta/5)$, we have $q(t) \to 0$ as $t \to \infty$, and this is a contradiction. Therefore, if $h < (3/2c_1)$, then (2.1) has no periodic solution for $-X_1 < x < X_2$.

(ii) For $0 < \gamma < (2\xi_0/5)$, let

$$f_2(x) = \begin{cases} f\left(-\dfrac{5\gamma}{2}\right), & x < -\dfrac{5\gamma}{2}, \\ f(x), & -\dfrac{5\gamma}{2} \leqq x \leqq \dfrac{5\gamma}{2}, \\ f\left(\dfrac{5\gamma}{2}\right), & x > \dfrac{5\gamma}{2}, \end{cases}$$

and let $\beta = (5\gamma/2)$. Then, using Theorem 6.1, in a similar way to the above, we can show that if $h < 3/(2c_2)$, then 0 is uniformly asymptotically stable with attraction radius γ for (6.9). On the other hand, since $|x(t, \phi)| \leqq (5\gamma/2) < \xi_0$ for all $t \geqq 0$ if $|\phi| \leqq \gamma$, the solution $x(t, \phi)$ of (6.9) for $|\phi| \leqq \gamma$ is also a solution of (2.1), and thus we obtain the same stability for (2.1).

(iii) Let $h < \min(3/(2c_2), 4\xi_0/(5B_1), 4\xi_0/(5B_2))$ be fixed. If we choose γ, $0 < \gamma < (2\xi_0/5)$, sufficiently near $(2\xi_0/5)$, then we have $h < \min(2\gamma/B_1, 2\gamma/B_2)$. Now, let $x(t, \phi)$ be a positive solution of the second kind of (2.1) such that $x(t, \phi) \to \infty$ as $t \to \infty$, and let n be the smallest integer for which we have $|\phi| \leqq -\gamma + n(X_1 + X_2)$. If we let $t_0 = \inf\{t: x(t, \phi) = \gamma + n(X_1 + X_2)\}$, then $|x(t) - n(X_1 + X_2)| \leqq 2\gamma$ for $t_0 - h \leqq t \leqq t_0$ because $\dot{x}(t) \leqq B_1$. On the other hand, since γ is an attraction radius of 0 for (2.1) by (ii) and since the extended $f(x)$ is a periodic function with period $X_1 + X_2$, γ is also an attraction radius of $n(X_1 + X_2)$ for (2.1), and hence we must have $x(t, \phi) \to n(X_1 + X_2)$ as $t \to \infty$. This contradicts the fact that $x(t, \phi) \to \infty$ as $t \to \infty$.

In a similar way to the above, we have a contradiction also in the case $x(t, \phi) \to -\infty$ as $t \to \infty$. Thus we can conclude that (2.1) for the extended $f(x)$ has no periodic solution of the second kind.

7. Applications of Corollary 6.1. 1. For equation (4.2), let $f(x) = \delta - \sin(x + \alpha)$. Let $y = -c(\alpha)x$ be tangential to the curve $y = f(x)$ at the point $(-\xi(\alpha), f(-\xi(\alpha)))$ for $-(\pi/2) - \alpha < -\xi(\alpha) < -\alpha$. Moreover, let $\zeta_1(\alpha) = \min(\xi(\alpha), (1 - \sin\alpha)/(\cos\alpha))$, $\zeta_2(\alpha) = \min(\xi(\alpha), \pi - 2\alpha)$, $H(x) = 3x/(2f(-x))$ $(x \neq 0)$, and let $H(0) = 3/2$. We obtain the following proposition by applying Corollary 6.1.

