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COMPACTIFICATIONS OF $C \times C^*$ AND $(C^*)^2$

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1. **Introduction.** By a compactification of a complex analytic manifold V we mean a compact complex analytic manifold S together with a 1-codimensional analytic subset C of S such that $S - C$ is biholomorphic to V. A compactification (S, C) of a non-singular complex analytic surface V will be called a *minimal normal compactification* of V, if it satisfies the following two conditions: (i) any singular point of C is an ordinary double point, (ii) no non-singular rational irreducible component of C with self-intersection number -1 has at most two intersection points with other components of C.

Hirzebruch [2] posed the problem of classifying the compactifications of the n-dimensional complex affine space C^n. Answering this in the case of $n = 2$, Kodaira [3] proved that every compactification of C^2 is a rational surface and Morrow [8] announced a list of all minimal normal compactifications of C^2.

In his lecture on Nevanlinna theory [4], letting $C^* = C - \{0\}$, Kodaira asked what the compactifications of $C \times C^*$ are. Recently, considering the problem of classifying the compactifications of $C \times C^*$ and $(C^*)^2$, Ueda [13] obtained the following two results: (a) any compactification of $C \times C^*$ is a rational surface, (b) any minimal normal compactification (S, C) of $(C^*)^2$ is of one of the following three types: (1) S is a rational surface, (2) S is a Hopf surface containing only one irreducible curve, (3) S is an algebraic surface (with $p_g = 0$, $b_1(S) = 2$) and C an irreducible non-singular elliptic curve on S, while Simha [12] proved that, in this third case (3), S is a projective line bundle over an elliptic curve admitting a global section C such that $S - C$ is an analytically non-trivial principal C-bundle. The proofs of the above results in [3] and [13] are based on the Nevanlinna theory generalized by Kodaira [3].

In this note, we shall consider rational compactifications of $C \times C^*$ and $(C^*)^2$. In order to formulate our results, let us associate to each curve C composed of non-singular rational curves crossing normally, a weighted graph $\Gamma(C)$ as follows: we represent each irreducible component of C by a circle $\circ$ ("vertex"), join these circles by straight lines as many times as the corresponding components intersect each other, and attach to each of these circles the number ("weight") equal to the self-intersection number of the corresponding irreducible component of C. We shall prove

THEOREM 1. *If (S, C) is a minimal normal rational compactification of $(C^*)^2$, then* (1) *S is the* (*complex*) *projective plane P^2 and C is the union of three lines in general position* (*see Figure* 1), *or* (2) *S is a projective line bundle F_n over a projective line P^1 and C is the union of its* 0*-section, ∞-section and two distinct fibers, where $n \geqq 0$, $n \neq 1$ and the self-intersection numbers of the* 0*-section and ∞-section are $-n$ and n respectively* (*see Figure* 2).

THEOREM 2. *If (S, C) is a minimal normal compactification of $C \times C^*$, then all the irreducible components of C are non-singular rational curves and the pair $(S, \Gamma(C))$ is of one of the types in Table* I *below* (*see the end of this Introduction*).

Since (S, C) of types 4_r and 5_r in Table I can be transformed into that of type 3 by successive applications of birational transformations of S which are biregular on $S - C$ as follows: $5_r \rightarrow 4_r \rightarrow 5_{r-1} \rightarrow 4_{r-1} \rightarrow \cdots \rightarrow 5_1 \rightarrow 4_1 \rightarrow 3$ (see n°4), we can deduce from Theorem 2 that: *for any compactification (S, C) of $C \times C^*$, there is a birational mapping of S to P^2 which biregularly maps $S - C$ to $P^2 -$ (two lines).*

The proof of Theorems 1 and 2 which we shall give below leans heavily on the theory of cluster sets at isolated essential singular points of holomorphic mappings into complex surfaces [10]. Note that, in Theorem 2, we shall prove at the same time the rationality of any compactification of $C \times C^*$, without the use of Nevanlinna theory, but in [10] which we need instead, Ahlfors's theory on covering surfaces is applied(*). We shall do in n°4 the analysis of $\Gamma(C)$ in a way analogous to that in Ramanujam [11] in some sense.

