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ON BANACH-LIE GROUPS ACTING ON FINITE DIMENSIONAL MANIFOLDS

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0° Introduction. A Banach-Lie group is a combined concept of a Banach manifold and a topological group. Namely, a topological group G is called a *Banach-Lie group* (modeled on a Banach space E), if G is a C^k-Banach manifold on E, $k \geqq 3$, and the group operations are of class C^k. As in the case of finite dimensional Lie groups, G carries a real analytic structure ([13]), and the tangent space $\mathfrak{g}$ at the identity being canonically identified with the model space E has a structure of a Lie algebra such that the Lie bracket product $[u, v]$ on $\mathfrak{g}$ is a bounded bilinear operator, i.e. there is a constant C such that $\|[u, v]\| \leqq C\|u\|\,\|v\|$, where by taking a suitable multiple of the norm, C may be taken to be unity or zero.

A normed linear space with a bounded bilinear Lie bracket product is called a *normed Lie algebra*, and if it is complete with respect to the norm topology, it is called a *Banach-Lie algebra*. A Banach-Lie algebra is called *enlargable* ([20]) if it is a Lie algebra of a Banach-Lie group. Finite dimensional Lie algebras are always enlargable. However, there exist non-enlargable Banach-Lie algebras ([20]), while every Banach-Lie algebra is a Lie algebra of a *local* Banach-Lie group (cf. [3] and [6]).

The existence of non-enlargable Lie algebra is, however, the only known fact with no finite dimensional analogue. Moreover, there are good criteria for enlargability, one of which is stated as follows: Let $\mathfrak{g}$, $\mathfrak{h}$ be Banach-Lie algebras such that there is a continuous Lie algebra monomorphism $\mathfrak{h}$ into $\mathfrak{g}$. If $\mathfrak{g}$ is enlargable, then so is $\mathfrak{h}$. (Cf. [20].) Most of the theorems hold, and indeed are proved by the classical procedures from the theory of finite dimensional Lie groups. (Cf. [3], [6] and [13].) The implicit function theorem and Frobenius theorem hold also in the category of Banach manifolds under the restriction that the considered linear space has a direct summand (splitting).

There are a lot of examples of Banach-Lie groups in operator calculus ([8] and the bibliography therein). Most of them are generalizations of classical groups with various topologies for spaces of operators. In many of those groups, separability does not hold anymore even if the group is connected. In addition to the closedness of the linear subspaces which we consider, a second point about which we must worry to generalize the theory of finite dimensional Lie groups to the infinite dimensional case is the lack of separability. Anyway, Banach-Lie groups are properly enlarged mathematical object which covers the classical theory of Lie groups.

So, the following seems to be a natural question: Is there a finite dimensional manifold on which an infinite dimensional Banach-Lie group acts effectively and transitively? The answer is "yes," but there might be few examples as we can see in this paper.

Throughout this paper, a manifold M means always a connected, separable, finite dimensional and C^∞-manifold without boundary. Let $\Gamma(TM)$ be the Lie algebra of all C^∞-vector fields on M with the C^∞-topology. By $\mathscr{D}(M)$ we denote the group of all C^∞-diffeomorphisms on M. Separability is not assumed for the Banach-Lie groups in this paper.

First of all, we shall prove the following theorem in 1°, which reduces our problem to the corresponding problem on Lie algebras:

THEOREM A. *If a connected Banach-Lie group G acts smoothly and effectively on a manifold M, then there is a continuous imbedding of the Lie algebra $\mathfrak{g}$ of G into $\Gamma(TM)$ satisfying the following*:

(*) *Every $u \in \mathfrak{g}$ is complete,* i.e. *there is a one parameter transformation group* $\exp tu$ *generated by the vector field u.*

(**) $\mathrm{Ad}(\exp tu)\mathfrak{g} = \mathfrak{g}$, *where for a smooth diffeomorphism φ of M,* $\mathrm{Ad}(\varphi)u$ *is defined by* $(\mathrm{Ad}(\varphi)u)(x) = d\varphi u(\varphi^{-1}x)$.

Conversely, let $\mathfrak{g}$ be a Banach-Lie algebra such that there is a continuous inclusion of $\mathfrak{g}$ into $\Gamma(TM)$ and that $\mathfrak{g}$ satisfies (*). *Then $\mathfrak{g}$ is enlargable. Indeed there is a Banach-Lie group G such that the Lie algebra of G is $\mathfrak{g}$ and that G is a subgroup of $\mathscr{D}(M)$. In particular, $\mathfrak{g}$ satisfies* (**).

The method of the proof of the above theorem yields also that G acts smoothly and transitively on a manifold, if and only if the action is ample, i.e. infinitesimal transitive at every point. (Cf. 1°.) Therefore by the implicit function theorem, the isotropy subgroup H of G is a closed Banach-Lie subgroup such that the manifold is diffeomorphic to the factor space G/H. Thus, our problem is reduced to the following: Find a pair of infinite dimensional Banach-Lie groups (G, H) such that (i) G is connected, (ii) H is a closed Banach-Lie subgroup of G, (iii) $\dim G/H < \infty$ and (iv) $\bigcap_{g \in G} gHg^{-1} = \{e\}$.

However, the following theorem shows that such examples are not so rich (cf. 2°):

THEOREM B. *If a connected Banach-Lie group G acts effectively, transitively and smoothly on a compact manifold, then G must be a finite dimensional Lie group.*

Moreover, the following has been known in the joint work with P. de la Harpe [16]:

THEOREM. *Let $\mathfrak{g}$ be the Lie algebra of an infinite dimensional Banach-Lie group G. Suppose $\mathfrak{g}$ has no proper closed finite codimensional ideal. Then the only possible smooth action of G on a finite dimensional manifold is trivial.*

The above theorem show that $\mathfrak{g}$ has by no means a character of simple Lie algebras. A Banach-Lie algebra $\mathfrak{g}$ will be called *solvable* if the descending series $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots$ of derived algebras $\mathfrak{g}_n = [\mathfrak{g}_{n-1}, \mathfrak{g}_{n-1}]^-$ ($-$means the closure) finishes at a finite stage $\mathfrak{g}_n$ (i.e. $\mathfrak{g}_{n+1} = 0$). $\mathfrak{g}$ will be called *almost solvable* if there is a finite codimensional closed ideal ρ of $\mathfrak{g}$ such that ρ is solvable. The following will sharpen the above theorem:

THEOREM C. *If a connected infinite dimensional Banach-Lie group G acts smoothly, effectively and transitively on a non-compact manifold, then G must be almost solvable.*

The above theorem will be proved in 3°.

In 4°, several examples of Banach-Lie groups acting effectively, smoothly and transitively on a manifold will be given.

The idea of the proof of both Theorems B and C is based on the following simple fact: Since $\mathfrak{g}$ is a Banach-Lie algebra, $\mathrm{ad}(u): \mathfrak{g} \mapsto \mathfrak{g}$ is a bounded linear operator for any $u \in \mathfrak{g}$. However since every $u \in \mathfrak{g}$ can be canonically identified with a smooth vector field on a manifold M, $\mathrm{ad}(u)$ must have a character of unbounded operators because $\mathrm{ad}(u)$ is a differential operator. Indeed, the character of unboundedness appears in various way. For instance, if $\mathfrak{g}$ contains $x(\partial/\partial x)$, $x^2(\partial/\partial x)$ and $x^3(\partial/\partial x)$ then $\mathfrak{g}$ contains $x^n(\partial/\partial x)$ for all $n \geqq 0$ and $[x(\partial/\partial x), x^n(\partial/\partial x)] = (n-1)x^n(\partial/\partial x)$. Thus, $\mathrm{ad}(x(\partial/\partial x)): \mathfrak{g} \to \mathfrak{g}$ can not be bounded in any norm.

By the above theorems and the above idea of the proof, it seems to be natural to conjecture that there exist few examples of infinite dimensional Banach-Lie groups acting smoothly, effectively and transitively on a finite dimensional manifold.

The main idea of making such examples is as follows: Though $\partial/\partial x$ is a differential operator, it is a bounded linear operator of $E = \{\sum a_n x^n;\ \sup n!|a_n| < \infty\}$ into itself, where the norm on E is defined by $\|u\| = \sup n!|a_n|$.

1° Some remarks on Banach-Lie groups and Banach-Lie algebras. In this section, the proof of Theorem A and some other remarks on Banach-Lie groups will be given.

The first half of Theorem A is easy to prove. Indeed, let G be a connected Banach-Lie group acting effectively and smoothly on a manifold M and let ρ be the action. $\rho: G \mapsto \mathscr{D}(M)$ is then a monomorphism. For any $u \in \mathfrak{g}$, there is a one parameter subgroup $\{\exp' tu; t \in \boldsymbol{R}\}$ of G generated by u defined by the unique solution of $(d/dt)x_t = u \cdot x_t$, $x_0 = e$, where $u \cdot g$ means the derivative of the right translation $R_g: G \to G$. Set $d\rho(u) = (d/dt)|_{t=0}\rho(\exp' tu)$. Then, $d\rho(u) \in \boldsymbol{\Gamma}(TM)$ such that $\exp td\rho(u) = \rho(\exp' tu)$. Thus, we see that $\mathfrak{g}$ satisfies $(*)$. $d\rho: \mathfrak{g} \mapsto \boldsymbol{\Gamma}(TM)$ is obviously a Lie monomorphism. For the proof that $\mathfrak{g}$ satisfies $(**)$, we have only to note the following identity:

$$\mathrm{Ad}\,(\exp d\rho(u))d\rho(v) = d\rho(\mathrm{Ad}\,(\exp' u)v)\ , \tag{1}$$

which is proved by showing that $\exp'(\mathrm{Ad}\,(\exp' u)v) = \exp' u \cdot \exp' v \cdot \exp' -u$ and $\rho(\exp'(\mathrm{Ad}\,(\exp' u(v)) = \exp d\rho(u) \cdot \exp d\rho(v) \cdot \exp - d\rho(u) = \exp(\mathrm{Ad}\,(\exp d\rho(u))d\rho(v))$.

The second half of Theorem A is proved in the following

PROPOSITION 1.1. *Let $\mathfrak{g}$ be a Lie algebra consisting of C^∞-vector fields on M with the property $(*)$. Suppose that $\mathfrak{g}$ is a Banach-Lie algebra under a stronger topology than the C^∞-topology on M. Then $\mathfrak{g}$ satisfies $(**)$ and enlargable.*

PROOF. Let G be the group generated by $\{\exp u; u \in \mathfrak{g}\}$. G is a subgroup of $\mathscr{D}(M)$. On the other hand, since $\mathfrak{g}$ is a Banach-Lie algebra, there is a local Banach-Lie group V with $\mathfrak{g}$ as the Lie algebra. For any $u \in \mathfrak{g}$, a local one parameter group $\exp' tu$ is uniquely defined in V as the solution of $(d/dt)x_t = u \cdot x_t$, $x_0 = e$. By the inverse mapping theorem, the exponential mapping $\exp'$ is a real analytic diffeomorphism of a neighborhood U' of 0 in $\mathfrak{g}$ onto a neighborhood V' of the identity e in V.

Define a mapping ρ: $V' \mapsto G$ by $\rho(\exp' u) = \exp u$. Then, $\rho(\exp' su \cdot \exp' tu) = \exp su \cdot \exp tu$. For $\exp' u \in V'$, define $\mathrm{Ad}\,(\exp' u)v$ by $(d/ds)|_{s=0} \exp' u \cdot \exp' sv \cdot \exp' -u$. Since $\mathfrak{g}$ is the tangent space at the identity e of the Banach manifold V', we get $\mathrm{Ad}\,(\exp' u)v \in \mathfrak{g}$, and hence $\mathrm{Ad}\,(\exp' u)\mathfrak{g} = \mathfrak{g}$. Now, note that $\mathrm{Ad}\,(\exp' tu)v$ is a unique solution of the equation

$$(d/dt)w_t = [u, w_t]\ , \qquad w_0 = v\ . \tag{2}$$

On the other hand, (2) can be regarded as an equation with respect to vector fields on M. The unique solution of (2) is given by $\mathrm{Ad}\,(\exp tu)v$.

Thus, we get

$$(3)\qquad \mathrm{Ad}\,(\exp' tu)v = \mathrm{Ad}\,(\exp tu)v\ , \qquad v \in \mathfrak{g},\ u \in U'\ .$$

Since G is generated by $\{\exp u; u \in U'\}$, $\mathfrak{g}$ satisfies $(**)$.

Next, we prove that ρ is a local homomorphism. At the first, we have

$$(4)\qquad ((d/dt)(\exp' tu \cdot \exp' v))(\exp' tu \cdot \exp' v)^{-1} = u\ .$$

Set $\exp' v_t = \exp' tu \cdot \exp' v$ and $\dot{v}_t = (d/dt)v_t$. Since $\exp'$ is differentiable, we get

$$\begin{aligned} ((d/dt)\exp' v_t)(\exp' v_t)^{-1} &= ((\partial/\partial s)|_{s=0}\exp'(v_t + s\dot{v}_t))(\exp' v_t)^{-1} \\ &= (\partial/\partial s)|_{s=0}\int_0^1 (\partial/\partial\theta)[\exp'(\theta(v_t + s\dot{v}_t))(\exp' \theta v_t)^{-1}]d\theta \\ (5)\qquad &= \int_0^1 (\partial/\partial s)|_{s=0} dL_{\exp' \theta(v_t+s\dot{v}_t)} s\dot{v}_t \exp' -\theta v_t d\theta \\ &= \int_0^1 \mathrm{Ad}\,(\exp' \theta v_t)\dot{v}_t d\theta\ , \end{aligned}$$

where dL_g is the derivative of the left translation L_g. By (3)~(5), we have

$$(6)\qquad u = \int_0^1 \mathrm{Ad}\,(\exp' \theta v_t)\dot{v}_t d\theta = \int_0^1 \mathrm{Ad}\,(\exp \theta v_t)\dot{v}_t d\theta\ .$$

On the other hand, the same computation as in (5) holds for vector fields and hence

$$(7)\qquad ((d/dt)\exp v_t)(\exp v_t)^{-1}(x) = \int_0^1 (\mathrm{Ad}\,(\exp \theta v_t)\dot{v}_t)(x)d\theta\ , \qquad x \in M\ .$$

Hence $\rho(\exp' v_t) = \exp v_t$ satisfies the equation

$$(8)\qquad (d/dt)\rho(\exp' v_t) = u \cdot \rho(\exp' v_t)\ , \quad \rho(\exp' v(0)) = \exp v\ .$$

Thus, $\rho(\exp' v_t) = \exp tu \cdot \exp v$, hence $\rho(\exp' u \exp' v) = \exp u \cdot \exp v$.