PROPOSITION 7.1. (i) *If $h < H(\zeta_1(\alpha))$, then (4.2) has no nontrivial periodic solution for $-\pi - 2\alpha < x < \pi - 2\alpha$.*

(ii) *If $h < H(\zeta_2(\alpha))$, then for any γ, $0 < \gamma < (2(\pi - 2\alpha)/5)$, 0 is uniformly asymptotically stable with attraction radius γ for (4.2).*

(iii) *If $h < \min(H(\zeta_2(\alpha)), (4(\pi - 2\alpha)/5(1 + \sin\alpha)))$, then (4.2) has no periodic solution of the second kind.*

PROOF. It is easy to see that $H(\zeta_1(\alpha)) \leqq 3/(2\cos\alpha)$. First, we prove that any periodic solution of (4.2) in $-\pi - 2\alpha < x < \pi - 2\alpha$ is greater than $(\sin\alpha - 1)/\cos\alpha$ if $h < 3/(2\cos\alpha)$. Now let $q(t)$ be a nontrivial periodic solution of (4.2) such that $-\pi - 2\alpha < q(t) < \pi - 2\alpha$ for $t \geqq 0$, and let $q(t_1)$ be its minimum value. Clearly $q(t_1) < 0$. By changing t_1 into a larger one if necessary, we may assume that $t_0 = \sup\{t < t_1: q(t) = 0\} \geqq 0$. For $\phi = \phi(\theta) = (\delta - 1)\theta$, $-h \leqq \theta \leqq 0$, we have $q(t_0 + \theta) \leqq \phi(\theta) < (\pi/2) - \alpha$ for all $\theta \in [-h, 0]$ since $3/(2\cos\alpha) < (\pi - 2\alpha)/2(1 - \delta)$ from (4.3), and consequently $q(t_1) \geqq x(h, \phi)$. It is sufficient to show $x(h, \phi) > (\sin\alpha - 1)/\cos\alpha$ only for $(1/\cos\alpha) \leqq h < 3/(2\cos\alpha)$. In this case, we have

$$x(h,\phi) > \int_{-h}^{-1/\cos\alpha}(\delta-1)ds + \int_{-1/\cos\alpha}^{0}\{-\sqrt{1-\delta^2}(\delta-1)s\}ds \geqq \frac{\sin\alpha-1}{\cos\alpha},$$

and thus $q(t_1) > (\sin\alpha - 1)/\cos\alpha$. Let k_0, $0 < k_0 < 1$, be a number such that $q(t) < (\pi - 2\alpha)k_0$, and let

$$f_1(x) = \begin{cases} f\left(\dfrac{\sin\alpha-1}{\cos\alpha}\right), & x < \dfrac{\sin\alpha-1}{\cos\alpha}, \\ f(x), & \dfrac{\sin\alpha-1}{\cos\alpha} \leqq x \leqq (\pi-2\alpha)k_0, \\ f((\pi-2\alpha)k_0), & x > (\pi-2\alpha)k_0. \end{cases}$$

Since $xf_1(x) < 0$ for $x \neq 0$ and $f_1(0) = 0$, $f_1(x)$ satisfies (6.3) for $\beta = (5/2)\max((1-\sin\alpha)/\cos\alpha, (\pi-2\alpha)k_0)$. Furthermore, $q(t)$ also is a solution of $\dot{x}(t) = f_1(x(t-h))$. Let $c = (f(-\zeta_1(\alpha))/\zeta_1(\alpha))$. Then clearly $c > 0$ and $(f_1(x)/x) \geqq -c$ for $x \neq 0$. From this, $f_1(\phi) = f_1(\phi(-h))$ satisfies (6.2) for the same c. Therefore, if $0 < h < H(\zeta_1(\alpha))$, then (4.2) has no nontrivial periodic solution in $-\pi - 2\alpha < x < \pi - 2\alpha$ by Corollary 6.1.(i). Similarly, (ii) and (iii) hold by (ii) and (iii) of Corollary 6.1, respectively.

