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NUMERICAL RANGES OF PRODUCTS AND TENSOR PRODUCTS

ELIAS S. W. SHIU

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In this paper we study the relationship between the numerical ranges of Hilbert space operators and those of their products and tensor products.

Let $\mathscr{B}(\mathscr{H})$ denote the set of bounded linear operators on a complex Hilbert space $\mathscr{H}$. For $T \in \mathscr{B}(\mathscr{H})$, $W(T)$ denotes its numerical range, $W(T) = \{(Tx, x): \|x\| = 1\}$. For $T_j \in \mathscr{B}(\mathscr{H}_j)$, $j = 1, 2$, it is clear that $W(T_1 \otimes T_2)$ contains the set $W(T_1)\cdot W(T_2) = \{z_1 z_2: z_j \in W(T_j), j = 1, 2\}$; by the convexity of the numerical range, $W(T_1 \otimes T_2)$ also contains its convex hull, $\mathrm{co}\,(W(T_1)\cdot W(T_2))$ [11, Lemma 6.2]. We are interested in the conditions that guarantee $W(T_1 \otimes T_2) = \mathrm{co}\,(W(T_1)\cdot W(T_2))$. We shall show that if either T_1 or T_2 is normal, then

$$(1) \qquad \bar{W}(T_1 \otimes T_2) = \overline{\mathrm{co}}(W(T_1)\cdot W(T_2)),$$

where the bars denote the closure of the sets. This result follows from: Let $A, B \in \mathscr{B}(\mathscr{H})$ be two commuting operators; if A or B is normal, then $\bar{W}(AB) \subseteq \overline{\mathrm{co}}(W(A)\cdot W(B))$.

Consider the operator $S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ on C^2. For $T \in \mathscr{B}(\mathscr{H})$, $T \otimes S$ has the representation $\begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}$ on $\mathscr{H} \oplus \mathscr{H}$. Since $W(S) = \{z \in C: |z| \leqq 1/2\}$ and $\bar{W}(T \otimes S) = \{z \in C: |z| \leqq \|T\|/2\}$, (1) holds if and only if T is a normaloid, i.e., its norm equals its numerical radius [6, p. 114]. In fact, if T is a normaloid, then $\bar{W}(T \otimes S) = \bar{W}(T)\cdot W(S)$, for $W(S)$ is a disc centered at the origin. This discussion shows that (1) does not hold in general.

The following results are proved in Section 3: (i) Let $A, B \in \mathscr{B}(\mathscr{H})$ such that A commutes with B and B^*. If the set $W(A)\cdot W(B)$ lies on one side of a line through the origin, then $W(AB)$ lies on the same side. (ii) Let $T_j \in \mathscr{B}(\mathscr{H}_j)$, $j = 1, 2$. Then $W(T_1)\cdot W(T_2)$ lies on one side of a line through the origin if and only if $W(T_1 \otimes T_2)$ lies on the same side.

With these results we derive a theorem of E. Asplund [1]: For $T \in \mathscr{B}(\mathscr{H})$ and an integer $n \geqq 2$, $|\mathrm{Arg}\,(Tx, x)| \leqq \pi/n$, $\forall x \in \mathscr{H}$, if and only

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if for each sequence $x_0, x_1, \cdots, x_{n-1}, x_n = x_0$ of n elements in $\mathscr{H}$, $\sum_{j=0}^{n-1} \operatorname{Re}(Tx_j, x_j - x_{j+1}) \geqq 0$.

1. Preliminaries. For $\Omega, \Omega_1 \subseteqq \boldsymbol{C}$, let $\operatorname{co}(\Omega)$ and $\partial\Omega$ denote the convex hull and the boundary of Ω, respectively, and $\Omega \cdot \Omega_1 = \{zz_1 : z \in \Omega, z_1 \in \Omega_1\}$. A proof of the following fact is given in [5, p. 683]: $\operatorname{co}(\Omega \cdot \Omega_1) = \operatorname{co}(\operatorname{co}(\Omega) \cdot \operatorname{co}(\Omega_1))$. The next result is obvious for compact Ω.