Now, assume for a while that there is a sequence $\{v_n\}$ in U' such that $\lim v_n = 0$ and $\rho(\exp' v_n) = e$ for every n. Then, $\{\exp tv_n; t \in \boldsymbol{R}\}$ is a circle group contained in the group of diffeomorphisms on M. Since $\{v_n\}$ converges to 0 in the C^∞-topology and $\exp v_n = e$, any neighborhood of e of $\mathscr{D}(M)$ contains a compact subgroup. However, this contradicts Theorem 2, [13] p. 208, namely there is no small compact subgroup in $\mathscr{D}(M)$. Thus, we see that there is a neighborhood V'' of e in V such that $\rho\colon V'' \mapsto G$ is a monomorphism.

To prove $\mathfrak{g}$ is enlargable is to make G a Banach-Lie group. However, $\rho(V'')$ has a structure of local Banach-Lie group by identifying with V'' through ρ, and G is generated by $\rho(V'')$. Thus, by a standard method similar to finite dimensional Lie groups, one can give uniquely a structure of Banach-Lie group on G which is compatible with that on $\rho(V'')$.

The above proposition completes the proof of Theorem A.

The following is known by Lemma 2.2 [12]:

LEMMA 1.2. *Let* $\mathfrak{g}$ *be a Lie algebra contained in* $\Gamma(TM)$ *and satisfying* $(*)$ *and* $(**)$ *of Theorem* A. *Let* G *be the group generated by* $\{\exp u; u \in \mathfrak{g}\}$. *Then, the orbit* $N = G(x)$ *of a point* $x \in M$ *is a* C^∞-*immersed submanifold of* M *such that* $T_yN = \mathfrak{g}(y)$, *where* T_yN *is the tangent space at* y *and* $\mathfrak{g}(y) = \{u(y); u \in \mathfrak{g}\}$. (For the countability axiom, see [4] p. 96.)

COROLLARY 1.3. *Let* G *be a connected Banach-Lie group acting smoothly on a manifold* M. *Then, the action is transitive, if and only if it is ample.*

PROOF. Let ρ be the action of G. Set $d\rho(u) = (d/dt)|_{t=0}\rho(\exp' tu)$. Then, $d\rho$ is a continuous Lie homomorphism of the Lie algebra $\mathfrak{g}$ of G into $\Gamma(TM)$ such that $\rho(\exp' tu) = \exp td\rho(u)$. Let $\tilde{\mathfrak{g}}$ be the image of $d\rho$. Then, $\tilde{\mathfrak{g}}$ is a Banach-Lie algebra contained in $\Gamma(TM)$ and by Theorem A, $\tilde{\mathfrak{g}}$ satisfies $(*)$ and $(**)$.

Since a connected Banach-Lie group G is generated by $\{\exp u; u \in \mathfrak{g}\}$, $\rho(G)$ is generated by $\{\exp d\rho(u); u \in \mathfrak{g}\}$. By the hypothesis, M is an orbit of $\rho(G)$. Thus, by Lemma 1.2, we get $T_yM = \tilde{\mathfrak{g}}(y) = d\rho(\mathfrak{g})(y)$. The converse can be easily obtained by using Hahn-Banach's theorem and the implicit function theorem.

LEMMA 1.4. *Let* $\mathfrak{g}$ *be the Lie algebra of a connected Banach-Lie group* G *and* $\mathfrak{h}$ *a finite codimensional closed Lie subalgebra of* $\mathfrak{g}$. *Then, there is a unique Banach-Lie subgroup* H *of* G *with the Lie algebra* $\mathfrak{h}$. *The closure* $\bar{H}$ *of* H *is also a Banach-Lie subgroup of* G. *If* $\mathfrak{h}$ *is an ideal, then* H *and* $\bar{H}$ *are normal subgroups of* G. *In particular,* $G/\bar{H}$ *is a connected (finite dimensional) Lie group and hence a separable space.*

PROOF. By Hahn-Banach's theorem, there is a finite dimensional subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ (direct sum). Let $\tilde{\mathfrak{h}} = \{\mathfrak{h} \cdot g; g \in G\}$ be the right invariant distribution on G. Since $\mathfrak{h}$ is a subalgebra, $\tilde{\mathfrak{h}}$ is involutive. By Frobenius' theorem, there exist a neighborhood U (resp. V) of the origin of $\mathfrak{h}$ (resp. $\mathfrak{m}$) and a smooth diffeomorphism Φ of $U \oplus V$ onto a neighborhood W of the identity e of G such that

(i) the derivative $(d\Phi)_0$ of Φ at the origin is the identity,

(ii) for each $v \in V$, $\Phi(U \oplus \{v\})$ is an integral submanifold of $\tilde{\mathfrak{h}}$.

Note that we do not assume the second countability axiom on G. Let H be the maximal integral submanifold of $\tilde{\mathfrak{h}}$ through the identity. H is a C^∞-Banach manifold and a group because $H \cdot h = H$ for any $h \in H$. For any $u \in \mathfrak{h}$, $\exp' u$ is contained in H, and in $\Phi(U + \{0\})$ for sufficiently small u, because $(d/dt) \exp' tu = u \cdot \exp' tu \in \tilde{\mathfrak{h}}$. Thus, the exponential mapping $\exp'$ is a C^∞-mapping and hence a C^∞-diffeomorphism of a connected open neighborhood U' of 0 of $\mathfrak{h}$ onto an open neighborhood $\tilde{U}'$ of e of H.

Now, by Lemma 2.1 [12], we have $\mathrm{Ad}(\exp' u)\mathfrak{h} = \mathfrak{h}$ for any $u \in \mathfrak{h}$. Let U_1 be a star-shaped neighborhood of 0 of $\mathfrak{h}$ such that $U_1 \subset U'$ and $(\exp' U_1)(\exp' U_1)^{-1} \subset W$. Let $F\colon \exp' U_1 \times \exp' U_1 \mapsto W$ be the mapping defined by $F(g, h) = gh^{-1}$. Since $(dF)_{(g,h)}(u \cdot g, v \cdot h) = (u - \mathrm{Ad}(g)\,\mathrm{Ad}(h)^{-1}v)gh^{-1} \in \tilde{\mathfrak{h}}$ for any $u, v \in \mathfrak{h}$, $g, h \in \exp' U_1$, we see that the image of F is contained in $\Phi(U \oplus \{0\})$, and hence F is a C^∞-mapping of $\exp' U_1 \times \exp' U_1$ into $\Phi(U \oplus \{0\})$. Therefore, the neighborhood $\tilde{U}'$ in H has a structure of a local Banach-Lie group.

Let H' be the group generated by $\tilde{U}'$. H' is then an open subset of H and a Banach-Lie group with the Lie algebra $\mathfrak{h}$. Indeed, it is proved by the standard method similar to that in finite dimensional Lie groups. Note the right translation $R_g\colon H \mapsto H$ is smooth. Therefore H' is also a closed subset of H. Since H is connected, we get $H = H'$, hence H is a Banach-Lie group. Remark that if G satisfies the second countability axiom, then the above argument can be replaced by a simpler one parallel to that of [4] p. 95.

Now, suppose H is not closed in G. Then, there exists a sequence $\{u_n\}_{n \in N}$ in $\mathfrak{m}$ such that $u_n \neq 0$, $\lim_{n\to\infty} u_n = 0$ and $\Phi(u_n) \in H$. Taking a subsequence if necessary, we assume that the sequence $\{u_n/\|u_n\|\}$ converges to an element $u \in \mathfrak{m}$. By a little careful argument, we can choose a C^1-curve $c(t)$ in $\mathfrak{m}$ such that $(d/dt)|_{t=0} c(t) = u$ and that the image of the curve contains infinitely many point of $\{u_n\}$. Taking again a subsequence, we may assume that for each n there is a value t_n of the parameter with $c(t_n) = u_n$ and $\Phi(c(t_n)) \in H$. Obviously, $\lim_{n\to\infty} t_n = 0$. Since $\Phi(c(t_n)) \in H$, we have $\mathrm{Ad}(\Phi(c(t_n)))\mathfrak{h} = \mathfrak{h}$ for all $n \in N$, so that $(d/dt)|_{t=0} \mathrm{Ad}(\Phi(c(t)))\mathfrak{h} \subset \mathfrak{h}$, because $\mathfrak{h}$ is closed. Since $(d/dt)|_{t=0} \mathrm{Ad}(\Phi(c(t)))v = \mathrm{ad}((d\Phi)_0 u)v$ and $(d\Phi)_0 u = u$, we have $[u, \mathfrak{h}] \subset \mathfrak{h}$. Thus, $\mathfrak{h}_1 = \boldsymbol{R} \cdot u \oplus \mathfrak{h}$ is a Banach-Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ as an ideal. Moreover, since $\Phi(c(t_n))^k \in H$ for any k, $\exp' tu$ is contained in $\bar{H}$. It is because of the fact that for any C^1-curve $F(t)$ in G with $F(0) = e$, $\{F(t/k)^k\}$ converges to $\exp' t\dot{F}(0)$.

Let H_1 be the Banach-Lie subgroup with the Lie algebra $\mathfrak{h}_1$. Then it is not hard to see that $\bar{H}_1 = \bar{H}$. Note that codim $\mathfrak{h}_1 <$ codim $\mathfrak{h}$. If H_1 is not closed in G, then one can make H_2 such that $\bar{H}_2 = \bar{H}_1$ and codim $\mathfrak{h}_2 <$ codim $\mathfrak{h}_1$ by the same procedure as above. Since, codim $\mathfrak{h}$ is finite, the above procedure must stop at some stage H_l. Namely, we have $H_l = \bar{H}_l = \bar{H}_{l-1} = \cdots = \bar{H}$. H_l is obviously a Banach-Lie group.

If $\mathfrak{h}$ is an ideal, then by Lemma 2.1 [12] we see $\mathrm{Ad}(\exp' u)\mathfrak{h} = \mathfrak{h}$ for any $u \in \mathfrak{g}$. Since $\exp' \mathrm{Ad}(\exp' u)v = \exp' u \cdot \exp' v \cdot \exp' -u$, the desired results can be easily obtained.

REMARK 1. Let G be a connected Banach-Lie group and H a closed Banach-Lie subgroup of finite codimension. Then it is trivial that G/H is a manifold with or without the separability axiom. However, the separability of G/H will be shown in the next section. So, G/H is in fact a finite dimensional manifold.

COROLLARY 1.5. *By the same notations as above, $\mathfrak{h}$ is an ideal of $\mathfrak{h}_l$. If $\mathfrak{h}$ is a proper maximal finite codimensional subalgebra which is not an ideal of* $\mathfrak{g}$, *then H is closed in G.*

LEMMA 1.6. *Let G be a connected Banach-Lie group with the Lie algebra* $\mathfrak{g}$. *For any closed subalgebra $\mathfrak{h}$ of* $\mathfrak{g}$, *there is an immersed Banach-Lie subgroup H of G having $\mathfrak{h}$ as the Lie algebra. Moreover, if $\mathfrak{h}$ is an ideal of* $\mathfrak{g}$, *then H is a normal subgroup of G.*

PROOF. By a criterion of enlargability (cf. 0°), there is a simply connected Banach-Lie group $\tilde{H}$ with the Lie algebra $\mathfrak{h}$. Since there is a continuous inclusion $\mathfrak{h} \subset \mathfrak{g}$, there is a smooth homomorphism $\tilde{\rho}$ of $\tilde{H}$ into G such that the kernel of $\tilde{\rho}$ is a discrete normal subgroup of $\tilde{H}$. Thus, $H = \tilde{H}/\mathrm{Ker}\,\tilde{\rho}$ is the desired group. The induced monomorphism $\rho: H \mapsto G$ is obviously an immersion.

Identifying H with $\rho(H)$, we see that for every $u \in \mathfrak{h}$, $\exp' tu$ is contained in H. Suppose now that $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. Then, $\mathrm{Ad}(\exp' tu)\mathfrak{h} = \mathfrak{h}$ for any $u \in \mathfrak{g}$ because $\mathrm{Ad}(\exp' tu)v$ is the unique solution of the equation (2) and $[u, \mathfrak{h}] \subset \mathfrak{h}$. Thus, by the same reasoning as in Lemma 1.4, H is a normal subgroup of G.

REMARK 2. Let G be a connected Banach-Lie group with the Lie algebra $\mathfrak{g}$ acting smoothly on a manifold M. Let ρ be the action. Then, $N = \mathrm{Ker}\,\rho$ is a normal and closed subgroup of G and $d\rho$ is a Lie homomorphism of $\mathfrak{g}$ into $\Gamma(TM)$. The kernel $\mathfrak{n}$ of $d\rho$ is a closed ideal of $\mathfrak{g}$. By Theorem A, the Banach-Lie algebra $\mathfrak{g}/\mathfrak{n}$ is enlargable, and indeed

$G/N \cong \rho(G)$ is a Banach-Lie group with the Lie algebra $\mathfrak{g}/\mathfrak{n}$. On the other hand, by 1.6, $\mathfrak{n}$ generates an immersed, normal Banach-Lie subgroup N' of G. Since the Lie algebra of N' is $\mathfrak{n}$ and $\rho(\exp' tu) = e$, we see that $N' \subset N$. However, it is not clear whether N' = the identity component of N or not. The reason of this difficulty is that one can not use the implicit function theorem. So, if $\mathfrak{n}$ has a direct summand in $\mathfrak{g}$ (for instance the case of $\operatorname{codim} \mathfrak{n} < \infty$) or if $\mathfrak{g}$ is a Hilbert space, then one can conclude $N' = N$.

2° **Proof of Theorem B and the separability of a factor space.** Let M be a compact manifold. Suppose G is a connected Banach-Lie group acting smoothly, transitively and effectively on M. By 1.3 and the implicit function theorem, the isotropy subgroup G_0 at $x_0 \in M$ is a closed Banach-Lie subgroup of G, and $M = G/G_0$. Thus, for the proof of Theorem B, it is enough to show the following:

PROPOSITION 2.1. *Suppose G is a connected Banach-Lie group and H a closed finite condimensional Banach-Lie subgroup of G. If the factor space G/H is compact, then $N = \bigcap_{g \in G} gHg^{-1}$ is a finite codimensional closed normal Banach-Lie subgroup of G.*

The above proposition will be proved in several steps below.