REMARK 1. Since $(1-\sin\alpha)/\cos\alpha \leqq (\pi-2\alpha)/\pi \leqq \pi - 2\alpha$ by (4.3), we have $\zeta_1(\alpha) \leqq \zeta_2(\alpha) \leqq \xi(\alpha)$ and consequently $H(\zeta_1(\alpha)) \geqq H(\zeta_2(\alpha))$. On the other hand, for any given $h > 0$, α, $0 < \alpha < (\pi/2)$, sufficiently near $(\pi/2)$, satisfies conditions in (i) and (ii), since $\lim_{\alpha\to\pi/2-0} H(\zeta_2(\alpha)) = \infty$.

REMARK 2. In the above proof, since $0 < c \leqq (\sin\alpha + 1)/(\alpha + 1)$, (6.2) holds for $c = (\sin\alpha+1)/(\alpha+1)$. Hence if $0 < h < 3(\alpha+1)/(2(\sin\alpha+1))$, then (i) and (ii) hold. Moreover, since $\xi(\alpha) \geqq \alpha$ and $(1-\sin\alpha)/\cos\alpha$ is decreasing, if we let $\alpha_0\cos\alpha_0 = 1 - \sin\alpha_0$, $(\pi/6) < 0.555 < \alpha_0 < 0.556 < (\pi/5)$, then $\zeta_1(\alpha) = (1-\sin\alpha)/\cos\alpha \leqq \alpha$ and consequently $H(\zeta_1(\alpha)) \geqq H(\alpha) = 3\alpha/(2\sin\alpha)$ for $\alpha_0 \leqq \alpha < (\pi/2)$. Therefore, if $0 < h < 3\alpha/(2\sin\alpha)$, then the condition in (i) holds for $\alpha_0 \leqq \alpha < (\pi/2)$. Similarly, if $0 < h < 3\alpha/(2\sin\alpha)$, then the condition in (ii) holds for $\alpha \geqq (\pi/3)$.

REMARK 3. If $0 < h < 3/(2\cos\alpha)$, then for sufficiently small $\gamma > 0$, 0 is uniformly asymptotically stable with attraction radius γ for (4.2).

2. For equation (5.2), if we let $f(x) = \delta - g(x+\delta)$, then $f(\phi) = f(\phi(-h))$ satisfies (6.2) and (6.3) for $c = 1$. Define $G(x)$ by

$$G(x) = \begin{cases} g(x), & -1 \leqq x \leqq 1, \\ g(2-x), & 1 < x < 3, \end{cases}$$

and extend $G(x)$ as a continuous periodic function with period 4. We denote this extended function by $G(x)$ again. For $F(x) = \delta - G(x+\delta)$, consider

$$\dot{x}(t) = F(x(t-h)) , \qquad t \geqq 0 . \tag{7.1}$$

For equations (5.2) and (7.1), we describe the following proposition without proof.

PROPOSITION 7.2. (i) *If* $h < (3/2)$, *then* 0 *is globally uniformly asymptotically stable for* (5.2). *Moreover, for any* γ, $0 < \gamma < (2(1-\delta)/5)$, 0 *is uniformly asymptotically stable with attraction radius* γ *for* (7.1).

(ii) *If* $h < (3/2)$, *then* (5.2) *has no nontrivial periodic solution, and* (7.1) *has no nontrivial periodic solution in* $-1-\delta < x < 1-\delta$.

(iii) *If* $h < \min((3/2), (4(1-\delta)/(5(1+\delta))))$, *then* (7.1) *has no periodic solution of the second kind.*

8. Existence of periodic solutions of the second kind.

In this section, we assume that $f(x)$ of (2.1) is a continuous and periodic function with period $X_1 + X_2$. Consider the equation

$$\dot{x}(t) = f(x(t-h)) + B , \qquad t \geqq 0 , \quad B > B_2 . \tag{8.1}$$

Employing the following theorem, which is found in [8], we show the existence of periodic solutions of the second kind for (2.1) and (8.1).