LEMMA [9, p. 295]. *Let $\Omega \subseteqq \boldsymbol{C}$ be bounded. Then*

$$\operatorname{co}(\Omega) = \left\{\sum_{j=1}^{\infty} \alpha_j z_j : \alpha_j \geqq 0, \sum_{j=1}^{\infty} \alpha_j = 1 \text{ and } z_j \in \Omega\right\}.$$

COROLLARY 1 (cf. [3, Lemma 1]). *Let $T_j \in \mathscr{B}(\mathscr{H}_j)$ such that $\sup_j \|T_j\| < \infty$. Then $\bigoplus_j T_j \in \mathscr{B}(\bigoplus_j \mathscr{H}_j)$ and $\operatorname{co}(\bigcup_j W(T_j)) = W(\bigoplus_j T_j)$.*

For $T \in \mathscr{B}(\mathscr{H})$, we say T has a dilation S if $S \in \mathscr{B}(\mathscr{K})$, $\mathscr{K}$ a Hilbert space containing $\mathscr{H}$ as a subspace, and $TP = PSP$, P being the orthogonal projection from $\mathscr{K}$ onto $\mathscr{H}$ ([6, Chapter 18], [11, §2]). Under these conditions T is called the compression of S to $\mathscr{H}$. Clearly, $W(T) \subseteqq W(S)$.

Let Ω be a closed subset of $\boldsymbol{C}$ containing the spectrum of T, $\sigma(T)$. Ω is said to be spectral for T (in the sense of von Neumann) if for each rational function q with poles outside Ω, $\|q(T)\| \leqq \sup_{z \in \Omega} |q(z)|$ ([6, p. 123], [11, p. 538])

An operator T is called a diagonal operator if there is an orthonormal basis of $\mathscr{H}$ consisting of eigenvectors of T([10, p. 23], [6, p. 29]). If $W(T) \subseteqq [0, \infty)$, we say T is nonnegative and write $T \geqq 0$; a nonnegative operator has a unique nonnegative square root by the spectral theorem [10, Theorem 1.12].

2. Main results.

THEOREM 1 [2, Theorem 2]. *Let $A, B \in \mathscr{B}(\mathscr{H})$ be two commuting operators. If $A \geqq 0$, then $W(AB) \subseteqq W(A) \cdot W(B)$.*

PROOF. $AB = A^{1/2}BA^{1/2}$.

THEOREM 2. *Let $A, B \in \mathscr{B}(\mathscr{H})$ be two commuting operators. If A is diagonal, then $W(AB) \subseteqq \operatorname{co}(W(A) \cdot W(B))$.*

PROOF. Let $A = \sum_j \lambda_j P_j$, where $\{P_j\}$ is a family of mutually orthogonal projections, i.e., $P_j^* = P_j$ and $P_j P_k = \delta_{jk} P_j$, and $\sum_j P_j = I$. Assume that the λ_j's are distinct complex numbers, then $B = \sum_j P_j B P_j$ (cf. [10, Corollary 0.14]). If B_j denotes the compression of B to $P_j\mathscr{H}$, then AB has the representation $\bigoplus_j \lambda_j B_j$ on $\bigoplus_j P_j\mathscr{H}$. Thus

$$W(AB) = \text{co}\,(\textstyle\bigcup_j \lambda_j W(B_j)) \qquad \text{Corollary 1}$$
$$\subseteqq \text{co}\,(\textstyle\bigcup_j \lambda_j W(B)) = \text{co}\,(W(A)\cdot W(B))\,.$$

The next result generalizes [7, Theorem 2.2] and the initial steps of their proofs are identical.

THEOREM 3. *Let $A, B \in \mathscr{B}(\mathscr{H})$ be two commuting operators. If A is normal, then $\bar{W}(AB) \subseteqq \overline{\text{co}}\,(W(A)\cdot W(B))$.*

PROOF. By the spectral theorem [10, Theorem 1.12] and the Fuglede's theorem [10, Theorem 1.16], A can be approximated uniformly by diagonal operators which commute with B. Since $\bar{W}(\cdot)$ and the multiplication of operators are both continuous with respect to the uniform operator topology [6, Problem 175 & Problem 91], the result follows from Theorem 2.

The finite-dimensional versions of the following three theorems are given in [8, Theorem 1 & Theorem 2].

THEOREM 1′. *Let $T_j \in \mathscr{B}(\mathscr{H}_j)$, $i = 1, 2$. If $T_1 \geqq 0$ or $T_2 \geqq 0$, then $W(T_1 \otimes T_2) = W(T_1)\cdot W(T_2)$.*

PROOF. $W(T_1 \otimes I) = W(T_1)$, $W(I \otimes T_2) = W(T_2)$.