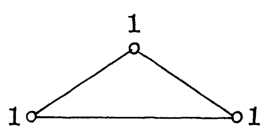

FIGURE 1

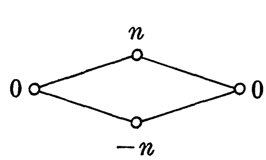

FIGURE 2

(*) The result of Ueda for compactifications of $(C^*)^2$ and the rationality of any compactification of C^2 mentioned above can also be proved by the aide of [10] instead of Nevanlinna theory in [3].

TABLE I

Type	S	$\Gamma(C)$	
1.	$\boldsymbol{P}^2$	Figure 3	
2.	F_n	Figure 4	($n \geqq 0$, $n \neq 1$ and m is any integer)
3.	F_n	Figure 5	($n \geqq 0$, $n \neq 1$)
4_r.		Figure 6	($r \geqq 0$, p_i's and q_i's are $\geqq 3$)
5_r.		Figure 7	($r \geqq 0$, p_i's and q_i's are $\geqq 3$)

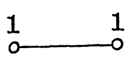

FIGURE 3

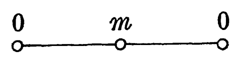

FIGURE 4

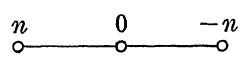

FIGURE 5

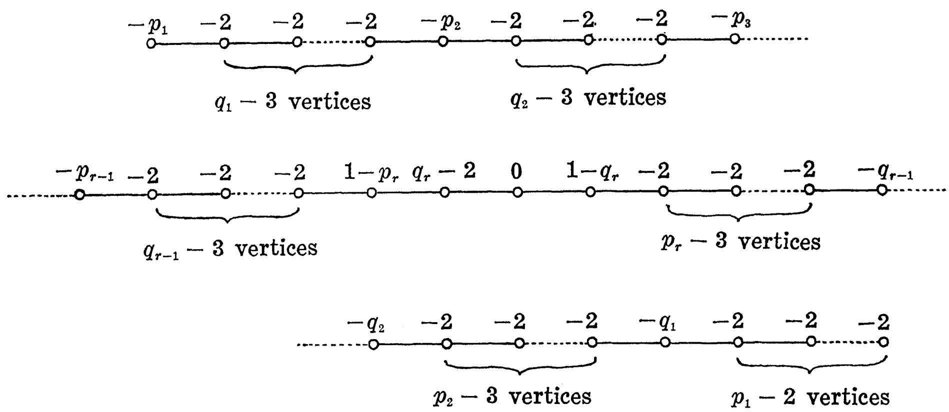

FIGURE 6

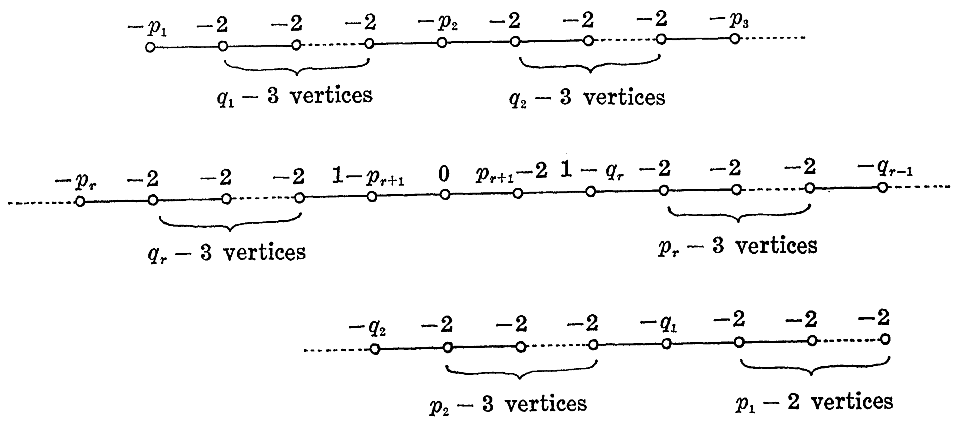

FIGURE 7

2. Preliminaries. 1° *A result of* [10]. Let S be a non-singular complex analytic surface and C be a connected compact (complex) analytic curve on S satisfying the above conditions (i) and (ii) in Introduction. Suppose further that, for each irreducible component C_i of C, there is a holomorphic mapping $f_i\colon \Delta^* \to S - C$ of a punctured disc $\Delta^* = \{z \in \boldsymbol{C} \mid 0 < |z| < 1\}$ into $S - C$ such that $C_i \subset f_i(0; S) \subset C$, where

$$f_i(0; S) = \bigcap_{\rho > 0} \overline{f_i(\Delta_\rho^*)}\,, \qquad \Delta_\rho^* = \{z \in \boldsymbol{C} \mid 0 < |z| < \rho\}$$

and $\overline{f_i(\Delta_\rho^*)}$ is the closure of $f_i(\Delta_\rho^*)$ in S. We have then, by [10] (Chapter II, n° 7),