Let $M = G/H$. By the hypothesis, M is a compact C^∞-manifold on which G acts smoothly and transitively. Let ρ be the action. We use the same notations as in Remark 2 in the previous section. Since $H \supset N$, we have only to show that $\dim \mathfrak{g}/\mathfrak{n} < \infty$ for the proof of 2.1.

The Banach-Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{n}$ is naturally identified with a subalgebra of $\Gamma(TM)$ and the inclusion mapping is continuous. Since M is compact, there are $u_1, \cdots, u_k \in \tilde{\mathfrak{g}}$ ($k < \infty$) such that $\{u_1(x), \cdots, u_k(x)\}$ spans the tangent space T_xM of M at every x. We set $D = \sum_{i=1}^k \operatorname{ad}(u_i)^2$.

LEMMA 2.2. *D is a strongly elliptic differential operator of order 2 of $\Gamma(TM)$ into $\Gamma(TM)$. Moreover, $D\tilde{\mathfrak{g}} \subset \tilde{\mathfrak{g}}$ and the mapping $D\colon \tilde{\mathfrak{g}} \mapsto \tilde{\mathfrak{g}}$ is a bounded operator.*

PROOF. Obviously, $D\tilde{\mathfrak{g}} \subset \tilde{\mathfrak{g}}$, and the mapping $D\colon \tilde{\mathfrak{g}} \mapsto \tilde{\mathfrak{g}}$ is bounded. Let $(x_1, \cdots, x_n)$ be a C^∞-local coordinate system of M at $x \in M$. By this coordinate system, every u_j is written in the form $u_j = \sum_{i=1}^n X_j^i(\partial/\partial x_i)$, $j = 1 \sim k$, where X_j^i are smooth functions in $x_1, \cdots, x_n$. Thus, for any $v \in \Gamma(TM)$ we have

$$(Dv)(x) = \sum_{i=1}^n \left\{\sum_{j=1}^k \sum_{a,b=1}^n X_j^a X_j^b (\partial^2/\partial x_a \partial x_b) v_i + (\text{lower order terms})\right\}(\partial/\partial x_i) \tag{9}$$

Thus, the symbol of D is given by

$$(10) \qquad \sigma(D)\xi = \Big(\sum_{j=1}^{k} \langle \xi, u_j \rangle^2\Big) I, \qquad \xi \in T^*M - \{0\},$$

where T^*M is the cotangent bundle, $I: TM \mapsto TM$ is the identity mapping and $\langle \xi, u_j \rangle$ means the natural pairing. Hence it is clear that $\sigma(D) = \sigma(D^*)$, where D^* is the formal adjoint operator of D with respect to an arbitrarily fixed C^∞-riemannian metric on M. Let $|\xi|$ be the length of ξ. Since $\{u_1(x), \cdots, u_k(x)\}$ spans the tangent space T_xM for every $x \in M$, there is a positive constant c such that $\sum_{j=1}^{k} \langle \xi, u_j \rangle^2 \geqq c|\xi|^2$. Hence, $\langle (\sigma(D)\xi - c|\xi|^2)X, X \rangle \geqq 0$ for any $X \in TM$, i.e. D is strongly elliptic.

Let TM^c be the complexification of TM, and $\Gamma(TM^c)$ the space of all C^∞-sections of TM^c with the C^∞-topology. Then, $\Gamma(TM^c)$ is the complexification of $\Gamma(TM)$, that is $\Gamma(TM^c) = \Gamma(TM) \otimes \boldsymbol{C}$. The complexification $\tilde{\mathfrak{g}}^c$ of $\tilde{\mathfrak{g}}$ is naturally imbedded in $\Gamma(TM^c)$, and the operator D can be regarded as a differential operator of $\Gamma(TM^c)$ into itself such that $D\tilde{\mathfrak{g}}^c \subset \tilde{\mathfrak{g}}^c$ and that $D: \tilde{\mathfrak{g}}^c \mapsto \tilde{\mathfrak{g}}^c$ is bounded. The following proposition is known in functional analysis: (For the proof, see the appendix of this paper.)

PROPOSITION 2.3. *Let E be a C^∞-complex, finite dimensional vector bundle over a compact riemannian manifold M and $\Gamma(E)$ the space of the C^∞-sections of E with the C^∞-topology. Let $D: \Gamma(E) \mapsto \Gamma(E)$ be a strongly elliptic differential operator of order 2 such that $\sigma(D) = \sigma(D^*)$. Then, there are countably many eigenvalues $\{\lambda_n\}_{n=1,2,\cdots}$ such that the following are satisfied*:

(1) $\dim E_{\lambda_n} < \infty$, *where E_{λ_n} are generalized eigenspaces, i.e. the linear space of the elements $v \in \Gamma(E)$ such that $(D - \lambda_n)^m v = 0$ for some integer m.*

(2) $\lim \operatorname{Re} \lambda_n = \infty$.

(3) *The generalized eigenspaces are complete in $\Gamma(E)$*, i.e. $\sum \oplus E_{\lambda_n}$ *is dense in $\Gamma(E)$.*

(4) *Setting $\mathfrak{F}_n = (\sum_{k \geqq n} \oplus E_{\lambda_k})^-$, we have $\bigcap \mathfrak{F}_n = \{0\}$.*

Now, let $\{\lambda_n\}_{n=1,2,\cdots}$ be the eigenvalues of D. Let $\tilde{\mathfrak{g}}_n = \tilde{\mathfrak{g}}^c \cap \mathfrak{F}_n$. Since the inclusion $\tilde{\mathfrak{g}}^c \subset \Gamma(TM^c)$ is continuous, the $\tilde{\mathfrak{g}}_n$ are closed finite codimensional subspaces of the Banach space $\tilde{\mathfrak{g}}^c$ such that $\tilde{\mathfrak{g}}^c = \tilde{\mathfrak{g}}_1 \supset \tilde{\mathfrak{g}}_2 \cdots \supset \tilde{\mathfrak{g}}_n \supset \cdots$ and $\bigcap \tilde{\mathfrak{g}}_n = \{0\}$. It is clear that $D\tilde{\mathfrak{g}}_n \subset \tilde{\mathfrak{g}}_n$ for every n.

Set $F_k = \tilde{\mathfrak{g}}_k / \tilde{\mathfrak{g}}_{k+1}$ and $F = \sum \oplus F_k$ (arbitrarily finite sum). We define a norm $\|\ \|$ on F by the following manner: For any $\hat{u} = \sum \hat{u}_k$, define $\|\hat{u}\| = \sum \|\hat{u}_k\|$ and $\|\hat{u}_k\| = \inf\{\|u_k\|; u_k \in \hat{u}_k\}$. F is a normed linear space, and D induces a linear operator $\hat{D}$ of F into itself.

LEMMA 2.4. *$\hat{D}: F \mapsto F$ is a bounded linear operator.*

PROOF. There is a positive constant c such that $||Du|| \leqq c||u||$. Thus, if $\hat{u} = \sum \hat{u}_k$, $\hat{u}_k \in F_k$, then

$$||\hat{D}\hat{u}|| = \sum ||\hat{D}\hat{u}_k|| \leqq \sum \inf\{||Du_k||;\ u_k \in \hat{u}_k\} \leqq \sum c||\hat{u}_k|| = c||\hat{u}|| .$$

On the other hand, since $\dim F_k < \infty$, there is an integer ν_k such that $(\hat{D} - \lambda_k I)^{\nu_k} F_k = \{0\}$, so that there is a non-trivial element $w_k \in F_k$ such that $\hat{D}w_k = \lambda_k w_k$. Since $\lim \lambda_n = \infty$, Lemma 2.4 shows that $F_k = \{0\}$ for sufficiently large k. Hence, $\tilde{\mathfrak{g}}_n = \{0\}$ for some n. Therefore, $\dim \tilde{\mathfrak{g}} < \infty$, and $\mathfrak{n}$ is a closed finite codimensional ideal of $\mathfrak{g}$. Hence by Remark 2 in 1°, N is a normal Banach-Lie subgroup with the Lie algebra $\mathfrak{n}$. This complete the proof of Proposition 2.1, and hence Theorem B.

By Theorem B, an infinite dimensional Banach-Lie group can act only on a non-compact manifold. However, such a Banach-Lie group seems to be severely restricted. The following was a main theorem of [16].

PROPOSITION 2.5. *Let G be a connected Banach-Lie group with the Lie algebra $\mathfrak{g}$. Suppose $\mathfrak{h}$ is a proper finite codimensional closed maximal subalgebra of $\mathfrak{g}$, Then, $\mathfrak{h}$ contains a finite codimensional ideal of $\mathfrak{g}$.*

The above result was proved in several steps in [16] by using the classification of infinite primitive Lie algebras. The theorem stated in the introduction is an immediate conclusion from the above result.

Now, if an infinite dimensional Banach-Lie group G acts smoothly, effectively and transitively on M, then the isotropy subgroup of G can not be a maximal subgroup. Moreover, we have the following:

LEMMA 2.6. *Let G be a connected Banach-Lie group and H a finite codimensional closed and connected Banach-Lie subgroup of G. Suppose the Lie algebra $\mathfrak{h}$ of H is not maximal in the Lie algebra $\mathfrak{g}$ of G. Then, there is a closed and connected Banach-Lie subgroup H' such that $H' \supsetneqq H$ and H contains a finite codimensional closed normal Banach-Lie subgroup N of H'. In particular, $\bigcap_{h \in H'} hHh^{-1}$ is a finite codimensional Banach-Lie subgroup of H'.*

PROOF. Let $\mathfrak{h}''$ be a subalgebra of $\mathfrak{g}$ such that $\mathfrak{g} \supsetneqq \mathfrak{h}'' \supsetneqq \mathfrak{h}$ and there is no non-trivial subalgebra between $\mathfrak{h}''$ and $\mathfrak{h}$. Since $\operatorname{codim} \mathfrak{h} < \infty$, $\mathfrak{h}''$ is a closed subspace of $\mathfrak{g}$. Let H'' be the Banach-Lie subgroup of G generated by $\mathfrak{h}''$. Since the inclusion $H'' \subset G$ is continuous, H can be regarded as a closed subgroup of H''. By Proposition 2.5, there is a closed normal Banach-Lie subgroup N'' of H'' contained in H and such that $\dim H''/N'' < \infty$. Let H' and N be the closures of H'' and N'' in G

respectively. By Lemma 1.4, these are Banach-Lie subgroups of G. Obviously, $H' \supsetneqq H$, $H \supset N$ and N is a normal subgroup of H'.

Now, $\tilde{N} = \bigcap_{h \in H'} hHh^{-1}$ contains N. Then, $\tilde{N}/N$ is a closed normal subgroup of the finite dimensional Lie group H'/N. Thus, $\tilde{N}/N$ is a Lie group. Since the canonical projection $\pi: H' \mapsto H'/N$ is smooth, we see by the implicit function theorem that $\tilde{N} = \pi^{-1}(\tilde{N}/N)$ is a Banach-Lie subgroup of G.

COROLLARY 2.7. *Notations and assumptions being as in the above lemma, there is an increasing series* $H = G_0 \subsetneqq G_1 \subsetneqq G_2 \subsetneqq \cdots \subsetneqq G_l = G$ *of closed and connected Banach-Lie subgroups satisfying the following*:

(1) *There is no non-trivial closed and connected Banach-Lie subgroup between* G_{i-1} *and* G_i *for each* $i = 1, 2, \cdots, l$.

(2) $N_i = \bigcap_{g \in G_{i+1}} gG_ig^{-1}$ *is a finite codimensional Banach-Lie subgroup of* G *and a normal subgroup of* G_{i+1}.

Proof is easy by using the above lemma.

COROLLARY 2.8. *Let* G *be a connected Banach-Lie group and* H *a finite codimensional closed Banach-Lie subgroup. Then,* G/H *is a (separable) smooth manifold.*

PROOF. G_i/G_{i-1} is a separable manifold, because G_i/N_i acts transitively on G_i/G_{i-1} and G_i/N_i is a finite dimensional Lie group and hence a separable manifold. Hence the total space G/H is separable.

3° **Almost solvable Banach-Lie groups.** In this section, the proof of Theorem C will be given.

A triple $\{G, H, K\}$ of connected Banach-Lie groups with the Lie algebras $\{\mathfrak{g}, \mathfrak{h}, \mathfrak{k}\}$ is provisionally said to be an *AS-triple system* if the following are fulfilled:

(i) $H \supsetneqq K$ and they are finite codimensional closed Banach-Lie subgroups of G.

(ii) Set $\mathfrak{n} = \bigcap_{g \in G} \mathrm{Ad}\,(g)\mathfrak{h}$. Then, $\mathfrak{g}/\mathfrak{n}$ is almost solvable. (Cf. 0°)

(iii) Set $\mathfrak{n}' = \bigcap_{h \in H} \mathrm{Ad}\,(h)\mathfrak{k}$. Then $\dim \mathfrak{h}/\mathfrak{n}' < \infty$.

By Corollary 2.7 combined with an induction, the proof of Theorem C is reduced immediately to the following:

PROPOSITION 3.1. *Let* $\{G, H, K\}$ *be an AS-triple system and let* $\mathfrak{n}'' = \bigcap_{g \in G} \mathrm{Ad}\,(g)\mathfrak{k}$. *Then* $\mathfrak{g}/\mathfrak{n}''$ *is almost solvable.*

The above proposition will be proved in several lemmas below. If $\mathfrak{g}$ is almost solvable, then we consider the class of all finite codimensional closed solvable ideals of $\mathfrak{g}$, and take a maximal element $\mathfrak{z}$. Then, $\mathfrak{z}$ is a closed ideal of $\mathfrak{g}$ and contains all solvable ideals of $\mathfrak{g}$. Indeed, let

$\mathscr{I}$ be a solvable ideal of $\mathfrak{g}$. Then, $\mathfrak{z} + \mathscr{I}/\mathfrak{z} = \mathscr{I}/\mathfrak{z} \cap \mathscr{I}$ is solvable and $\mathfrak{z} + \mathscr{I}$ is a closed ideal of $\mathfrak{g}$, because $\mathfrak{z} + \mathscr{I}/\mathfrak{z}$ is closed in the finite dimensional space $\mathfrak{g}/\mathfrak{z}$. Hence $\mathfrak{z} + \mathscr{I}$ is a finite codimensional closed solvable ideal of $\mathfrak{g}$, so that $\mathfrak{z} + \mathscr{I} = \mathfrak{z}$. The maximal element $\mathfrak{z}$ is called the *radical* of $\mathfrak{g}$. It is clear that $\mathfrak{g}/\mathfrak{z}$ is a finite dimensional semi-simple Lie algebra.