THEOREM 2′. *Let $T_j \in \mathscr{B}(\mathscr{H}_j)$, $j = 1, 2$. If T_1 or T_2 is diagonal, then $W(T_1 \otimes T_2) = \text{co}\,(W(T_1)\cdot W(T_2))$.*

THEOREM 3′. *Let $T_j \in \mathscr{B}(\mathscr{H}_j)$, $j = 1, 2$. If T_1 or T_2 is normal, then (1) holds.*

REMARK. Theorem 2 can be derived from Theorem 2′, because $A\otimes B$ is a dilation of AB: Let $\{\mu_k\}$ be an enumeration of $\{\lambda_j\}$ with each λ_j repeated according to its multiplicity, i.e., the rank of P_j. Then

$$B \otimes A \cong \bigotimes_k \mu_k B = \bigoplus_k \bigoplus_j \mu_k B_j\,.$$

If $T_i \in \mathscr{B}(\mathscr{H}_i)$ has a dilation S_i, $i = 1, 2$, then $S_1 \otimes S_2$ is a dilation of $T_1 \otimes T_2$. Applying Theorem 3′, we have

THEOREM 4. *Let $T_i \in \mathscr{B}(\mathscr{H}_i)$, $i = 1, 2$. If T_1 has a normal dilation N, then $\bar{W}(T_1 \otimes T_2) \subseteqq \overline{\text{co}}\,(W(N)\cdot W(T_2))$.*

COROLLARY 2. *Let $T_i \in \mathscr{B}(\mathscr{H}_i)$, $i = 1, 2$. If Ω is spectral for T_1, then $\bar{W}(T_1 \otimes T_2) \subseteqq \overline{\text{co}}\,(\Omega\cdot W(T_2))$.*

PROOF. Assume Ω is compact. By the Berger-Foias-Lebow Theorem [11, Corollary 2.3], there is a (strong) normal dilation N of T_1 with $\sigma(N) \subseteqq \partial\Omega$.

Let $\mathscr{N}$ denote the set of operators $\{T: T$ has a normal dilation N such that $\bar{W}(T) = \bar{W}(N)\}$. By Theorem 4, (1) holds if T_1 or T_2 belongs to $\mathscr{N}$. We note that the subnormal operators [6, p. 322] and the Toeplitz operators [6, p. 349] belong to $\mathscr{N}$. Moreover, if $\bar{W}(T)$ is spectral for T, then $T \in \mathscr{N}$ by Corollary 2; in fact, it is shown by M. Schreiber that $\bar{W}(T)$ is spectral for T if and only if there exists a strong normal dilation N of T such that $\bar{W}(T) = \bar{W}(N)$ [11, Theorem 2.4].

Let $T_j \in \mathscr{B}(\mathscr{H}_j)$, $j = 1, 2$. It follows from a result of A. Brown and C. Pearcy [11, Theorem 6.1] that $\sigma(T_1 \otimes T_2) = \sigma(T_1)\cdot\sigma(T_2)$. Thus (1) holds whenever $T_1 \otimes T_2$ is convexoid [11, Theorem 6.2]. If T_1 and T_2 are hyponormal, a simple computation shows that $T_1 \otimes T_2$ is also hyponormal and hence (1) holds [11, Corollary 6.2].

CONJECTURE. Let $T_j \in \mathscr{B}(\mathscr{H}_j)$, $j = 1, 2$. If T_1 or T_2 is hyponormal, then (1) holds.

3. Sectorial operators. In this section we are concerned with the operators whose numerical ranges are contained in half-planes supported at the origin.

For $T \in \mathscr{B}(\mathscr{H})$, let $\Theta(T)$ denote the closure of the set $\{(Tx, x)\}$. Since the numerical range of an operator is convex, either $\Theta(T)$ is the entire complex plane or it is a closed sector with vertex at the origin and with angular opening at most equal to π. We note that $\Theta(T) = \Theta(S^*TS)$ whenever S is invertible. If $\mathscr{H}$ is finite dimensional and $0 \in W(T)$, then $\Theta(T)$ coincides with the angular field introduced in [13]. For $\alpha \in [0, \pi/2]$, let $\Phi(\alpha)$ denote the symmetric sector $\{\rho e^{i\theta}: \rho \geqq 0, -\alpha \leqq \theta \leqq \alpha\}$.

THEOREM 5. *Let $A, B \in \mathscr{B}(\mathscr{H})$ and suppose A commutes with B and B^*. If* $\mathrm{co}\,(\Theta(A)\cdot\Theta(B)) \neq \boldsymbol{C}$, *then* $\Theta(AB) \subseteqq \Theta(A)\cdot\Theta(B)$.

