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NOTE ON GENERALIZED INFORMATION FUNCTION

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Abstract. Solution of a functional equation, connected with entropy, directed divergence, inaccuracy and their generalization of type β etc. is obtained without the assumption of any regularity condition on the functions involved.

1. Introduction. In [3], the following functional equation

$$f(x) + (1-x)^{\beta} g(y/(1-x)) = h(y) + (1-y)^{\beta} k(x/(1-y)) , \tag{1.1}$$

for $x, y \in [0, 1[$ with $x + y \in [0, 1]$, $\beta \neq 1$, was considered, giving the details of its association with entropy of type β [2], directed divergence of type β [9] and inaccuracy of type β [10]. For $\beta \to 1$, these measures reduce to Shannon's entropy [11], directed divergence [6] and inaccuracy [5].

In this paper, the functional equation (1.1) is solved by simple and direct method, without any further assumption on the functions, by reducing (1.1) to a similar form involving only one function,

$$u(x) + (1-x)^{\beta} u(y/(1-x)) = u(y) + (1-y)^{\beta} u(x/(1-y)) . \tag{1.2}$$

REMARK 1. If $u: [0, 1] \to R$ (reals) is a solution of (1.2), then $v(x) = u(x) - u(1)x^{\beta}$ is also a solution of (1.2) with $v(x) = v(1-x)$, that is, with v symmetric. Then the solution of (1.2) can be obtained from [1, 8], for $\beta \neq 1$. So, no generality is lost in considering the symmetric solution of (1.2). Thus the general solution of (1.2) for $\beta \neq 1$ is given by $u(x) = A[x^{\beta} + (1-x)^{\beta} - 1] + Bx^{\beta}$.

2. Solution of (1.1). Let $f, h: [0, 1[\to R$, $g, k: [0, 1] \to R$, satisfy the functional equation (1.1) for $x, y \in [0, 1[$ with $x + y \in [0, 1]$, where $\beta (\neq 2)$ is positive.

For $x = 0$, (1.1) gives

$$h(y) = g(y) + b_1(1-y)^{\beta} + b_2, \quad \text{for} \quad y \in [0, 1[, \tag{2.1}$$

where b_1, b_2 are constants.

With $y = 1 - x$ in (1.1), (1.1) becomes with the help of (2.1),

$$f(x) = g(1-x) + c_1 x^{\beta} + c_2(1-x)^{\beta} + b_2, \quad \text{for} \quad x \in]0, 1[, \tag{2.2}$$

where c_1 and c_2 are constants.

Now (2.1), (2.2), and (1.1) with $y = 0$, yield

$$k(x) = g(1-x) + c_1 x^\beta + d_1(1-x)^\beta + d_2, \quad \text{for} \quad x \in]0, 1[, \tag{2.3}$$

where d_1 and d_2 are constants.

Thus (1.1) can be rewritten as

$$\begin{aligned} g(1-x) + (1-x)^\beta g(y/(1-x)) &= g(y) + (1-y)^\beta g(1-x/(1-y)) \\ &+ d_1(1-x-y)^\beta - c_2(1-x)^\beta + (b_1 + d_2)(1-y)^\beta , \end{aligned} \tag{2.4}$$

for $x \in]0, 1[$, $y \in [0, 1[$ with $x + y \in]0, 1[$.

Putting $y = 0$, (2.4) gives

$$g(0) = d_1 - c_2 = -b_1 - d_2 . \tag{2.5}$$

Define

$$m(x) = g(x) + d_1 x^\beta - g(0), \quad \text{for} \quad x \in [0, 1] . \tag{2.6}$$

From (2.4), (2.5), and (2.6), we get

$$m(1-x) + (1-x)^\beta m(y/(1-x)) = m(y) + (1-y)^\beta m(1-x/(1-y)) \tag{2.7}$$

for $x \in]0, 1[$, $y \in [0, 1[$, with $x + y \in]0, 1[$.

Interchanging x and y in (2.7) and defining

$$N(x) = m(x) - m(1-x), \quad \text{for} \quad x \in]0, 1[\tag{2.8}$$

we have

$$N(x) = -N(1-x), \quad \text{for} \quad x \in]0, 1[, \tag{2.9}$$

$$N(1/2) = 0 \tag{2.10}$$

and

$$N(x) + N(y) = (1-x)^\beta N(y/(1-x)) + (1-y)^\beta N(x/(1-y)) , \quad \text{for} \quad x, y, x + y \in]0, 1[. \tag{2.11}$$

Now $x = y$ in (2.11) gives,

$$N(x) = (1-x)^\beta N(x/(1-x)), \quad \text{for} \quad x \in]0, 1/2[. \tag{2.12}$$

Also $y = 1/2$ in (2.11) results in

$$N(x) = (1-x)^\beta N(1/2(1-x)) + N(2x)/2^\beta, \quad \text{for} \quad x \in]0, 1/2[. \tag{2.13}$$