Now, to prove Proposition 3.1, we start with the following:

LEMMA 3.2. *Let* $\mathfrak{g}$ *be a Banach-Lie algebra and* $\mathscr{I}$ *a closed ideal of* $\mathfrak{g}$. *Then* $\mathfrak{g}$ *is almost solvable, if and only if* $\mathscr{I}$ *and* $\mathfrak{g}/\mathscr{I}$ *are almost solvable.*

PROOF. Let $\mathfrak{z}_0$ be the radical of $\mathscr{I}$. For any $\tilde{u} \in \mathfrak{g}/\mathfrak{z}_0$, $\operatorname{ad}(\tilde{u})$ induces a derivation $A(\tilde{u})$ of $\mathscr{I}/\mathfrak{z}_0$. Since $\mathscr{I}/\mathfrak{z}_0$ is semi-simple, there is $\tilde{v} \in \mathscr{I}/\mathfrak{z}_0$ such that $A(\tilde{u}) = \operatorname{ad}(\tilde{v})$. $\tilde{v}$ is uniquely determined by $\tilde{u}$. Thus, there is a Lie homomorphism $\rho\colon \mathfrak{g}/\mathfrak{z}_0 \mapsto \mathscr{I}/\mathfrak{z}_0$, $\rho(\tilde{u}) = \tilde{v}$, such that $\rho(\tilde{v}) = \tilde{v}$ for any $\tilde{v} \in \mathscr{I}/\mathfrak{z}_0$. Hence the exact sequence

$$0 \mapsto \mathscr{I}/\mathfrak{z}_0 \mapsto \mathfrak{g}/\mathfrak{z}_0 \mapsto \mathfrak{g}/\mathscr{I} \mapsto 0$$

splits. Thus, $\mathfrak{g}/\mathfrak{z}_0$ is almost solvable. Consider the exact sequence

$$0 \to \mathfrak{z}_0 \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{z}_0 \to 0 .$$

The full inverse of the radical of $\mathfrak{g}/\mathfrak{z}_0$ is also a finite codimensional solvable ideal of $\mathfrak{g}$, hence $\mathfrak{g}$ is almost solvable. The converse is easy to prove.

Now, let $\{G, H, K\}$ be an AS-triple system with the Lie algebras $\{\mathfrak{g}, \mathfrak{h}, \mathfrak{k}\}$. Consider the disjoint union $\bigcup_{gH \in G/H} \mathfrak{n}/\operatorname{Ad}(g)(\mathfrak{n} \cap \mathfrak{n}')$. By Lemma 1.6 $\operatorname{Ad}(g)\mathfrak{n} = \mathfrak{n}$ for any $g \in G$. Hence $\operatorname{Ad}(g)(\mathfrak{n} \cap \mathfrak{n}')$ depends only on $gH \in G/H$. Thus $V = \bigcup_{gH \in G/H} \mathfrak{n}/\operatorname{Ad}(g)(\mathfrak{n} \cap \mathfrak{n}')$ makes sence and V is a smooth finite dimensional vector bundle over G/H with the fibre $\mathfrak{n}/\mathfrak{n} \cap \mathfrak{n}'$ and the group of the automorphisms of $\mathfrak{n}/\mathfrak{n} \cap \mathfrak{n}'$ as the transition functions. The fibre of V is a finite dimensional Lie algebra and hence the space of the smooth sections $\Gamma(V)$ becomes a Lie algebra by the point-wise Lie bracket product. Define a Lie homomorphism $\dot{\sigma}\colon \mathfrak{n} \mapsto \Gamma(V)$ by $\dot{\sigma}(w)(xH) = w + \operatorname{Ad}(x)(\mathfrak{n} \cap \mathfrak{n}') \in \mathfrak{n}/\operatorname{Ad}(x)(\mathfrak{n} \cap \mathfrak{n}')$.

Let V_0 be the subbundle given by the radicals of the fibers. V_0 is then a smooth subbundle of V and there is a projection $\pi\colon \Gamma(V) \mapsto \Gamma(V/V_0)$. We set $\dot{\sigma}_\pi(w) = \pi\dot{\sigma}(w)$, $w \in \mathfrak{n}$. Let $\mathfrak{n}''$ be the kernel of $\dot{\sigma}$. Then, obviously $\mathfrak{n}'' = \bigcap_{x \in G} \operatorname{Ad}(x)(\mathfrak{n} \cap \mathfrak{n}') = \bigcap_{x \in G} \operatorname{Ad}(x)\mathfrak{k}$.

Now, assume for a while that $\dim \dot{\sigma}_\pi(\mathfrak{n}) < \infty$. Let $\mathfrak{n}_1$ be the kernel of $\dot{\sigma}_\pi$. Since $\dot{\sigma}(\mathfrak{n}_1) \subset \Gamma(V_0)$ and the fibre of V_0 is solvable, we get that $\dot{\sigma}(\mathfrak{n}_1) \cong \mathfrak{n}_1/\mathfrak{n}''$ is solvable. Thus, $\mathfrak{n}/\mathfrak{n}''$ is almost solvable. Since $\mathfrak{g}/\mathfrak{n}$ is assumed to be almost solvable, Lemma 3.2 shows that $\mathfrak{g}/\mathfrak{n}''$ is almost solvable. Thus, for the proof of 3.1, we have only to show $\dim \mathring{\sigma}_\pi(\mathfrak{n}) < \infty$.

Let ρ_0 be the radical of $\mathfrak{n}/\mathfrak{n}\cap\mathfrak{n}'$ and ρ_0' the full inverse of ρ_0 by the natural projection of $\mathfrak{n}$ onto $\mathfrak{n}/\mathfrak{n}\cap\mathfrak{n}'$. The factor bundle V/V_0 is then given by the disjoint union $\bigcup_{xH\in G/H}\mathfrak{n}/\mathrm{Ad}(x)\rho_0'$. Let $\mathscr{U}$ be an open neighborhood of xH in G/H such that there exists a local smooth section $\mathscr{A}\colon \mathscr{U}\mapsto G$ of the fibre bundle $\{G; H, G/H\}$. (The existence of a smooth section is ensured by the implicit function theorem.) Now, for any $yH\in\mathscr{U}$, $\mathrm{Ad}(\mathscr{A}(yH))$ is an isomorphism of $\mathfrak{n}$ onto itself, hence induces an isomorphism $A_{yH}\colon \mathfrak{n}/\rho_0'\mapsto\mathfrak{n}/\mathrm{Ad}(y)\rho_0'$. Thus, we get a local, smooth trivialization $\tau\colon\mathscr{U}\times\mathfrak{n}/\rho_0'\mapsto V/V_0$ defined by $\tau(yH, w)=(yH, A_{yH}w)$. Since the A_{yH} are Lie algebra isomorphisms, a local section of V/V_0 on $\mathscr{U}$ can be naturally identified with a smooth mapping of $\mathscr{U}$ into $\mathfrak{n}/\rho_0'$. The Lie bracket product of $\Gamma(V/V_0)$ is translated into the pointwise Lie bracket product.

For any $w\in\mathfrak{n}$, we denote by $\mu(w)$ the smooth mapping of $\mathscr{U}$ into $\mathfrak{n}/\rho_0'$ defined by $\mu(w)(yH)=A_{yH}^{-1}(w+\mathrm{Ad}(y)\rho_0')$. For any $u\in\mathfrak{g}$, we denote by X_u the smooth vector field on G/H defined by u, i.e. $X_u(yH)=(d/dt)|_{t=0}\exp tu\cdot yH$. For any $v\in\mathfrak{h}$, $\mathrm{ad}(v)$ leaves $\mathfrak{n}$ and $\mathfrak{n}'$ invariant, hence induces a derivation of $\mathfrak{n}/\mathfrak{n}\cap\mathfrak{n}'$. Since $\rho_0'/\mathfrak{n}\cap\mathfrak{n}'$ is the radical of $\mathfrak{n}/\mathfrak{n}\cap\mathfrak{n}'$, $\mathrm{ad}(v)$ induces also a derivation $\delta(v)$ of $\mathfrak{n}/\rho_0'$. Since $\mathfrak{n}/\rho_0'$ is a semi-simple Lie algebra, there is $v'\in\mathfrak{n}/\rho_0'$ such that $\delta(v)=\mathrm{ad}(v')$. Thus, we get a Lie homomorphism ϑ of $\mathfrak{h}$ into $\mathfrak{n}/\rho_0'$ such that $\vartheta(v)=v'$. ϑ: $\mathfrak{h}\to\mathfrak{n}/\rho_0'$ is of course a bounded linear mapping.

LEMMA 3.3. $\mu(w)(yH)=A_{yH}^{-1}(w+\mathrm{Ad}(y)\rho_0')=\mathrm{Ad}(\mathscr{A}(yH))^{-1}w+\rho_0'\in\mathfrak{n}/\rho_0'$, $w\in\mathfrak{n}$, $yH\in\mathscr{U}$. *Moreover, there is a smooth mapping* $\lambda\colon\mathfrak{g}\times\mathscr{U}\mapsto\mathfrak{h}$ *depending on the local section* $\mathscr{A}$ *such that for every fixed* $yH\in\mathscr{U}$, $\lambda(*, yH)\colon\mathfrak{g}\mapsto\mathfrak{h}$ *is a bounded linear operator and such that* $\mu([u, w])(yH)=(-X_u\mu(w))(yH)+\mathrm{ad}(\vartheta(\lambda(u, yH))\mu(w)(yH)$, $u\in\mathfrak{g}$, $w\in\mathfrak{n}$. (*Note that the second term does not involve differentiation.*)

PROOF. The first one is easy to obtain by definitions. The second one is obtained by the following computation:

$$\mu([u, w])(yH)=\mu\Big(\frac{d}{dt}\Big|_{t=0}\mathrm{Ad}(\exp tu)w\Big)(yH)=\frac{d}{dt}\Big|_{t=0}\mu(\mathrm{Ad}(\exp tu)w)(yH)$$

$$=\frac{d}{dt}\Big|_{t=0}\mathrm{Ad}(\mathscr{A}(yH)^{-1}\exp tu\cdot\mathscr{A}(\exp -tu\cdot yH)\cdot\mathscr{A}(\exp -tu\cdot yH)^{-1})w+\rho_0'$$

$$=\frac{d}{dt}\Big|_{t=0}\mu(w)(\exp -tu\cdot yH)$$

$$+\frac{d}{dt}\Big|_{t=0}\{\mathrm{Ad}(\mathscr{A}(yH)^{-1}\exp tu\cdot\mathscr{A}(\exp -tu\cdot yH)\cdot\mathscr{A}(yH)^{-1})w+\rho_0'\}$$

$$= (-X_u\mu(w))(yH) + \frac{d}{dt}\Big|_{t=0}\{\mathrm{Ad}\,(\mathscr{A}(yH)^{-1}\exp tu\cdot\mathscr{A}(\exp -tu\cdot yH))\,\mathrm{Ad}\,(\mathscr{A}(yH)^{-1})w+\rho_0'\}\,.$$

Since $\exp - tu\cdot\mathscr{A}(yH)H \ni \mathscr{A}(\exp - tu\cdot yH)$, we have that $(d/dt)|_{t=0}\mathscr{A}(yH)^{-1}\times\exp tu\cdot\mathscr{A}(\exp - tu\cdot yH)$ is contained in $\mathfrak{h}$. We denote it by $\lambda(u, yH)$. $\lambda(*, yH)\colon \mathfrak{g}\to\mathfrak{h}$ is then a bounded linear operator, because $\mathscr{A}(yH)^{-1}\times\exp u\cdot\mathscr{A}(\exp - u\cdot yH)$ is smooth with respect to u and yH. Therefore,

$$\mu([u, w])(yH) = (-X_u\mu(w))(yH) + \delta(\lambda(u, yH))\mu(w)(yH)$$
$$= \{-X_u\mu(w) + \mathrm{ad}\,(\vartheta(\lambda(u, *))\mu(w)\}(yH)\,.$$

Now, let $\Gamma(\mathscr{U}, \mathfrak{n}/\rho_0')$ be the Lie algebra of the smooth mappings of $\mathscr{U}$ into $\mathfrak{n}/\rho_0'$ with the pointwise Lie bracket product. $\mu\colon \mathfrak{n}\mapsto\Gamma(\mathscr{U}, \mathfrak{n}/\rho_0')$ is then a Lie homomorphism. Taking the complexification $\mathfrak{n}^c = \mathfrak{n}\oplus\sqrt{-1}\,\mathfrak{n}$, $(\mathfrak{n}/\rho_0')^c = \mathfrak{n}/\rho_0'\oplus\sqrt{-1}\,\mathfrak{n}/\rho_0'$, μ can be regarded as a complex Lie homomorphism of $\mathfrak{n}^c$ into $\Gamma(\mathscr{U}, (\mathfrak{n}/\rho_0')^c)$. Let $(x_1, \cdots, x_n)$ be a smooth local coordinate system on $\mathscr{U}$, and define a following filtration on $\mu(\mathfrak{n}^c)$: Let $\tilde{\mathfrak{n}}_0 = \mu(\mathfrak{n}^c)$, $\tilde{\mathfrak{n}}_k = \{\tilde{w}\in\tilde{\mathfrak{n}}_0;\ j^{k-1}_{xH}\tilde{w} = 0\}$, where j^s_{xH} means the s-th jet at $xH\in\mathscr{U}$. Obviously, $[\tilde{\mathfrak{n}}_k, \tilde{\mathfrak{n}}_l]\subset\tilde{\mathfrak{n}}_{k+l}$. Set $F_k = \tilde{\mathfrak{n}}_k/\tilde{\mathfrak{n}}_{k+1}$ and $F = \sum_{k\geq 0}\oplus F_k$. For every $u\in\mathfrak{g}$, $X_u - \mathrm{ad}\,(\vartheta(\lambda(u, *)))$ can be regarded as a mapping of $\mu(\mathfrak{n}^c)$ into itself, and induces a linear mapping $\tilde{X}_u$ of F into itself such that if $X_u(xH)\neq 0$, then $\tilde{X}_uF_k\subset F_{k-1}$. Indeed, if $w = \sum_{|\alpha|=k} w_\alpha x^\alpha$, $w_\alpha\in(\mathfrak{n}/\rho_0')^c$, is an element of F_k, then $\tilde{X}_u w = \sum_{i=1}^n\sum_{|\alpha|=k} a_i w_\alpha(\partial/\partial x_i)x^\alpha$, where $X_u(xH) = \sum a_i(\partial/\partial x_i)$, $a_i\in\boldsymbol{R}$ and $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$ and $|\alpha| = \alpha_1+\alpha_2+\cdots+\alpha_n$.