THEOREM 8.1. *Suppose* K *is a cone (or a truncated cone),* A *is positive with respect to* K, *is completely continuous and* F *is the set of positive eigenvectors of* A. *If*

(i) *for any open set* $G \subset C$, $0 \in G$,

$$\inf_{\phi \in \partial G \cap K} |A\phi| > 0 ,$$

(ii) *there exists an* $M > 0$ *such that* $\phi \in F$, $|\phi| = M$, $A\phi = \mu\phi$ *imply* $\mu < 1$,

(iii) *there exists an open neighborhood* H *of zero,* $\bar{H} \subset B(M)$, *such that* $\phi \in \partial H \cap F$, $A\phi = \mu\phi$ *imply* $\mu > 1$,
then A *has a fixed point in* $K \cap (B(M) \backslash \bar{H})$.

First, we consider equation (2.1). Let ξ_0, ξ_1, $-X_1 < \xi_0 < \xi_1 < 0$, be fixed, and let

$$K_0 = \{\psi \in C: \psi(-h) = \xi_0, \psi(\theta) \text{ is nondecreasing on } [-h, 0], \psi(0) \leqq \xi_1\} ,$$

and

$$K = \{\phi \in C: \phi = \psi - \zeta^{\xi_0} \text{ for some } \psi \in K_0\} ,$$

where $\zeta^\nu(\theta) = \nu$, $-h \leqq \theta \leqq 0$. Then K is a truncated cone. We assume the following conditions.

(I) Any solution $x(t, \psi)$ of (2.1) reaches $\xi_0 + X_1 + X_2$ in finite time uniformly for $\psi \in K_0$.

(II) For $t_0 = \inf\{t: x(t, \psi) = \xi_0 + X_1 + X_2\}$, $X_2 \leqq x(t, \psi) \leqq \xi_0 + X_1 + X_2$ for $t_0 - h \leqq t \leqq t_0$.

(III) For k_0, $0 < k_0 < 1$, sufficiently near 1, $x(t_0 + h, \psi) - x(t_0, \psi) \leqq (\xi_1 - \xi_0)k_0$ uniformly for $\psi \in K_0$.

For $\phi \in K$ and $\psi = \phi + \zeta^{\xi_0}$, let $\tau(\phi) = t_0 + h$, and let $A: \phi \longrightarrow \tilde{\phi} = x_\tau(\psi) - \zeta^{\xi_0 + X_1 + X_2}$. Under assumptions (I), (II), and (III), A is a positive mapping with respect to K, is continuous and takes closed bounded sets into bounded sets, and we have the following lemma.

LEMMA 8.1. *Under assumptions* (I) *and* (II), *there exists an* $\eta > 0$ *such that* $|A\phi| \geqq \eta$ *uniformly for* $\phi \in K$.

PROOF. Suppose not. Then there is a subsequence $\{\phi_n\}$ in K such that $|A\phi_n| \longrightarrow 0$ as $n \longrightarrow \infty$. We can assume that $\tau(\phi_n) \longrightarrow \tau_0 > 2h$ as $n \longrightarrow \infty$. Since $\{x(t, \psi_n)\}$, $\psi_n = \phi_n + \zeta^{\xi_0}$, is uniformly bounded and equicontinuous on $[0, \tau_0]$, there exists a uniformly convergent subsequence. For the simplicity, we assume $x(t, \psi_n) \longrightarrow y(t)$ uniformly for $t \in [0, \tau_0]$ as $n \longrightarrow \infty$. Then $y(t)$ is a solution of (2.1) on $[h, \tau_0]$. Since $\dot{y}(t) = 0$ for $\tau_0 - h \leqq t \leqq \tau_0$, we have $f(y(s)) = 0$, namely, $y(s) \leqq X_2$, for $\tau_0 - 2h \leqq s \leqq \tau_0 - h$. This contradicts the continuity of $y(t)$ on $[0, \tau_0]$, and hence the conclution holds.

Under assumptions (I), (II), and (III), condition (i) in Theorem 8.1 is satisfied by Lemma 8.1. Condition (ii) is satisfied for $M > (\xi_1 - \xi_0)k_0$, and condition (iii) holds for $H = B(\eta/2)$ by Lemma 8.1. Thus we have the following theorem.