Using (2.12), (2.13), and (2.9), we have

$$(1-x)^\beta[-N((1-2x)/(1-x)) + N((1-2x)/2(1-x))] = N(2x)/2^\beta , \quad \text{for} \quad 0 < x < 1/2 ,$$

which by the substitution $(1-2x)/(1-x) = t$, yields

$$[-N(t) + N(t/2)]/(2-t)^\beta = N(2(1-t)/(2-t))/2^\beta, \quad \text{for} \quad t \in]0, 1[,$$

which with the use of (2.12) for $x = t/2$ and (2.9) leads to

$$N(t) = 2N(t/2), \quad \text{for} \quad t \in]0, 1[. \tag{2.14}$$

Finally from (2.13), (2.14), (2.12), and (2.9), we obtain

$$N(x) = (1-x)^\beta N(1/2(1-x)) + N(x)/2^{\beta-1}, \quad \text{for} \quad x \in]0, 1/2[,$$

that is,

$$(1 - (1/2^{\beta-1}))N(x) = (1-x)^\beta N(1/2(1-x)), \quad \text{for} \quad x \in]0, 1/2[$$

that is,

$$(1 - (1/2^{\beta-1}))(1-x)^\beta N(1 - x/(1-x)) = (1-x)^\beta N(1 - (1/2(1-x))) ,$$

which by putting $t = (1-2x)/(1-x)$, yields

$$(1 - (1/2^{\beta-1}))N(t) = N(t/2), \quad \text{for} \quad t \in]0, 1[,$$

that is,

$$(1 - (1/2^{\beta-2}))N(t/2) = 0, \quad \text{for} \quad t \in]0, 1[,$$

so that, since $\beta \neq 2$

$$N(t) = 0 \quad \text{for all} \quad t \in]0, 1[.$$

Thus from (2.8), we get $m(x) = m(1-x)$ and from (2.7) we obtain

$$m(x) + (1-x)^\beta m(y/(1-x)) = m(y) + (1-y)^\beta m(x/(1-y)) , \qquad \text{for} \quad x, y, x+y \in]0, 1[,$$

which is the same as (1.2), with $\beta \neq 2$.

REMARKS 2. Then, it follows from [1, 8], that if β is also not equal to 1,

$$m(x) = AS_\beta(x) = A[x^\beta + (1-x)^\beta - 1] , \qquad x \in]0, 1[,$$

so that from (2.6), (2.2), (2.1), and (2.3), we get

$$\begin{aligned} f(x) &= AS_\beta(x) + c_1 x^\beta - d_3(1-x)^\beta + c_3 \\ g(y) &= AS_\beta(y) - d_1 y^\beta + d_3 \\ h(x) &= AS_\beta(x) - d_1 x^\beta + b_1(1-x)^\beta + c_3 \\ k(y) &= AS_\beta(y) + c_1 y^\beta - b_1 , \end{aligned}$$

for $x \in [0, 1[$, $y \in [0, 1]$, where A, b_1, c_1, c_3, d_1, d_3 are arbitrary constants. From this it is easy to see that in this case no further assumptions on the functions are necessary to solve (1.2) and hence (1.1).

REMARK 3. In case $\beta = 1$, then it is necessary to impose some further conditions on any one and hence on all of the functions involved in (1.1) to get the solution of (1.2) [refer 12, 4, 6, 7 etc.]. Further the above procedure gives a simpler method of solving the functional equation (1.2) when $\beta = 1$, dealt in [13]. This method greatly simplifies the proof given in [13] regarding (1.2) for $\beta = 1$.

Thus, we have proved the following theorem.

THEOREM. *If* $f, h: [0, 1[\to R$, $g, k: [0, 1] \to R$ *satisfy* (1.1) *for* $x, y \in [0, 1[$ *with* $x + y \in [0, 1]$, *where* $\beta(>0) \neq 2$, *then* f, g, h, k *are given by* (2.2), (2.6), (2.1), *and* (2.3) *where* m *is a solution of* (1.2).

REMARK 4. It is now clear to see that no further assumption on F is necessary [refer 9, 10] to obtain the solution of the functional equation

$$\begin{aligned} &F(x, y) + (1 - x)^\beta (1 - y)^\gamma F(u/(1 - x), v/(1 - y)) \\ &\quad = F(u, v) + (1 - u)^\beta (1 - v)^\gamma F(x/(1 - u), y(1 - v)) , \end{aligned}$$

in the form $F(x, y) = A[x^\beta y^\gamma + (1 - x)^\beta (1 - y)^\gamma - 1] + Bx^\beta y^\gamma$, [refer 3].

REMARK 5. An open question. Is it possible to obtain the solution of (1.1), by reducing (1.1) to that of (1.2), in case $\beta = 2$?

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