LEMMA 3.4. *There is a norm* $\|\ \|$ *on* F *such that* (1) F *is a normed Lie algebra and* (2) $\tilde{X}_u\colon F\mapsto F$ *is a bounded linear operator for any* $u\in\mathfrak{g}$.

PROOF. Let $\mathfrak{n}_0' = \mathfrak{n}^c$ and $\mathfrak{n}_k' = \mu^{-1}(\mathfrak{n}_k)$, $k=1, 2, 3, \cdots$. Then $\mathfrak{n}_0'\supset\mathfrak{n}_1'\supset\mathfrak{n}_2'\supset\cdots$ is a filtration such that $[\mathfrak{n}_k', \mathfrak{n}_l']\subset\mathfrak{n}_{k+l}'$. Set $F_k' = \mathfrak{n}_k'/\mathfrak{n}_{k+1}'$ and $F' = \sum\oplus F_k'$. Since $F_k'\cong F_k$ by the natural way, F is a normed Lie algebra by the same norm as in the phrase just above Lemma 2.4. Note that $\mathrm{ad}\,(u)\colon \mathfrak{n}^c\mapsto\mathfrak{n}^c$ is a bounded operator. Then, by the same reasoning as in the proof of Lemma 2.4, we get that the operator $\tilde{X}_u\colon F\mapsto F$ is bounded.

LEMMA 3.5. *Notations and assumptions being as above, we get* $F_1 = F_2 = \cdots = F_k = \cdots = \{0\}$, *i.e.* $F_0 = \mu(\pi^c)$.

PROOF. Let $\mathscr{H}$ be a Cartan subalgebra of $(\mathfrak{n}/\rho_0')^c$ and $\mathscr{H}\oplus\sum_{r\in\Delta}\boldsymbol{C}\cdot e_r$ the root decomposition, where Δ is the root system and e_r is an element such that $[h, e_r] = r(h)e_r$ for any $h\in\mathscr{H}$ and that $h_r = [e_r, e_{-r}]$ is an element of $\mathscr{H}$ with $r(h_r) = \pm 1$.

Now, assume that $F_k \neq \{0\}$ for some $k \geqq 1$. Then, there is a non-trivial $u \in F_k$ such that $u = \sum p_i(x)h_i + \sum_{r\in\Delta} q_r(x)e_r$, where $h_1, \cdots, h_l$ is a basis of $\mathscr{H}$ and p_i, q_r are homogeneous polynomials of degree k.

Assume at first that the $q_r = 0$ for any $r \in \Delta$. Then, there exists $r' \in \Delta$ such that $\mathrm{ad}\,(e_{r'})u \neq 0$, because otherwise $\mathrm{ad}\,(u) = 0$. $\mathrm{ad}\,(e_{r'})u$ is written in the form $p(x)e_{r'}$. In what follows, we show that F_k contains a non-trivial element written in the form $p(x)e_r$. Assume secondly that in the expression of the above u there is r such that $q_r \neq 0$. In this case we may assume $p_1 = \cdots = p_l = 0$ by applying $\mathrm{ad}\,(h_r)$. On the other hand, $\mathrm{ad}\,(e_{r'})u = \sum_{r\in\Delta} q_r(x)\varepsilon_{r,r'}e_{r+r'}$, where $\varepsilon_{r,r'} \neq 0$ if and only if $r + r' \in \Delta$ or $= 0$. (We use the convension $e_0 = \pm h_r$.) Since $2r' \notin \Delta$, the number of non-zero terms of $\mathrm{ad}\,(e_{r'})u$ can be reduced by one by a suitable choice of r'. Now, applying $\mathrm{ad}\,(h_r)$ for some r, we may assume that the e_0-components are zero. We repeat the above procedure for an appropriately chosen series of $\mathrm{ad}\,(e_{r_1})$, $\mathrm{ad}\,(h_{r_2})$, $\mathrm{ad}\,(e_{r_3})$, $\cdots$. Then, consequently, we have that there exists a non-trivial element $u \in F_k$ written in the form $p(x)e_r$.

Since $\{X_u(xH),\ u \in \mathfrak{g}\}$ spans the tangent space of G/H at xH, we get a non-trivial element x_je_r for some j by applying $\tilde{X}_{u_1}, \tilde{X}_{u_2}, \cdots$ $(u_i \in \mathfrak{g})$ repeatedly.

Since $x_je_r \in F_1$ and $[F_0, F_1] = F_1$ because of semi-simplicity, we get $x_jh_r, x_je_{-r} \in F_1$ and hence $x_j^ke_r, x_j^kh_r, x_j^ke_{-r} \in F_k$ for every $k \geqq 1$.

Let $v \in \mathfrak{g}$ be an element such that $X_v(xH) = (\partial/\partial x_j)$. Then $\mathrm{ad}\,(x_jh_r)\tilde{X}_v(x_j^ke_r) = \pm k \cdot x_j^ke_r$. On the other hand, $\mathrm{ad}\,(x_jh_r)\tilde{X}_v\colon F \mapsto F$ is a bounded operator, and hence we get a contradiction.

Since $xH \in G/H$ is arbitrary, the above lemma shows that $\dot{\sigma}_\pi(w)$, $w \in \mathfrak{n}$, must be a locally constant section of V/V_0. Hence we have $\dim \dot{\sigma}_\pi(\mathfrak{n}) < \infty$. This complete the proof of Proposition 3.1 and hence Theorem C in the introduction. Moreover, the above argument shows also that the transition function of V/V_0 must be reduced to a discrete group.

4° **B-triple systems and examples.** In this section, we shall give several examples of Banach-Lie groups acting effectively, smoothly and transitively on finite dimensional manifolds. Taking Corollary 2.7 into account, we call $\{G, H, K\}$ a *B-triple system*, if G is a connected Banach-Lie group and H, K are finite codimensional closed and connected Banach-Lie subgroup such that $H \supsetneqq K$ and $N = \bigcap_{g\in G} gHg^{-1}$, $N' = \bigcap_{h\in H} hKh^{-1}$ are finite codimensional Banach-Lie subgroups of G. A B-triple system is an AS-triple system in the previous section. A B-triple system $\{G, H, K\}$ will be called to be *finite type*, if $N'' = \bigcap_{g\in G} gKg^{-1}$ is a finite co-dimensional Banach-Lie subgroup of G. Obviously, $N'' = \bigcap_{g\in G} g(N \cap N')g^{-1}$.

If $N'' = \{e\}$, then $\{G, H, K\}$ will be called *effective*.

Let $\{G, H, K\}$ be a B-triple system. Then, G acts smoothly and transitively on G/K. By Corollary 2.8, G/K is a connected manifold. Moreover G acts as fibre preserving diffeomorphisms of a smooth fibre bundle $\{G/K; H/K, G/H\}$ with the fibre H/K and the base space G/H. The normal subgroup N acts as diffeomorphisms leaving each fibre invariant, and $g(N \cap N')g^{-1}$ is a finite codimensional normal Banach-Lie subgroup of N acting trivially on the fibre gH/K of the bundle.

The kernel of the action of G on G/K is given by N''. Hence, G/N'' is a Banach-Lie group with the Lie algebra $\mathfrak{g}/\mathfrak{n}''$, where $\mathfrak{n}'' = \bigcap \mathrm{Ad}\,(g)\mathfrak{k}$ and $\mathfrak{k}$ is the Lie algebra of K (cf. Proposition 1.1 and Remark 2). Therefore $\{G/N'', H/N'', K/N''\}$ is an effective B-triple system.

Let $\{G, H, K\}$ be a B-triple system. Note that $g(N \cap N')g^{-1} = gh(N \cap N')h^{-1}g^{-1}$ for any $h \in H$, hence the group $N/g(N \cap N')g^{-1}$ depends only on the point $gH \in G/H$. Let $\mathscr{F}$ be the disjoint union $\bigcup_{gH \in G/H} N/g(N \cap N')g^{-1}$. Then, $\mathscr{F}$ is a smooth fibre bundle over G/H with the fibre $N/N \cap N'$ and the automorphism group of $N/N \cap N'$ as the transition functions. Each fibre of $\mathscr{F}$ is a finite dimensional Lie group.

LEMMA 4.1. *Let $\mathscr{F} \times G/K$ be the fibrewise product of $\mathscr{F}$ and the bundle $\{G/K; H/K, G/H\}$. Then there is a smooth fibre preserving mapping ρ of $\mathscr{F} \times G/K$ onto G/K such that ρ gives the canonical group action on each fibre.*

PROOF. Let $ng(N \cap N')g^{-1}$ be a point of the fibre of $\mathscr{F}$ at $gH \in G/H$, and let ghK be a point of the fibre of G/K at $gH \in G/H$. We define $\rho(ng(N \cap N')g^{-1}, ghK) = ng(N \cap N')g^{-1}ghK = nghK = gn'hK$, $n' = g^{-1}ng \in N \subset H$. It is easy to see that ρ is a smooth action of $N/g(N \cap N')g^{-1}$ on gH/K and hence ρ is smooth.

LEMMA 4.2. *Let $\Gamma(\mathscr{F})$ be the space of all smooth sections of $\mathscr{F}$. Then, by the fibrewise product $\Gamma(\mathscr{F})$ is an infinite dimensional group. There is a homomorphism σ of N into $\Gamma(\mathscr{F})$ such that the kernel N'' of σ is given by $\bigcap_{g \in G} g(N \cap N')g^{-1}$.*

PROOF. The first statement is easy to prove. For an element $n \in N$, $ng(N \cap N')g^{-1}$ can be regarded as an element of $N/g(N \cap N')g^{-1}$. Hence n defines a smooth section $\sigma(n)$ of $\mathscr{F}$ such that $\sigma(n)(gH) = ng(N \cap N')g^{-1}$. Obviously $\sigma(n) = e$ if and only if $n \in N''$.

LEMMA 4.3. *There is a homomorphism Λ of G into the group of automorphism of $\Gamma(\mathscr{F})$ such that $\Lambda(g)\sigma(n) = \sigma(gng^{-1})$ for any $n \in N$.*

PROOF. Let $\tilde{f}$ be a section of $\mathscr{F}$. For every point $xH \in G/H$, $\tilde{f}(xH)$ is an element of $N/x(N \cap N')x^{-1}$. So, we write $\tilde{f}(xH) = f(xH)x(N \cap N')x^{-1}$, $f(xH) \in N$. Define $\Lambda(g)\tilde{f}$ by $(\Lambda(g)\tilde{f})(yH) = gf(g^{-1}yH)g^{-1}y(N \cap N')y^{-1}$. In particular, $(\Lambda(g)\sigma(n))(yH) = gng^{-1}y(N \cap N')y^{-1} = \sigma(gng^{-1})(yH)$. It is easy to see that $\Lambda(gg') = \Lambda(g)\cdot\Lambda(g')$ and $\Lambda(g)$ is an automorphism.

Let $\mathfrak{n}$, $\mathfrak{n}'$ be the Lie algebras of N, N' respectively. Let V be the vector bundle over G/H defined by the disjoint union $\bigcup_{gH \in G/H} \mathfrak{n}/\mathrm{Ad}\,(g)(\mathfrak{n} \cap \mathfrak{n}')$ and $\Gamma(V)$ the space of all smooth sections of V (cf. the previous section). Each fibre of V is the Lie algebra of the fibre of $\mathscr{F}$ at the same base point, and $\Gamma(V)$ is a Lie algebra by the fibrewise Lie bracket product. We define the exponential mapping $\exp\colon \Gamma(V) \mapsto \Gamma(\mathscr{F})$ by $(\exp \tilde{f})(xH) = \exp \tilde{f}(xH)$, $\tilde{f} \in \Gamma(V)$. The mapping $\dot{\sigma}$ defined in the previous section is related to σ as follows:

$$\dot{\sigma}(w)(xH) = (d/dt)|_{t=0}\sigma(\exp tw)(xH)\,, \tag{11}$$

where exp in the right hand member is the exponential mapping of $\mathfrak{n}$ into N. $\dot{\sigma}\colon \mathfrak{n} \mapsto \Gamma(V)$ is a Lie homomorphism and the kernel $\mathfrak{n}''$ of $\dot{\sigma}$ is given by $\bigcap_{g \in G} \mathrm{Ad}\,(g)(\mathfrak{n} \cap \mathfrak{n}')$. Moreover, we have $\exp \dot{\sigma}(w) = \sigma(\exp w)$, $w \in \mathfrak{n}$.

For every $g \in G$, define a mapping $\lambda(g)\colon \Gamma(V) \mapsto \Gamma(V)$ by

$$(\lambda(g)\tilde{w})(xH) = (d/dt)|_{t=0}(\Lambda(g)\exp t\tilde{w})(xH)\,, \quad \tilde{w} \in \Gamma(V)\,, \quad xH \in G/H\,. \tag{12}$$

Then, $\lambda(g)$ is an isomorphism of the Lie algebra $\Gamma(V)$ onto itself. For every $u \in \mathfrak{g}$ (the Lie algebra of G), define a mapping $\alpha(u)\colon \Gamma(V) \mapsto \Gamma(V)$ by

$$(\alpha(u)\tilde{w})(xH) = (d/dt)|_{t=0}(\lambda(\exp tu)\tilde{w})(xH)\,. \tag{13}$$

$\alpha(u)$ is then a derivation of $\Gamma(V)$.

LEMMA 4.4. *Notations being as above, we have the following identities*:

(a) $\lambda(g)\dot{\sigma}(v) = \dot{\sigma}(\mathrm{Ad}\,(g)v)$, $v \in \mathfrak{n}$, $g \in G$.

(b) $\alpha(u)\dot{\sigma}(v) = \dot{\sigma}(\mathrm{ad}\,(u)v)$, $v \in \mathfrak{n}$, $u \in \mathfrak{g}$, *where* $\mathrm{ad}\,(u)v = [u, v]$.