LEMMA 1. *The curve C must be of one of the types* (α) *to* (ε) *in the following Table* II, *in which, for the types β_b* ($b \geqq 2$), *γ, γ', δ and ε, each irreducible component of C is a non-singular rational curve and the assigned Figures* (8–12) *represent the weighted graph $\Gamma(C)$.*

TABLE II

	Name of Type	Explication of C
α	$\alpha(n)$	an irreducible non-singular elliptic curve with the self-intersection number $(C^2) = n \geqq 0$.
β_1	$\beta(n)$	an irreducible rational curve with only one ordinary double point and $(C^2) = n \geqq 0$.
β_b	$\beta(n_1, n_2, \cdots, n_b)$ $(b \geqq 2)$	Figure 8, all $n_i = -2$ or $\max\{n_1, n_2, \cdots, n_b\} \geqq 0$.
γ	$\gamma(n_1, n_2, \cdots, n_b)$ $(b \geqq 1)$	Figure 9, all $n_i = -2$ or $\max\{n_1+1, n_2, \cdots, n_{b-1}, n_b+1\} \geqq 0$.
γ'	$\gamma'(n_1, n_2, \cdots, n_b)$ $(b \geqq 2)$	Figure 10, $\max\{n_1 + 1, n_2, \cdots, n_b\} \geqq 0$.
δ	$\delta(n_0; q_1/l_1, q_2/l_2, q_3/l_3)$	Figure 11, (i) $n_0 \geqq 2$, (ii) $(l_1, l_2, l_3) = (3, 3, 3)$, $(2, 4, 4)$ or $(2, 3, 6 - m)$ with $m = 0, 1, 2, 3$, (iii) for each $i = 1, 2, 3$, (l_i, q_i) is a pair of coprime integers such that $0 < q_i < l_i$ and that $l_i/q_i = n_{i,1} - 1 \rfloor n_{i,2} - \cdots - 1 \rfloor n_{i,r_i}$ (continued fraction expansion), where $n_{i,j} \geqq 2$ are the integers appearing in Figure 11.
ε	$\varepsilon(n_1, n_2, \cdots, n_b)$ $(b \geqq 1)$	Figure 12, $\max\{n_1, n_2, \cdots, n_b\} \geqq 0$.

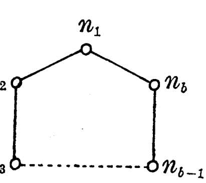

FIGURE 8

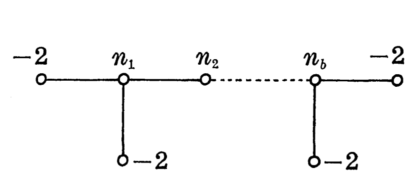

FIGURE 9

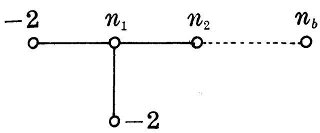

FIGURE 10

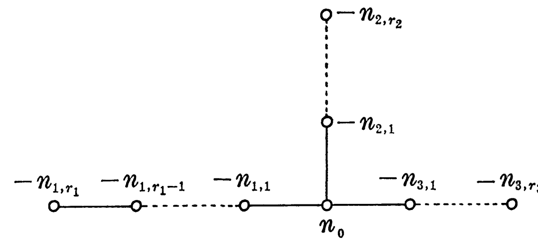

FIGURE 11

FIGURE 12

The following three remarks can be checked easily and left to the reader.