THEOREM 8.2. *Under assumptions* (I), (II), *and* (III), (2.1) *has a periodic solution of the second kind.*

Next we consider equation (8.1). If we define

$$K_1 = \{\psi \in C: \psi(-h) = A_2,\ \psi(\theta) \text{ is nondecreasing on } [-h, 0],\ \phi(0) \leqq X_1 + X_2 - A_1\},$$

and

$$K = \{\phi \in C: \phi = \psi - \zeta^{A_2} \text{ for some } \psi \in K_1\},$$

then K is a truncated cone.

LEMMA 8.2. *Any solution* $x(t, \psi)$ *of* (8.1) *for* $\psi \in K_1$ *is increasing, and reaches* $X_1 + X_2 + A_2$ *till the time* $(X_1 + X_2)/(B - B_2)$. *Moreover, if*

$$(8.2) \qquad (A_1 + A_2)(B + B_1) + \int_{-A_1}^{0} p(s)ds < B_1(X_1 + X_2) + \frac{A_2 B_2}{2}$$

and if

$$(8.3) \qquad h < \frac{1}{B+B_1}\left(X_1 + X_2 + \frac{A_2B_2}{2B_1} - \frac{1}{B_1}\int_{-A_1}^{0} p(s)ds\right),$$

then for k_0, $0 < k_0 < 1$, *sufficiently near* 1, *we have*

$$(B - B_2)h \leqq x(t_0 + h, \psi) - x(t_0, \psi) \leqq (X_1 + X_2 - A_1 - A_2)k_0$$

uniformly for $\psi \in K_1$, *where* $x(t_0, \psi) = X_1 + X_2 + A_2$.

PROOF. Since $\dot{x}(t) \geqq B - B_2 > 0$, it is clear that $x(t) = x(t, \psi)$ reaches $X_1 + X_2 + A_2$ till the time $(X_1 + X_2)/(B - B_2)$. Let $x(t_0) = X_1 + X_2 + A_2$. By (8.2), there exists an h which satisfies $h = (A_1 + A_2)/B_1$ and (8.3), and for such an h, we have

$$\begin{aligned} x(t_0 + h) - x(t_0) &\leqq Bh + \int_{-h}^{-(A_1+A_2)/B_1} B_1 ds + \int_{-(A_1+A_2)/B_1}^{-A_2/B_1} p(B_1 s + A_2)ds \\ &\quad + \int_{-A_2/B_1}^{0}\left\{-\frac{B_2}{A_2}(B_1 s + A_2)\right\}ds \\ &\leqq (B + B_1)h - A_1 - A_2 + \frac{1}{B_1}\int_{-A_1}^{0} p(s)ds \\ &\quad - \frac{A_2B_2}{2B_1} < X_1 + X_2 - A_1 - A_2 . \end{aligned}$$

From this, for k_0, $0 < k_0 < 1$, sufficiently near 1, we obtain $x(t_0 + h) - x(t_0) \leqq (X_1 + X_2 - A_1 - A_2)k_0$ uniformly for $\psi \in K_1$. We have the same estimate for $h < (A_1 + A_2)/B_1$. Finally, it holds clearly that $x(t_0 + h) - x(t_0) \geqq (B - B_2)h$ uniformly for $\psi \in K_1$.

For $\phi \in K$ and $\psi = \phi + \zeta^{A_2}$, let $\tau(\phi) = t_0 + h$, and let $A: \phi \to \tilde{\phi} = x_\tau(\psi) - \zeta^{X_1+X_2+A_2}$. Under the conditions of Lemma 8.2, A is a positive mapping with respect to K, is continuous and takes closed bounded sets into bounded sets. Furthermore, condition (i) in Theorem 8.1 is satisfied from $|A\phi| \geqq (B - B_2)h > 0$ uniformly for $\psi \in K_1$. (ii) and (iii) hold for $M > (X_1 + X_2 - A_1 - A_2)k_0$ and $H = B(\eta)$, $0 < \eta < (B - B_2)h$, respectively. Thus we obtain the following theorem.