PROOF. Since $\mathrm{Ad}\,(g)\mathfrak{n} = \mathfrak{n}$, the right hand member of (a) is will-defined. By Lemma 4.3 and (11), we have

$$\begin{aligned}\lambda(g)\dot{\sigma}(v) &= (d/dt)|_{t=0}\Lambda(g)\sigma(\exp tv) = (d/dt)|_{t=0}\sigma(g\cdot\exp tv\cdot g^{-1})\\ &= (d/dt)|_{t=0}\sigma(\exp t\,\mathrm{Ad}\,(g)v) = \dot{\sigma}(\mathrm{Ad}\,(g)v)\,.\end{aligned}$$

Taking the derivative of (a), we have $\alpha(u)\dot{\sigma}(v) = (d/dt)|_{t=0}\dot{\sigma}(\mathrm{Ad}\,(\exp tu)v)$. Since $w \rightsquigarrow \dot{\sigma}(w)(xH)$ is a continuous linear mapping of $\mathfrak{n}$ into $\mathfrak{n}/\mathrm{Ad}\,(x)(\mathfrak{n} \cap \mathfrak{n}'))$, we have $(d/dt)|_{t=0}\dot{\sigma}(\mathrm{Ad}\,(\exp tu)v)(xH) = \dot{\sigma}((d/dt)|_{t=0}\,\mathrm{Ad}\,(\exp tu)v)(xH) = \dot{\sigma}(\mathrm{ad}\,(u)v)(xH)$. Thus, we get the identity (b).

LEMMA 4.5. *For any* $u \in \mathfrak{g}$, $\alpha(u): \boldsymbol{\Gamma}(V) \mapsto \boldsymbol{\Gamma}(V)$ *is a differential operator of order at most one. If* $\alpha(u): \boldsymbol{\Gamma}(V) \mapsto \boldsymbol{\Gamma}(V)$ *is of order* 0, *then* $u \in \mathfrak{n}$.

PROOF. This is done by an essentially same computation as in the proof of Lemma 3.3. Here we shall do it by using an arbitrarily fixed C^∞-connection on V. Let xH be an arbitrary point of G/H. The fibre of V at xH is given by $\mathfrak{n}/\mathrm{Ad}(x)(\mathfrak{n} \cap \mathfrak{n}')$. By Hahn-Banach theorem, there is a finite dimensional linear subspace $\mathfrak{m}$ of $\mathfrak{n}$ such that $\mathfrak{n} = \mathfrak{m} \oplus \mathrm{Ad}(x)(\mathfrak{n} \cap \mathfrak{n}')$. $\mathfrak{m}$ can be identified (as a linear space) with the fibre of V at xH.

For any $u \in \mathfrak{g}$, $(\exp - su)xH$ $(s \in [0, \infty))$ is a smooth curve in G/H. Let τ_t be the parallel displacement along the above curve from the point $(\exp - tu)xH$ to xH. For any $v \in \mathfrak{m}$, $\pi_s(v)$ is an element of the fiber of V at $(\exp - su)xH$ defined by $v + \mathrm{Ad}((\exp - su)x)(\mathfrak{n} \cap \mathfrak{n}')$. We set $A(s)v = \tau_s \pi_s v$. Then, $A(s): \mathfrak{m} \mapsto \mathfrak{m}$ is a linear mapping such that $A(0) = I$, and hence a linear isomorphism for sufficiently small s.

Let $\tilde{w} \in \boldsymbol{\Gamma}(V)$. We set $w(s) = A(s)^{-1}\tau_s\tilde{w}((\exp - su)xH)$. Note that $\pi_s w(s) = \tilde{w}((\exp - su)xH)$. Now, we have

$$\begin{aligned}(\alpha(u)\tilde{w})(xH) &= (d/ds)|_{s=0}(\lambda(\exp su)\tilde{w})(xH)\\ &= (\partial/\partial s)|_{s=0}(\partial/\partial t)|_{t=0}(\Lambda(\exp su)\exp t\tilde{w})(xH)\\ &= (\partial/\partial s)|_{s=0}(\partial/\partial t)|_{t=0}\exp su \cdot \exp t\tilde{w}((\exp - su)xH)\cdot \exp - su\\ &= (\partial/\partial s)|_{s=0}(\partial/\partial t)|_{t=0}\exp su \cdot \exp tw(s)\cdot \exp - su \cdot x(N \cap N')x^{-1}\\ &= (d/ds)|_{s=0}\dot{\sigma}(\mathrm{Ad}(\exp su)w(s))(xH)\\ &= \dot{\sigma}([u, w(0)]) + (d/ds)|_{s=0}A(s)^{-1}\tau_s\tilde{w}((\exp - su)xH)\\ &= \dot{\sigma}([u, \tilde{w}(xH)]) - ((d/ds)|_{s=0}A(s))\tilde{w}(xH)\\ &\quad + (\nabla/ds)|_{s=0}\tilde{w}((\exp - su)xH)\ ,\end{aligned}$$

where ∇/ds means the covariant derivative.

The last term contains a differentiation. If $\alpha(u)$ is of order 0, then the last term must be zero for all $\tilde{w} \in \boldsymbol{\Gamma}(V)$. Thus, $(\exp - su)xH = xH$ and hence $\exp su \in \bigcap_{x \in G} xHx^{-1}$. Therefore, $u \in \bigcap_{x \in G}\mathrm{Ad}(x)\mathfrak{h} = \mathfrak{n}$. The converse is of course true.

PROPOSITION 4.6. *Suppose* $\{G, H, K\}$ *is a B-triple system such that* G/H *is compact. Then,* $\{G, H, K\}$ *is of finite type.*

PROOF. Let $\{\tilde{u}_1, \cdots, \tilde{u}_l\}$ be a basis of $\mathfrak{g}/\mathfrak{n}$. Every $\tilde{u}_i$ can be regarded as a smooth vector field on G/H. Since G/N acts transitively on G/H, $\{\tilde{u}_1(yH), \cdots, \tilde{u}_l(yH)\}$ spans the tangent space of G/H at every $yH \in G/H$. Let u_i be an element of $\mathfrak{g}$ such that $u_i + \mathfrak{n} = \tilde{u}_i$. Consider the differential operator $D = \sum_{i=1}^{l} \alpha(u_i)^2 \colon \Gamma(V) \mapsto \Gamma(V)$.

We fix an arbitrary riemannian structure on $M = G/H$ and on the bundle V. First of all, we shall prove that $\sigma(D) = \sigma(D^*)$ and D is a strongly elliptic differential operator of order 2.

Let $(y_1, \cdots, y_n)$ be a local coordinate system on G/H. Taking a local trivialization, $\tilde{w}$ can be written as an m-tuple of smooth functions

$$(\tilde{w}_1(y_1, \cdots, y_n), \cdots, \tilde{w}_\alpha(y_1, \cdots, y_n), \cdots, \tilde{w}_m(y_1, \cdots, y_n))$$

$m = \dim \mathfrak{n}/\mathfrak{n} \cap \mathfrak{n}'$.

Let $\sum_{i=1}^{n} X_j^i(\partial/\partial y_i)$ be the local expression of the vector field $\tilde{u}_j$. Then, by Lemma 4.5, $D\tilde{w} = ((D\tilde{w})_1, \cdots, (D\tilde{w})_\alpha, \cdots, (D\tilde{w})_m)$ is written in the form

$$(D\tilde{w})_\alpha = \sum_{j=1}^{l} \sum_{a,b=1}^{n} X_j^a X_j^b (\partial^2/\partial y_a \partial y_b)\tilde{w}_\alpha + \text{lower order terms.} \tag{14}$$

Thus, the symbol $\sigma(D)$ is given by $\sigma(D)\xi = \sum_{j=1}^{l} \{\xi(u_j(yH))\}^2 I$, where ξ is a cotagent vector at $yH \in G/H$ and $I\colon V \mapsto V$ is the identity mapping. Therefore we see $\sigma(D) = \sigma(D^*)$. Since G/H is assumed to be compact, there is a positive constant c such that $\sum_{j=1}^{l} \{\xi(u_j(yH))\}^2 \geqq c|\xi|^2$ for any $yH \in G/H$ hence D is strongly elliptic.

Thus, by Proposition 2.3 and by the same reasoning as in 2°, we get that $\dim \dot{\sigma}(\mathfrak{n}) < \infty$. The kernel $\mathfrak{n}''$ of $\dot{\sigma}$ is given by $\bigcap_{x \in G} \mathrm{Ad}\,(x)(\mathfrak{n} \cap \mathfrak{n}')$.

COROLLARY 4.7. *The conclusion* (1) *of Corollary* 2.7 *can be replaced that G_{i+1}/G_i are non-compact manifold for $i \geqq 1$.*

In what follows we shall give several examples of effective B-triple systems.

Let G be a Banach-Lie group, possibly finite dimensional, and G_0 a finite codimensional closed Banach-Lie subgroup of G. Let E be an infinite dimensional Banach space and f a smooth representation of G on E, where "smooth" means that the mapping $f\colon G \times E \to E$, $(g, u) \rightsquigarrow f(g)u$, is smooth. By Lemma 1.3.4 [17], f is a smooth mapping of G into $GL(E)$. We assume the following property:

(P) $\bigcap_{g \in G} gG_0g^{-1} = \{e\}$ and there exists a finite codimensional $f(G_0)$-invariant closed subspaces E_0 of E such that $\bigcap_{g \in G} f(g)E_0 = \{0\}$.

Define the semi-direct product $G \circ E$ as follows: For (g, u), $(h, v) \in G \times E$, define a multiplication by $(g, u) \circ (h, v) = (gh, u + f(g)v)$. This makes $G \circ E$ a Banach-Lie group. Similarly, $G_0 \circ E$ and $G_0 \circ E_0$ are closed finite codimensional Banach-Lie subgroups of $G \circ E$. It is easy to see that $(g, 0) \circ (h, v) \circ (g^{-1}, 0) = (ghg^{-1}, f(g)v)$. Hence $\{G \circ E, G_0 \circ E, G_0 \circ E_0\}$ is an effective B-triple system. Therefore, $G \circ E$ acts effectively, smoothly on $G/G_0 \times E/E_0$.

Let $\mathfrak{g}$, $\mathfrak{g}_0$ be the Lie algebras of G, G_0 respectively. Let f' be the Lie homomorphism of $\mathfrak{g}$ into $\mathfrak{gl}(E)$ induced from f. Then, the Lie algebra of $G \circ E$ is $\mathfrak{g} \oplus E$ with the bracket product $[(t, u), (t', v)] = ([t, t'], f'(t)v - f'(t')u)$. By the assumption (P), we have $f'(\mathfrak{g}_0)E_0 \subset E_0$ and E_0 contains no non-trivial $f'(\mathfrak{g})$-invariant subspace. It is clear that the condition $\bigcap_{g \in G} f(g)E_0 = \{0\}$ is equivalent with that E_0 contains no non-trivial $f'(\mathfrak{g})$-invariant subspace.

EXAMPLE 1. $G =$ the additive group of the complex 2-plane $\boldsymbol{C}^2$, $G_0 = \{e\}$ and E is the Hilbert space given by the double infinite series $u = \sum_{n=-\infty}^{\infty} a_n e_n$ such that $\sum_{n=-\infty}^{\infty} |a_n|^2 < \infty$, $a_n \in \boldsymbol{C}$. Let $E_0 = \{u \in E; a_0 = 0\}$. Consider the bilateral shift $\sigma: E \to E$, $\sigma(e_n) = e_{n+1}$. Define a representation $f: \boldsymbol{C}^2 \mapsto GL(E)$ by $f(t, t') = \exp(t\sigma + t'\sigma^{-1})$. It is easy to see that there is no $\{\sigma, \sigma^{-1}\}$-invariant subspace in E_0, and hence $\bigcap_{t \in \boldsymbol{C}^2} f(t, t')E_0 = \{0\}$. Thus, $\{\boldsymbol{C}^2 \circ E, E, E_0\}$ is an effective B-triple system.

EXAMPLE 2. $G = \boldsymbol{C}$ (additive), $G_0 = \{e\}$, $E = \{u(z) = \sum_{n=0}^{\infty} a_n z^n; a_n \in \boldsymbol{C}, \sum_{n=0}^{\infty} |a_n|^2 (n!)^2 < \infty\}$. The representation f of $\boldsymbol{C}$ on E is given by $(f(t)u)(z) = u(z + t)$. Since $f'(1) = (d/dz)u$, it is clear that $\|f'(1)u\| \leqq \|u\|$, where $\|u\|^2 = \sum_{n=0}^{\infty} |a_n|^2 (n!)^2$ and $f(t) = \exp tf'(1)$. Let $E_0 = \{u \in E; a_0 = 0\}$. Then there is no non-trivial $f'(1)$-invariant subspace in E_0, and hence $\{\boldsymbol{C} \circ E, E, E_0\}$ is an effective B-triple system.

EXAMPLE 3. $\mathfrak{g} = \{\sum_{i=1}^{n} a_i(\partial/\partial x_i) + \sum_{j<i} b_j^i x_j(\partial/\partial x_i); a_i, b_j^i \in \boldsymbol{C}\}$, $\mathfrak{g}_0 = \{\sum_{j<i} b_j^i x_j(\partial/\partial x_i); b_j^i \in \boldsymbol{C}\}$ and $E = \{u = \sum_{|\alpha| \geqq 0} A_\alpha x^\alpha; A_\alpha \in \boldsymbol{C}, \sup_{|\alpha| \geqq 0} (\sum_{i=1}^{n} i\alpha_i)! |A_\alpha| < \infty\}$, where $\alpha = (\alpha_1 \cdots \alpha_i \cdots \alpha_n)$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. Then, E is a Banach space on which the nilpotent Lie algebra $\mathfrak{g}$ acts by the usual way. Obviously, $\mathfrak{g}$ is a subalgebra of the Lie algebra $\mathfrak{gl}(E)$ of the bounded linear operators. Let $E_0 = \{u \in E; A_0 = 0\}$. E_0 is a $\mathfrak{g}_0$-invariant subspace of E and E_0 contains no non-trivial $\mathfrak{g}$-invariant subspace. Let G and G_0 be the Lie group generated by $\mathfrak{g}$, $\mathfrak{g}_0$ respectively. Then, $\{G \circ E, G_0 \circ E, G_0 \circ E_0\}$ is an effective B-triple system.