REMARK 1. For the types $\beta(-2, -2, \cdots, -2)$ with $b \geqq 2$ and $\gamma(-2, -2, \cdots, -2)$, the intersection matrix $((C_i \cdot C_j))$ is degenerated and negative semi-definite.

REMARK 2. For the type $\gamma(n)$, we have: $\det((C_i \cdot C_j)) = 16(n + 2)$; therefore, if the matrix $((C_i \cdot C_j))$ is degenerated ($n = -2$), it is negative semi-definite.

REMARK 3. For the type δ, the determinant of the matrix $(-(C_i \cdot C_j))$ is of the form:

$$\det(-(C_i \cdot C_j)) = a \cdot \left(-n_0 - \sum_{k=1}^{3} q_k/l_k\right), \qquad a \geqq 0 .$$

Then, if $\det(-(C_i \cdot C_j)) = \pm\det((C_i \cdot C_j)) = 0$, combining this with the conditions (ii) and (iii) in Table II, we have the following six types:

$$\delta(-1; 1/2, 1/3, 1/4), \quad \delta(-1; 1/2, 1/4, 1/4), \quad \delta(-1; 1/3, 1/3, 1/3),$$
$$\delta(-2; 1/2, 2/3, 5/6), \quad \delta(-2; 1/2, 3/4, 3/4), \quad \delta(-2; 2/3, 2/3, 2/3).$$

And for these six types, $((C_i \cdot C_j))$ is negative semi-definite.

$2°$ *The proof of the existence of* f_i. Let S be a compact complex analytic surface, C a compact analytic curve on S, and suppose that $S - C$ is biholomorphic to V, where V is one of the following:

(1) $$\boldsymbol{C}^2 : |x| < \infty, \quad |y| < \infty,$$

(2) $$\boldsymbol{C} \times \boldsymbol{C}^* : |x| < \infty, \quad 0 < |y| < \infty,$$

(3) $$(\boldsymbol{C}^*)^2 : 0 < |x| < \infty, \quad 0 < |y| < \infty,$$

x, y being two complex variables. Then,

LEMMA 2. *For each irreducible component C_i of C, there exists a holomorphic mapping $f_i: \Delta^* \to S - C$ of a punctured disc $\Delta^*: 0 < |z| < 1$ into $S - C$ such that $C_i \subset f_i(0; S) \subset C$.*

PROOF. We will prove this for the case (3) $V = (\boldsymbol{C}^*)^2$. The proof for the other cases are similar. Let us take five distinct points P_k ($k = 1, 2, \cdots, 5$) of C_i which are regular points of C, and let $\varphi: S - C \to V$ be an analytic isomorphism. For each k, we can find a sequence of points $\{P_{kn}\}_{n=1,2,\dots}$ of $S - C$ which converges to P_k and such that both $\lim_{n\to\infty} x(\varphi(P_{kn}))$, $\lim_{n\to\infty} y(\varphi(P_{kn}))$ exist, where we allow ∞ as a limit. Let $x_{kn} = x(\varphi(P_{kn}))$, $y_{kn} = y(\varphi(P_{kn}))$. Since φ is an isomorphism, at least one of the following four conditions holds for each $k \in \{1, 2, \cdots, 5\}$: $\lim x_{kn} = \infty$, $\lim x_{kn} = 0$, $\lim y_{kn} = \infty$, $\lim y_{kn} = 0$. Hence, one of them holds for two k's. Replacing x by y or $1/x$ and renumbering $\{P_k\}$, if necessary, we may assume $\lim_{n\to\infty} x_{1n} = \lim_{n\to\infty} x_{2n} = \infty$. Further, we may assume $x_{kn} \neq x_{jm}$ for every $(k, n) \neq (j, m)$, after a slight variation of $\{P_{kn}\}$. By Lemma 3 below, we can find a holomorphic function $h(x)$ with no zeros on a punctured disc $\Delta_\rho^*: 1/\rho < |x| < \infty$ $(0 < \rho < \infty)$ such that $h(x_{kn}) = y_{kn}$ $(\neq 0)$ for all (k, n), $k = 1, 2$; $n = 1, 2, \cdots$. Let $f_i(x) = \varphi^{-1}(x, h(x))$, then we have a holomorphic mapping $f_i: \Delta_\rho^* \to S - C$ such that $f_i(\infty; S)$ $(\subset C)$ contains two regular points P_1, P_2 of C on C_i. Therefore, by Proposition 3 of [10], we have $C_i \subset f_i(\infty; S) \subset C$.