THEOREM 8.3. *Under the conditions of Lemma* 8.2, (8.1) *has a periodic solution of the second kind.*

9. Applications. First, consider equation (4.2) for $0 < \delta \leqq 1$. Let $-(\pi/2) - 2\alpha < \xi_0(\alpha) < -(\pi/2) - \alpha$. For a fixed α, consider K_0 and the corresponding K in Section 8 for $\xi_0 = \xi_0(\alpha)$ and $\xi_1 = -(\pi/2)$. The following lemma holds.

LEMMA 9.1. *There exist* $\xi_0 = \xi_0(\alpha)$ *and* δ_0, $0 < \delta_0 < 1$, *such that for* $\delta_0 \leqq \delta \leqq 1$,

$$(9.1)\quad \frac{\pi}{2(\delta+\sqrt{1-\delta^2})} < \min\left(\frac{\xi_0+2\pi}{1+\delta}+\frac{\pi+2\alpha-2\cos\alpha}{2(1+\delta)^2}, \frac{\pi+2\xi_0}{2(\sin(\xi_0+\alpha)-\delta)}\right),$$

and

$$(9.2)\quad \pi+\frac{6(\pi-2\alpha)(1-\delta)}{3\delta+\pi-3\sqrt{3}} < 2\alpha+\frac{(\pi-2\alpha+\delta)\delta}{2(\delta+\sqrt{1-\delta^2})}.$$

Furthermore, if h satisfies

$$(9.3)\quad \frac{\pi}{2(\delta+\sqrt{1-\delta^2})} \leqq h < \min\left(\frac{\xi_0+2\pi}{1+\delta}+\frac{\pi+2\alpha-2\cos\alpha}{2(1+\delta)^2}, \frac{\pi+2\xi_0}{2(\sin(\xi_0+\alpha)-\delta)}\right),$$

then any solution $x(t, \psi)$ of (4.2) reaches $\xi_0+2\pi$ in finite time uniformly for $\psi\in K_0$. Moreover, $\pi-2\alpha\leqq x(t,\psi)\leqq\xi_0+2\pi$ for $t_0-h\leqq t\leqq t_0$, where $t_0=\inf\{t: x(t,\psi)=\xi_0+2\pi\}$, and consequently $x_{t_0+h}(\theta)$ is nondecreasing, and for some $\eta=\eta(\delta,h)>0$ and k_0, $0<k_0<1$, sufficiently near 1, we have $\eta\leqq x(t_0+h,\psi)-x(t_0,\psi)\leqq-((\pi+2\xi_0)k_0/2)$ uniformly for $\psi\in K_0$.

PROOF. Since (9.1) and (9.2) hold for $\xi_0=-(\pi/2)-(7\alpha/4)$ and $\delta=1$, for δ_0, $0<\delta_0<1$, sufficiently near 1, they do for $\delta_0\leqq\delta\leqq 1$ also. Any solution $x(t)=x(t,\psi)$, $\psi\in K_0$, reaches $-(\pi/2)$ till the time $2\alpha/(\delta+\sqrt{1-\delta^2})$. Since $\dot{x}(t)\geqq\delta+\sqrt{1-\delta^2}$ for $t_1\leqq t\leqq t_1+h$, where $t_1=\inf\{t: x(t)=-(\pi/2\}$, if h satisfies (9.3), then we obtain $x(t_1+h)\geqq 0$. In the case $\delta=1$, $x(t)$ is increasing and clearly reaches $\xi_0+2\pi$ in finite time uniformly for $\psi\in K_0$. Next, let $\delta_0\leqq\delta<1$ and $t_2=\inf(t: x(t)=0\}$. Then $x(t_2+h)$ is a maximal value, and it is estimated by (4.8) for $h\geqq(\pi+2\alpha)/2(1+\delta)$. For $h<(\pi+2\alpha)/2(1+\delta)$, we have

$$(9.4)\quad \begin{aligned} x(t_2+h) &\leqq \int_{-(\pi+2\alpha)/2(1+\delta)}^{0} f((1+\delta)s)ds \leqq \frac{(\pi+2\alpha)\delta+2\cos\alpha}{2(1+\delta)} \\ &\leqq \frac{3\pi}{2}-2\alpha<\xi_0+2\pi. \end{aligned}$$