EXAMPLE 4. $\mathfrak{g} = \{a(\partial/\partial x) + b(x(\partial/\partial x) - (\partial/\partial y)); a, b \in \boldsymbol{C}\}$, $\mathfrak{g}_0 = \{0\}$, $E = \{f(x, y) = \sum_{i=0}^{N} f_i(y)x^i; f_i(y) = \sum_{n=0}^{\infty} a_n^i y^n, a_n^i \in \boldsymbol{C}, \sum_{i=0}^{N} \sum_{n=0}^{\infty} |a_n^i|^2 (n!)^2 < \infty\}$ and $E_0 = \{f \in E; a_0^0 = 0\}$. Then, $\mathfrak{g} \subset \mathfrak{gl}(E)$ and E_0 contains no non-trivial $\mathfrak{g}$-invariant subspace. $\mathfrak{g}$ is a two dimensional solvable Lie algebra. Let G be the group generated by $\mathfrak{g}$. Then, $\{G \circ E, E, E_0\}$ is an effective B-triple system.

REMARK. By the remark of p. 336 of [9], $\partial/\partial x; E \mapsto E$ must be nilpotent. Therefore the finiteness of N is necessary in this case.

5° **Appendix.** Let E be a C^∞-complex, finite dimensional vector bundle over a closed C^∞-(real) riemannian manifold M and $\Gamma(E)$ the space of the C^∞-sections of E with the C^∞-topology. For $u, v \in \Gamma(E)$, the notation $\langle u, v\rangle_0$ means the hermitian inner product given by

$$\langle u, v\rangle_0 = \int_M \langle u(x), v(x)\rangle d\mu(x) , \tag{1}$$

where $d\mu(x)$ is a C^∞-volume element on M and $\langle u(x), v(x)\rangle$ means the hermitian inner product of the fiber of E.

Suppose we have a differential operator D of $\Gamma(E)$ into itself. Let D^* be the formal adjoint operator of D, namely the differential operator satisfying $\langle Du, v\rangle_0 = \langle u, D^*v\rangle_0$ for any $u, v \in \Gamma(E)$. A complex number λ is called an *eigenvalue* of D, if there is $u \in \Gamma(E)$, $u \neq 0$, such that $Du = \lambda u$. The *generalized eigenspace* E_λ of the eigenvalue λ is the linear space of the elements $v \in \Gamma(E)$ such that $(D - \lambda)^m v = 0$ for some positive integer m. Obviously, $DE_\lambda \subset E_\lambda$.

The goal of this section is the following:

PROPOSITION 5.1. *Let $D: \Gamma(E) \mapsto \Gamma(E)$ be a differential operator of order* 2. *Suppose the symbol $\sigma(D)$ satisfies $\sigma(D) = \sigma(D^*)$ and that there is a positive constant c such that $\langle(\sigma(D)\xi - c|\xi|^2)X, X\rangle \geqq 0$ for any element $X \in E$ and any cotangent vector $\xi \in T^*M$, $\xi \neq 0$ with the same base point of X,* i.e. *$\sigma(D)\xi - c|\xi|^2$ is positive semi-definite. Then, there are countably many eigenvalues $\{\lambda_n\}_{n=1,2,\dots}$ such that $\lim_{n\to\infty} \operatorname{Re}\lambda_n = \infty$, $\dim E_{\lambda_n} < \infty$ and the generalized eigenspaces are complete in $\Gamma(E)$,* i.e. *$\sum_{n=1}^\infty \oplus E_{\lambda_n}$ is dense in $\Gamma(E)$. Moreover, setting $\mathscr{F}_n = (\sum_{k\geqq n}^\infty \oplus E_{\lambda_k})^-$, we have $\bigcap \mathscr{F}_n = \{0\}$.*

The above proposition is well-known if $D = D^*$ or M is a bounded domain of a euclidean space $\boldsymbol{R}^n$ (cf. [1] and [5, p. 1746]). Moreover, since M has no boundary, the proof is much easier and straightforward application of standard results of functional analysis. Indeed, the above fact is well-known for the people who are familiar to both functional analysis and differential grometry. Thus, in this section we will give only a rough sketch of the proof.

We denote by ∇ the riemannian connection on E. For any $u, v \in \Gamma(E)$, define a hermitian inner product $\langle u, v\rangle_k$ by

$$\langle u, v\rangle_k = \sum_{s=0}^k \int_M \langle(\nabla^s u)(x), (\nabla^s v)(x)\rangle d\mu(x) , \tag{2}$$

where $(\nabla^s u)(x)$ means the s-times covariant differentiation of u at $x \in M$. Denote by $\Gamma^k(E)$ the completion of $\Gamma(E)$ by the norm $\|u\|_k = \langle u, u\rangle_k^{1/2}$.

Thus, we get a series of separable Hilbert spaces

$$\Gamma^0(E) \supset \Gamma^1(E) \supset \cdots \supset \Gamma^k(E) \supset \Gamma^{k+1}(E) \supset \cdots . \tag{3}$$

Obviously, $\Gamma^{k+1}(E)$ is dense in $\Gamma^k(E)$, and by Rellich's theorem combined with partition of unity, the inclusions $\Gamma^{k+1}(E) \subset \Gamma^k(E)$ are compact operators. The well-known Sobolev's lemma is stated as follows in our situation:

LEMMA 5.2 (Sobolev). *Let $n = \dim M$. If $k = [n/2] + 1 + r$, then $\Gamma^k(E)$ can be regarded as a subspace of $\tilde{\Gamma}^r(E)$, the space of all C^r-sections of E with the C^r-topology. Moreover the inclusion is bounded.*

COROLLARY 5.3. *If $l \geqq [n/2] + 1$, then the inclusions $\Gamma^{k+l} \subset \Gamma^k$ are of Hilbert-Schmidt class for every k.*

PROOF. For every $v \in \Gamma^{k+l}(E)$, we have $v \in \tilde{\Gamma}^k(E)$ by Sobolev's lemma. Thus, for an element $X_s \in E \otimes T^*\underbrace{M \otimes \cdots \otimes T^*M}_{s}$ with a base point $x \in M$, the mapping $v \rightsquigarrow \langle (\nabla^s v)(x), X_s \rangle$ is a bounded linear mapping of $\Gamma^{k+l}(E)$ into $\boldsymbol{C}$ for every $s \leqq k$. By Riesz's theorem, there is an element $\varphi_s(x, X_s) \in \Gamma^{k+l}(E)$ such that $\langle (\nabla^s v)(x), X_s \rangle = \langle v, \varphi_s(x, X_s) \rangle_{k+l}$. Since $(\nabla^s v)(x)$ is continuous in x, $|\langle (\nabla^s v)(x), X_s \rangle|$ is bounded if X_s is restricted in the unit sphere bundle of $E \otimes T^*M \otimes \cdots \otimes T^*M$. Therefore, by the resonance theorem ([21, p. 69]) there exists a finite constant K_s such that $\|\varphi_s(x, X_s)\|_{k+l} \leqq K_s$ for each $x \in M$ and X_s in the unit sphere bundle.

Let $f_1, \cdots, f_m$ be an orthonormal basis of $E_x \otimes T_x{}^*M \otimes \cdots \otimes T_x{}^*M$. Then $|(\nabla^s v)(x)|^2 = \sum_{i=1}^m \langle (\nabla^s v)(x), f_i \rangle^2 = \sum_{i=1}^m \langle v, \varphi_s(x, f_i) \rangle_{k+l}^2$. Now, if $\{e_n\}_{n=1,2\cdots}$ is a complete orthonormal basis of $\Gamma^{k+l}(E)$, then

$$\begin{aligned} \sum_{n=1}^{\infty} \|e_n\|_k^2 &= \sum_{n=1}^{\infty} \sum_{s=0}^{k} \int_M |(\nabla^s e_n)(x)|^2 d\mu(x) \\ &= \sum_{s=0}^{k} \sum_{i=1}^{m} \sum_{n=1}^{\infty} \int_M \langle e_n, \varphi_s(x, f_i) \rangle_{k+l}^2 d\mu(x) \\ &= \sum_{s=0}^{k} \sum_{i=1}^{m} \int_M \|\varphi_s(x, f_i)\|_{k+l}^2 d\mu(x) \\ &\leqq \sum_{s=0}^{k} \sum_{i=1}^{m} K_s \int_M d\mu(x) < \infty . \end{aligned} \tag{4}$$

This implies that the inclusion $\Gamma^{k+l}(E) \subset \Gamma^k(E)$ is of Hilbert-Schmidt class.

Now, let $L: \boldsymbol{\Gamma}(E) \mapsto \boldsymbol{\Gamma}(E)$ be a differential operator of order 2 such that the symbol $\sigma(L)$ satisfies $|\sigma(L)\xi| \geqq c|\xi|^2$ (elliptic) for any $\xi \in T^*M$ ($c > 0$). By Gårding's inequality, we have

$$\|Lu\|_k \geqq (c/2)\|u\|_{k+2} - D_k\|u\|_{k+1} , \qquad u \in \boldsymbol{\Gamma}(E), k \geqq 0 , \tag{5}$$

where c is the some constant as above and D_k is a positive constant depending on k.

Let $\Gamma^{-1}(E)$ be the dual space of $\Gamma^1(E)$. Then, it is easy to see that $\Gamma^1(E) \subset \Gamma^0(E) \subset \Gamma^{-1}(E)$ and $\boldsymbol{\Gamma}(E)$ is dense in $\Gamma^{-1}(E)$. The differential operator L can be extended to an operator L_{-1} defined on some domain $\mathscr{D}(L_{-1})$ into $\Gamma^{-1}(E)$, where in fact $\mathscr{D}(L_{-1}) \supset \Gamma^1(E)$. The following regularlity lemma shows that the spactum of L does not depend on k.

LEMMA 5.4. *If there is a complex number z_0 such that the resolvent $R(z_0, L_{-1})$ induces an isomorphism of $\Gamma^{-1}(E)$ onto $\Gamma^1(E)$, then any resolvent $R(z, L_{-1})$ induces an isomorphism of $\Gamma^{k-1}(E)$ onto $\Gamma^{k+1}(E)$ for every $k \geqq 0$. The spectral set of L_{-1} consists of point spectra and the generalized eigenspaces E_λ of L_{-1} are contained in $\boldsymbol{\Gamma}(E)$. There are countably many point spectra (eigenvalues) $\{\lambda_n\}$ of L_{-1} such that $\lim_{n\to\infty} |\lambda_n| = \infty$, and if $\sum \bigoplus E_{\lambda_n}$ is dense in $\Gamma^{-1}(E)$, then so also is in $\boldsymbol{\Gamma}(E)$.*

PROOF. By the assumption, $L - z_0 I: \boldsymbol{\Gamma}(E) \mapsto \boldsymbol{\Gamma}(E)$ can be extended to an isomorphism of $\Gamma^1(E)$ onto $\Gamma^{-1}(E)$. Since the inclusion $\Gamma^1(E) \subset \Gamma^0(E)$ is compact, the resolvent $R(z_0, L_{-1}): \Gamma^{-1}(E) \mapsto \Gamma^{-1}(E)$ is a compact operator. Hence the spectral set consists of countably many point spectra $\{\lambda_n\}$ such that $\lim_{n\to\infty} |\lambda_n| = \infty$ and $\dim E_{\lambda_n} < \infty$.

Let $\rho(L_{-1})$ be the resolvent set of L_{-1}. Since a resolvent $R(z, A)$ of $A = R(z_0, L_{-1})$ is an isomorphism of $\Gamma^{-1}(E)$ onto itself, $AR(z, A): \Gamma^{-1} \mapsto \Gamma^1$ is an isomorphism. On the other hand, using the identity

$$zI - A = (zA^{-1} - I)A = z\{(z_0 - (1/z))I - L_{-1}\}A \, ,$$

we have $AR(z, A) = (1/z)R(z_0 - 1/z, L_{-1})$. Thus, $z_0 - 1/z \in \rho(L_{-1})$ if and only if $z \in \rho(A)$, and $R(z_0 - 1/z, L_{-1}): \Gamma^1(E) \mapsto \Gamma^1(E)$ is an isomorphism. By the inequality (5), if $\{(L - zI)u_n\}$ and $\{u_n\}$ ane Cauchy sequences in $\Gamma^k(E)$ and $\Gamma^{k+1}(E)$ respectively, then $\{u_n\}$ is a Cauchy sequence in $\Gamma^{k+2}(E)$. Therefore by induction we get $R(z, L_{-1})$ induces an isomorphism of $\Gamma^{k-1}(E)$ onto $\Gamma^{k+1}(E)$ for every $k \geqq 0$, $z \in \rho(L_{-1})$.

Since $\dim E_{\lambda_n} < \infty$, we have $R(z, L_{-1})^k E_{\lambda_n} = E_{\lambda_n}$ for any $z \in \rho(L_{-1})$ and $k \geqq 0$. Since $R(z, L_{-1})E_{\lambda_n} \subset \Gamma^{2k-1}$, we get $E_{\lambda_n} \subset \boldsymbol{\Gamma}(E)$. All others are easy to prove.

By the above lemma, we have only to consider the operator L_{-1} for the proof of Proposition 5.1. The following reduces the problem to Hilbert-Schmidt class.

LEMMA 5.5. *Notations and assumptions being as above, if the generalised eigenspaces of L^m are complete in $\boldsymbol{\Gamma}(E)$ for some positive* *integer* m, *then the generalized eigenspaces of* L *are complete in* $\Gamma(E)$.

PROOF. By the spectral mapping theorem ([5] p. 604), we have $\sigma(L^m_{-1}) = \sigma(L_{-1})^m$, where $\sigma(L)$ means the spectral set of L. For any $\lambda \in C - \{0\}$, we denote by E_λ, F_λ the generalized eigenspaces of L_{-1}, L^m_{-1} respectively. As a matter of course $E_\lambda = \{0\}$ if $\lambda \notin \sigma(L_{-1})$. Now for the proof, it is enough to show that $F_{\lambda m} = E_{\omega_1\lambda} \oplus E_{\omega_2\lambda} \oplus \cdots \oplus E_{\omega_m\lambda}$, where $\omega_1, \cdots, \omega_m$ are the m-th roots of unity. Let $\nu_0 = \dim F_{\lambda m}$. ν_0 is finite because L^m is elliptic.