PROOF. Without loss of generality, assume $\Theta(A) = \Phi(\alpha)$, $\alpha \in [0, \pi/2]$. Thus $\operatorname{Re} A = (A + A^*)/2 \geqq 0$. By the spectral theorem, $\operatorname{Re} A$ has a nonnegative square root Q. If $\operatorname{Re} A$ is invertible, then $A = \operatorname{Re} A + i \operatorname{Im} A = QNQ$, where N is the normal operator $I + iQ^{-1}(\operatorname{Im} A)Q^{-1}$. Since B commutes with Q, $\Theta(AB) = \Theta(QNBQ) = \Theta(NB)$. By Theorem 3 and the hypothesis that $\Theta(A)\cdot\Theta(B) = \Theta(N)\cdot\Theta(B)$ lies on one side of a line through the origin, we have $\Theta(NB) \subseteqq \Theta(N)\cdot\Theta(B) = \Theta(A)\cdot\Theta(B)$. Thus the theorem is proved if $\operatorname{Re} A$ is invertible. In general, consider $A + \varepsilon I$, $\varepsilon > 0$, instead of A. Now the result follows from [6, Problem 175 & Problem 91].

REMARK. If A and B commute and if A commutes with BB^* or

B^*B, then we have the following inequality for numerical radii: $w(AB) \leqq \|B\|\, w(A)$ [3, p. 217].

THEOREM 5′ [12, Theorem 2]. *Let* $T_j \in \mathscr{B}(\mathscr{H}_j)$, $j = 1, 2$. *If* $\Theta(T_1 \otimes T_2) \neq \boldsymbol{C}$ *or if* $\mathrm{co}\,(\Theta(T_1)\cdot\Theta(T_2)) \neq \boldsymbol{C}$, *then* $\Theta(T_1)\cdot\Theta(T_2) = \Theta(T_1 \otimes T_2)$.

PROOF. Since $W(T_1)\cdot W(T_2) \subseteqq W(T_1 \otimes T_2)$, we have $\Theta(T_1)\cdot\Theta(T_2) \subseteqq \Theta(T_1 \otimes T_2)$.

4. Application. Let $S, T \in \mathscr{B}(\mathscr{H})$ and $A, B \in \mathscr{B}(l_2)$, $A = (a_{jk})$, $B = (b_{jk})$. Let $\boldsymbol{x} = (x_k) \in \bigoplus_k \mathscr{H} \cong \mathscr{H} \otimes l_2$. Then

$$\begin{aligned}\sum_j \Big(\sum_k b_{jk} T x_k, \sum_k a_{jk} S x_k\Big)_{\mathscr{H}} &= ((b_{jk}T)(x_k), (a_{jk}S)(x_k))_{\oplus_j \mathscr{H}} \\ &= ((T \otimes B)\boldsymbol{x}, (S \otimes A)\boldsymbol{x})_{\mathscr{H}\otimes l_2} \\ &= ((S^*T \otimes A^*B)\boldsymbol{x}, \boldsymbol{x})_{\mathscr{H}\otimes l_2}\,.\end{aligned}$$

The following is a result of E. Asplund [1, Theorem 3] (also see [12, Theorem 1] and [4, p. 118]).

THEOREM 6. *Let* $T \in \mathscr{B}(\mathscr{H})$, *and* n *is an integer,* $n \geqq 2$. *Then* $\Theta(T) \subseteqq \Phi(\pi/n)$ *if and only if for each sequence* $x_0, x_1, \cdots, x_{n-1}, x_n = x_0$ *of* n *elements in* $\mathscr{H}$, $\sum_{j=0}^{n-1} \mathrm{Re}\,(Tx_j, x_j - x_{j+1}) \geqq 0$.

PROOF. Let A denote the $n \times n$ matrix (a_{jk}),

where $\qquad a_{jj} = 1\,, \qquad j = 1, 2, \cdots, n\,,$

$\qquad\qquad a_{jj+1} = a_{n1} = -1\,, \qquad j = 1, 2, \cdots, n-1\,,$

and $\qquad a_{jk} = 0$ elsewhere.

A is normal and its eigenvalues are $1 - \exp(2\pi i m/n)$, $m = 1, 2, \cdots, n$. Thus $\Theta(A^*) = \Phi(\pi/2 - \pi/n)$. Consequently, $\mathrm{Re}\,(T \otimes A^*) \geqq 0$ if and only if $\Theta(T) \subseteqq \Phi(\pi/n)$, by Theorem 2′ or Theorem 5′.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MANITOBA
WINNIPEG, MANITOBA
R3T 2N2 CANADA