LEMMA 3. *Let $e = \{z_n\}_{n=1,2,\dots}$ be a discrete subset of the z-plane $\boldsymbol{C}$ and $a(z_n) = a_n$ be an arbitrary complex valued function on e. Then, there exists an entire function $f(z)$ such that $f(z_n) = a_n$. Furthermore, if $a(z_n)$ does not take zero, we can find $f(z)$ which does not take zero on $\boldsymbol{C}: |z| < \infty$.*

PROOF. (due to H. Cartan [1]). By Weierstrass's theorem, there is an entire function $g(z)$ of the form: $g(z) = \prod_{n=1}^{\infty} (z - z_n)e^{P_n(z)}$, where $P_n(z)$ are polynomials of z. By Mittag-Leffler's theorem, we can find a meromorphic function $\psi(z)$ on $|z| < \infty$ which has the same principal part as $a_n/g(z)$ at each point z_n, $n = 1, 2, \cdots$. Then, the product $f(z) = g(z) \cdot \psi(z)$ is an entire function which is of the form

$$f(z) = a_n + g(z) \cdot (\text{holomorphic function})$$

at each z_n, so that $f(z_n) = a_n$. As for the second assertion of Lemma 3, it suffices to apply the above result to find an entire function $h(z)$ such that $h(z_n) = \log a_n$. Putting $f(z) = e^{h(z)}$, we obtain a desired function.

3° *Rational ruled surfaces.* Let us recall some facts on rational ruled surfaces. Let S be a compact non-singular complex analytic surface and C be an irreducible non-singular rational curve on S with the self-intersection number $(C^2) = 0$. Since $H^0(C, \mathcal{O}) \cong \boldsymbol{C}$, $H^1(C, \mathcal{O}) = 0$ and the normal bundle of C is analytically trivial, there exists, by Kodaira and Spencer [7], a holomorphic function h in a neighborhood U of C such that $h^{-1}(0) = C$ (regular fiber). Therefore, S contains infinitely many rational curves $h^{-1}(t)$, and by Kodaira [5, Theorem 5.1], S is an algebraic surface, so that, again by Kodaira and Spencer [7], we obtain a holomorphic mapping π of S onto a non-singular analytic curve R which has C as a regular fiber. Thus, S is a ruled surface. Let us assume that there exists another rational curve $C' \neq C$ on S which intersects C. Since $\pi|_{C'}: C' \to R$ is not constant, R is then isomorphic to a projective line $\boldsymbol{P}^1$, and S is a rational ruled surface. By Nagata [9], S can be obtained, from a projective line bundle F_n over $\boldsymbol{P}^1$ which has the 0-section with the self-intersection number $-n$ $(n \geqq 0)$, by a finite succession of quadratic transformations $Q_{P_1}, Q_{P_2}, \cdots, Q_{P_r}$ $(r \geqq 0)$, in such a way that

$$S = Q_{P_r} Q_{P_{r-1}} \cdots Q_{P_1}(F_n) \quad \text{and} \quad \pi = \pi_0 \circ Q_{P_1}^{-1} \circ \cdots \circ Q_{P_r}^{-1},$$

where $\pi_0: F_n \to \boldsymbol{P}^1$ is the projection. Therefore, each fiber $\pi^{-1}(z)$ $(z \in \boldsymbol{P}^1)$ is a curve with no loop composed of non-singular rational curves crossing normally. Since rank $H_2(F_n; \boldsymbol{R}) = 2$, we have

LEMMA 4. rank $H_2(S; \boldsymbol{R}) = r + 2$.