Thus it follows from (4.8) and (9.4) that $x(t_2+h)<\xi_0+2\pi$. On the other hand, by (9.2), and (9.3) we have

$$(9.5)\quad \begin{aligned} x(t_2+h) &\geqq \int_{-h}^{-\pi/(2a)} a\,ds+\int_{-\pi/(2a)}^{0} f(x(s+t_2))ds \\ &> ah-\frac{\pi}{2}+\int_{-\pi/(2a)}^{-\alpha/a}\delta\,ds+\int_{-\alpha/a}^{-(\alpha-\delta)/a}(-as-\alpha+\delta)ds \\ &\geqq \frac{(\pi-2\alpha+\delta)\delta}{2a}\geqq\pi-2\alpha, \end{aligned}$$

where $a=\delta+\sqrt{1-\delta^2}$. From (9.5), since $f(x(t-h))>0$ for t such that $\delta-\alpha\leqq x(t-h)<0$, we have

$$\dot{x}(t) \geqq f(\delta - \alpha) \geqq \frac{3\delta + \pi - 3\sqrt{3}}{6},$$

for t such that $0 \leqq x(t) \leqq \pi - 2\alpha$. Thus if $\delta > (3\sqrt{3} - \pi)/3$, then $x(t)$ passes through the strip region $0 \leqq x \leqq \pi - 2\alpha$ during the time interval of length $(6(\pi - 2\alpha)/(3\delta + \pi - 3\sqrt{3}))$. If we let $t_3 = \inf\{t: x(t) = \pi - 2\alpha\}$, then $x(t_3 + h)$ is a maximal value. Since we have $\dot{x}(t) \geqq \delta - 1$, it follows from (9.2) and (9.5) that

$$\begin{aligned} x(t_3 + h) &\geqq x(t_2 + h) + \int_{t_2+h}^{t_3+h} \dot{x}(s)ds \\ &> \frac{(\pi - 2\alpha + \delta)}{2(\delta + \sqrt{1 - \delta^2})} - \frac{6(\pi - 2\alpha)(1 - \delta)}{2\delta + \pi - 3\sqrt{3}} \geqq \pi - 2\alpha . \end{aligned} \tag{9.6}$$

Thus there exists an $\varepsilon > 0$ uniformly for $\psi \in K_0$ such that

$$x(t_3 + h) \geqq \pi - 2\alpha + \varepsilon . \tag{9.7}$$

Therefore $x(t)$ reaches $\xi_0 + 2\pi$ in finite time uniformly for $\psi \in K_0$, and $x_{t_0+h}(\theta)$ is nondecreasing. Moreover, by (9.3) and (9.7),

$$x(t_0 + h) - x(t_0) \leqq \int_{-h}^{0} f(x(t_0 + s))ds < f(\xi_0)h \leqq -\frac{\pi + 2\xi_0}{2}$$

uniformly for $\psi \in K_0$, and consequently for k_0, $0 < k_0 < 1$, sufficiently near 1, we obtain $x(t_0 + h) - x(t_0) \leqq ((\pi + 2\xi_0)k_0/2)$ uniformly for $\psi \in K_0$. Finally, by Lemma 8.1, there exists an $\eta = \eta(\delta, h) > 0$ such that $x(t_0 + h) - x(t_0) \geqq \eta$.