Note that $L^m_{-1} - \lambda^m I = \prod_{i=1}^m (L_{-1} - \omega_i \lambda I)$. Let $p_j(z) = \prod_{i \neq j} (z - \omega_i \lambda)^{\nu_0}$. Since there is no common zero of $p_1(z), \cdots, p_m(z)$, there are polynomials $q_j(z)$ such that $1 \equiv \sum q_i(z) p_i(z)$. Therefore $u \equiv \sum q_i(L_{-1}) p_i(L_{-1}) u$ for any $u \in \Gamma(E)$. Set $u_i = q_i(L_{-1}) p_i(L_{-1}) u$. It is easy to check that $u \in F_{\lambda m}$ if and only if $u_i \in E_{\omega_i \lambda}$. This implies $F_{\lambda m} = E_{\omega_1\lambda} \oplus E_{\omega_2\lambda} \oplus \cdots \oplus E_{\omega_m\lambda}$.

PROOF OF THE FIRST HALF OF PROPOSITION 5.1. Set $H = (D + D^*)/2$. Then, H is an elliptic hermitian operator. By the assumption, we get the following using Garding's inequality:

$$(6) \quad c'' \|u\|_1^2 - D'' \|u\|_0^2 \leqq \langle Hu, u \rangle_0 \leqq c' \|u\|_1^2 + D' \|u\|_0^2 , \quad u \in \Gamma(E) .$$

Thus, by Friedricks extension theorem, there is a positive constant a such that $H + aI: \Gamma(E) \mapsto \Gamma(E)$ can be extended to an isomorphism of $\Gamma^1(E)$ onto $\Gamma^{-1}(E)$ and $c^{-1} \|u\|_1 \leqq \langle (H + aI)u, u \rangle_0 \leqq c \|u\|_1$ for some positive constant c: Namely $\langle (H + aI)u, u \rangle_0$ gives an hermitian inner product which is equivalent with $\langle\ ,\ \rangle_1$.

By Lemma 5.4, the resolvent $R(z, H_{-1})$ induces an isomorphism of $\Gamma^{k-1}(E)$ onto $\Gamma^{k+1}(E)$ for every $k \geqq 0$. Since $\langle Hu, u \rangle_0$ is real, the resolvent $R(a, H_0): \Gamma^0(E) \mapsto \Gamma^0(E)$ is self-adjoint, where $H_0: \mathscr{D}(H_0) = \Gamma^2(E) \mapsto \Gamma^0(E)$ is the Friedrichs extension of H. Therefore, $\sigma(H_0) \subset \{x > -a\}$ and the eigenspaces of H is complete in $\Gamma(E)$. $R(a, H_0)$ is also the restriction of $R(a, H_{-1})$.

On the other hand, $D: \Gamma(E) \mapsto \Gamma(E)$ can be extended to bounded linear operator $D_0: \Gamma^2(E) \mapsto \Gamma^0(E)$.

LEMMA 5.6. *Every resolvent* $R(z, D_0): \Gamma^0(E) \mapsto \Gamma^0(E)$ *induces an isomorphism of* $\Gamma^k(E)$ *onto* $\Gamma^{k+2}(E)$ *for any* $k \geqq 0$, *and on any ray* $\{re^{i\theta}; r \geqq 0\}$ *with a fixed* θ *such that* $e^{i\theta} \neq \pm 1$, $R(z, D_0)$ *exists and satisfies* $\|R(z, D_0)\|_0 \leqq C_\theta(1/z)$, $z = re^{i\theta}$, *for sufficiently large* r.

PROOF. Set $D = H + A$. Then A is a differential operator of order $\leqq 1$ and A can be extended to a bounded linear operator A_0 of $\Gamma^1(E)$ into $\Gamma^0(E)$. Since H_0 self-adjoint, we get $\|R(z, H_0)\|_0 \leqq 1/\mathrm{Im}\, z$. Since there is a positive constant C such that $\|A_0 u\|_0^2 \leqq C\langle (H_0 + aI)u, u \rangle_0 \leqq C \|(H_0 + aI)u\|_0 \|u\|_0$, we have

$$||A_0 u||_0 \leqq \varepsilon ||(H_0 + aI)u||_0 + (C/2\varepsilon)||u||_0$$

for any $\varepsilon > 0$. Thus, we get

$$||A_0 R(z, H_0)u||_0 \leqq \{\varepsilon + (\varepsilon a + \varepsilon|z| + C/2\varepsilon)(1/|z|\sin\theta)\}||u||_0$$

Therefore, if z is on a ray $\{re^{i\theta}; r \geqq 0\}$ with $e^{i\theta} \neq \pm 1$, then

$$||A_0 R(z, H_0)||_0 \leqq \{\varepsilon(1 + 1/\sin\theta) + ((1/|z|)\cdot(1/\sin\theta))(\varepsilon a + C/2\varepsilon)\} .$$

Take ε so that it may satisfy $\varepsilon(1 + 1/\sin\theta) < 1/2$. Then for sufficiently large z on the ray we have $||A_0 R(z, H_0)||_0 < 1$ and there is a constant K_θ such that $||(I - A_0 R(z, H_0))^{-1}||_0 \leqq K_\theta$. Thus, we get the existence of $R(z, D_0)$ for such z and $||R(z, D_0)||_0 \leqq (1/|z|)C_\theta$ by using the identity

$$zI - D_0 = (I - A_0 R(z, H_0))(zI - H_0) , \qquad z \notin \sigma(H_0) .$$

Moreover, since $(I - A_0 R(z, H_0))^{-1}$ is an isomorphism of $\Gamma^0(E)$ onto itself, we see that $R(z, D_0)$ induces an isomorphism of $\Gamma^0(E)$ onto $\Gamma^2(E)$. Thus, by the same reasoning as in Lemma 5.4, we have that every resolvent induces an isomorphism of $\Gamma^k(E)$ onto $\Gamma^{k+2}(E)$ for every $k \geqq 0$.

Now, by Corollary 5.3 and the equality $R(z^m, D_0^m) = \prod_{i=1}^m R(z\omega_i, D_0)$, there is a positive integer m such that the resolvent $R(z, D_0^m)$ of D_0^m is of Hilbert-Schmidt class. By Lemma 5.6, we see that $||R(z, D_0^m)||_0 \leqq (1/|z|)K_\theta$ for sufficiently large z on every ray such that $e^{im\theta} \neq 1$, where K_θ is a constant depending on θ.

Using the resolvent equation ([5] p. 600) and applying the completeness theorem ([5] p. 1041), we get the generalized eigenspaces are complete in $\Gamma^0(E)$, and hence in $\Gamma(E)$ by Lemma 5.4. (See also the proof of the next corollary.)

Let $\{\lambda_n\}_{n=1,2,\dots}$ be the eigenvalues of D. By the compactness of the resolvent, we see that $\lim|\lambda_n| = \infty$. However, since $|\arg\lambda_n| < \pi/4$ for sufficiently large n, we see that $\lim \operatorname{Re}\lambda_n = \infty$. This complete the proof of the first half of Proposition 1.1.

The second half is given by the following:

COROLLARY 1.7. *Let* $\{\lambda_n\}_{n=1,2,\dots}$ *be the eigenvalues of* D *such that* $|\lambda_1| \leqq |\lambda_2| \leqq \cdots$. *Let* $\mathscr{F}_n$ *be the closure of* $\sum_{k=n}^\infty \oplus E_{\lambda_k}$ *in* $\Gamma(E)$. *Then,* $\Gamma(E) = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_{n-1}} \oplus \mathscr{F}_n$ *for any* $n \geqq 1$, *and* $\bigcap_{n=1}^\infty \mathscr{F}_n = \{0\}$.

PROOF. Let α be a complex number such that the resolvent $R(\alpha, D_0)$ exists. We set $A = -R(\alpha, D_0)$, and $\mu_n = (\lambda_n - \alpha)^{-1}$. By the resolvent equation $-R(\mu^{-1}+\alpha, D_0) = \mu^2 R(\mu, A) - \mu I$, we have easily that $\{\mu_n\}_{n=1,2,\dots} \cup \{0\}$ is the spectral set of A. Let E'_{μ_n} be the generalized eigenspace of

A of eigenvalue μ_n. Then, plainly, $E'_{\mu_n} = E_{\lambda_n}$.

Let c_n (resp. c'_n) be a smooth simply closed curve in $\boldsymbol{C}$ such that the interior of c_n (resp. c'_n) contains the eigenvalue μ_n (resp. the eigenvalues $\{\mu_k\}_{k\geqq n}$). We set $\varepsilon_n = (1/2\pi i)\oint_{c_n} R(z, A)dz$, $\varepsilon'_n = (1/2\pi i)\oint_{c'_n} R(z, A)dz$. Then, by Theorem 10 [5] p. 568, we have $\varepsilon_n^2 = \varepsilon_n$, $\varepsilon'^2_n = \varepsilon'_n$, $\varepsilon_n\varepsilon_m = \varepsilon_m\varepsilon_n = 0$ $(n \neq m)$, $\varepsilon'_n\varepsilon_j = \varepsilon_j\varepsilon'_n = 0$ $(j < n)$ and $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{n-1} + \varepsilon'_n = I$. Note that $\varepsilon_n\Gamma^0(E) = E'_{\mu_n} = E_{\lambda_n}$. Hence, we get $\Gamma^0(E) = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_{n-1}} \oplus \varepsilon'_n\Gamma^0(E)$.

Since A^{-1} induces an isomorphism of $\Gamma^{k+2}(E)$ onto $\Gamma^k(E)$ for any $k \geqq 0$ and the spectral set does not depend on k, ε_n and ε'_n are also projection operators on $\Gamma^k(E)$ for all k. Thus, we get $\Gamma^k(E) = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_{n-1}} \oplus \varepsilon'_n\Gamma^k(E)$ for every k, and hence $\boldsymbol{\Gamma}(E) = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_{n-1}} \oplus \varepsilon'_n\boldsymbol{\Gamma}(E)$. Remark that $\varepsilon'_n\boldsymbol{\Gamma}(E)$ is A-invariant and hence D-invariant. Consider the restriction D: $\varepsilon'_n\boldsymbol{\Gamma}(E) \mapsto \varepsilon'_n\boldsymbol{\Gamma}(E)$. Then, applying the first half of Proposition 5.1, we have that $\varepsilon'_n\boldsymbol{\Gamma}(E)$ is the closure of $\sum_{k\geqq n} \oplus E_{\lambda_k}$, becuase the same estimate of the resolvent holds for the restricted operator.

Let $\mathscr{F}'_n$ be the closure of $\mathscr{F}_n$ in $\Gamma^0(E)$. We have only to show that $\bigcap \mathscr{F}'_n = \{0\}$. Moreover, it is enough to show the desired one for the Hilbert-Schmidt operator A^m, because the relation of generalized eigenspaces of L^m_{-1} and L_{-1} given in the proof of Lemma 5.5 holds by replacing L_{-1} by A. Thus, we consider the Hilbert-Schmidt operator A^m in what follows.

Let $N = \bigcap \mathscr{F}'_n$ and $B = A^m|N$. Then, B: $N \mapsto N$ is a quasi-nilpotent Hilbert-Schmidt operator. Since the same estimate holds for the resolvent of the restricted operator B, we have that $||R(z, B)|| = O(|z|^{-1})$ for sufficiently small z on any ray $\{re^{i\theta}; r \geqq 0\}$ with $e^{im\theta} \neq 1$. Thus, by Phragmen-Lindelöf's theorem, $zR(z, B)u$ is a $\Gamma^0(E)$-valued entire function. Hence by Liouville's theorem, $||zR(z, B)u||_0$ is constant. Using Schwarz's theorem, we get $R(z, B)u = v/z$, $v \in N$.

On the other hand, if z is sufficiently large, then by Neumann series, $R(z, B) = I/z + B/z^2 + B^2/z^3 + \cdots$. Therefore, we get $Bu = 0$. Since u is an arbitrary element of N, we have $A^mN = \{0\}$. Thus, $N = 0$ because otherwise $(D_0 - \alpha I)^m$ can not be defined as an operator.

References

[1] S. Agmon, On the eigenfunctions and the eigenvalues of general elliptic boundary value problems, Comm. Pure Appl. Math., XV (1962), 119–147.

[2] L. Auslander, L. Green and F. Hahn, Flows on homogeneous spaces, Ann. Math. Studies 53, (1963).

[3] G. Birkhoff, Analytical groups, Trans. Amer. Math. Soc. 43 (1938), 61–101.

[4] C. Chevalley, Theory of Lie groups I, Princeton Univ. Press, 1946.
[5] N. Dunford and J. T. Schwartz, Linear operators, I & II, Interscience, 1963.
[6] E. B. Dynkin, Normed Lie algebras and analytic groups, Amer. Math. Soc. Translations, 97, 1953.
[7] V. W. Guillemin, Infinite dimensional primitive Lie algebras, J. Diff. Geom. 4 (1970), 257-282.
[8] P. de la Harpe, Classical Banach-Lie groups and Banach-Lie algebras of operators in Hilbert spaces, Lecture Notes in Math. Springer-Verlag, 285, 1970.
[9] P. R. Halmos, A Hilbert space problem book, Americanbook, 1967.
[10] N. Jacobson, Lie algebras, Interscience, 1962.
[11] S. Kobayashi and T. Nagano, On filtered Lie algebras and geometric structures I & III, J. Math. Mech. 13 (1964), 875-908, and 14, 679-706.
[12] A. Koriyama, Y. Maeda and H. Omori, On Lie algebras of vector fields, Trans. A.M.S., 226 (1977), 89-117.
[13] B. Maissen, Lie Gruppen mit Banachraümen als Parameterräum, Acta Math. 108 (1962), 229-270.
[14] D. Montgomery and L. Zippin, Topological transformation groups, Interscience, 1955.
[15] T. Morimoto and N. Tanaka, The classification of real primitive infinite Lie algebras, J. Math. Kyoto Univ. 10 (1970), 207-243.
[16] H. Omori and P. de la Harpe, About interactions between Banach-Lie groups and finite dimensional manifolds, J. Math. Kyoto Univ. 12 (1972), 543-570.
[17] H. Omori, Infinte dimensional Lie transformation groups, Lecture Notes, in Math. 427 Springer-Verlag, 1974.
[18] S. Shnider, The classification of real primitive infinite Lie algebras J. Diff. Geom. 4 (1970), 81-89.
[19] I. M. Singer and S. Sternberg, On the infinite groups of Lie and Cartan I, J. Anal. Math. 15 (1965), 1-114.
[20] W. T. VanEst and T. J. Korthagen, Non-enlargible Lie algebras, Indag. Math. 26 (1964), 15-31.
[21] K. Yoshida, Functional analysis, Springer-Verlag, 1966.

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