For $\phi \in K$ and $\psi = \phi + \zeta^{\xi_0}$, let $\tau(\phi) = t_0 + h$, and let $A: \phi \longrightarrow \tilde{\phi} = x_\tau(\psi) - \zeta^{\xi_0+2\pi}$. Then, under the assumptions of Lemma 9.1, A satisfies the assumptions of Theorem 8.1. Thus we have the following proposition.

PROPOSITION 9.1. *Uuder the assumptions of Lemma* 9.1, (4.1) *has a periodic solution of the second kind.*

Next, consider equation (4.1) for $\delta > 1$. Let $x(t)$ be a periodic solution of the second kind of (4.1), and let $T > 0$ be the smallest period. Then it is easy to see that $x(t + T) - x(t) = 2p\pi$ for some integer p. We consider the case $p = 1$. Then T must be less than $2\pi/(\delta - 1)$ because $\delta > 1$, and consequently it is sufficient to consider only $h < 2\pi/(\delta - 1)$. Consider K_1 and the corresponding K for $A_2 = (\pi/2)$ and $X_1 + X_2 - A_1 = (3\pi/2)$.

LEMMA 9.2. *Let* $\delta > 1$ *in* (4.1). *Then any solution* $x(t, \psi)$ *of* (4.1) *is increasing and reaches* $(5\pi/2)$ *till the time* $2\pi/(\delta - 1)$. *If* $h < (\pi(2 + \delta)/(1 + \delta)^2)$, *then for* k_0, $0 < k_0 < 1$, *sufficiently near* 1, $(\delta - 1)h \leqq$

$x(t_0+h, \psi) - x(t_0, \psi) \leqq k_0\pi$ *uniformly for* $\psi \in K_1$, *where* $x(t_0, \psi) = (5\pi/2)$.

PROOF. Since $\dot{x}(t) \geqq \delta - 1 > 0$, it is clear that $x(t)$ is increasing and reaches $(5\pi/2)$ till the time $2\pi/(\delta-1)$. Let $x(t_0) = (5\pi/2)$. If $\pi/(1+\delta) \leqq h < (\pi(2+\delta)/(1+\delta)^2)$, then we have

$$x(t_0+h) - x(t_0) \leqq \int_{-h}^{-\pi/(1+\delta)} (1+\delta)ds + \int_{-\pi/(1+\delta)}^{0} \left\{\delta - \sin\left((1+\delta)s + \frac{5\pi}{2}\right)\right\}ds$$

$$\leqq (1+\delta)h - \frac{\pi}{1+\delta} < \pi$$

uniformly for $\psi \in K_1$. Similarly, for $h < \pi/(1+\delta)$,

$$x(t_0+h) - x(t_0) \leqq \int_{-\pi/(1+\delta)}^{0} \left\{\delta - \sin\left((1+\delta)s + \frac{5\pi}{2}\right)\right\}ds \leqq \pi - \frac{\pi}{(1+\delta)} .$$

Consequently, for k_0, $0 < k_0 < 1$, sufficiently near 1, we have $x(t_0+h) - x(t_0) \leqq k_0\pi$ uniformly for $\psi \in K_1$. It is clear that $x(t_0+h) - x(t_0) \geqq (\delta-1)h$ holds.

If we define the mapping A similarly to the case $\delta \leqq 1$, A satisfies the assumptions of Theorem 8.1 by Lemma 9.2, and hence we have the following proposition.

PROPOSITION 9.2. *Under the assumptions of Lemma* 9.2, (4.1) *has a periodic solution of the second kind.*

REMARK. (i) (9.2) is true for $\delta_0 = \sin(3\pi/7)$, where $0.974 < \delta_0 < 0.975$. (9.1) holds for a wider region of δ than (9.2) for $\xi_0(\alpha) = -(\pi/2) - (7\alpha/4)$.

(ii) For the case $\delta = 0.8$ and $h = 2$, Ueda and his colleagues have observed the existence of a periodic solution of the second kind.

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