On the other hand, if one denotes the number of irreducible components of $\pi^{-1}(z)$ by $1 + \alpha(z)$ for each $z \in \boldsymbol{P}^1$, we have

LEMMA 5. $\sum_{z \in \boldsymbol{P}^1} \alpha(z) = r$.

Now let $C_1, C_2, \cdots, C_n$ be irreducible non-singular rational curves on a non-singular projective algebraic surface S such that $C = \bigcup_{i=1}^n C_i$ is simply connected. We have then,

LEMMA 6. (a) *If there is a pair α, β $(1 \leqq \alpha < \beta \leqq n)$ such that $(C_\alpha^2) \geqq 0$, $(C_\beta^2) \geqq 0$ and $C_\alpha \cap C_\beta = \emptyset$, then $(C_\alpha^2) = (C_\beta^2) = 0$ and there is only one C_γ $(\gamma \neq \alpha, \beta)$ which intersects $C_\alpha \cup C_\beta$. Further, for this C_γ, we have $(C_\alpha \cdot C_\gamma) = (C_\beta \cdot C_\gamma) = \nu \geqq 1$. (The graph $\Gamma(C)$ looks like Figure* 13).

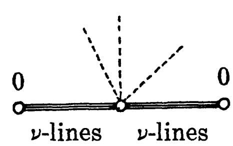

FIGURE 13

(b) *Assume that any singular point of C is an ordinary double point and that there is a pair α, β $(1 \leqq \alpha < \beta \leqq n)$ such that $(C_\alpha^2) > 0$, $(C_\beta^2) > 0$. Then $n = 2$ and $(C_1^2) = (C_2^2) = 1$. ($\Gamma(C)$ looks like Figure 3.)*

PROOF. (a) Blow up $n_\alpha = (C_\alpha^2)$ points $\{P_k\}$ of C_α and $n_\beta = (C_\beta^2)$ points $\{Q_k\}$ of C_β to obtain C'_α, C'_β in S' such that $(C'^2_\alpha) = (C'^2_\beta) = 0$, $C'_\alpha \cap C'_\beta = \varnothing$. Since the image C' of C in S', composed of non-singular rational curves, is connected, S' has a structure of ruled surface $\pi: S' \to \boldsymbol{P}^1$ over a projective line $\boldsymbol{P}^1$: $|z| \leqq \infty$ with an inhomogeneous coordinate z such that $C'_\alpha = \pi^{-1}(0)$, $C'_\beta = \pi^{-1}(\infty)$. If $n_\alpha > 0$, the image P'_k of P_k in S' is a compact (rational) curve and the function $z \circ \pi|_{P'_k}$ is not constant. Therefore it takes all values $|z| \leqq \infty$ and $P'_k \cap \pi^{-1}(\infty) \neq \varnothing$. This is contradictory to $C_\alpha \cap C_\beta = \varnothing$. Therefore, $n_\alpha = 0$. In the same way, $n_\beta = 0$. We have thus a rational ruled surface structure $S \xrightarrow{\pi} \boldsymbol{P}^1$ on S such that $\pi^{-1}(0) = C_\alpha$, $\pi^{-1}(\infty) = C_\beta$ (regular fibers). In the same way as above, we see that any component C_γ $(\gamma \neq \alpha, \beta)$ such that $C_\gamma \cap (C_\alpha \cup C_\beta) \neq \varnothing$ intersects both C_α, C_β. Since C is simply connected, this C_γ is unique and $C_\alpha \cap C_\gamma$ (resp. $C_\beta \cap C_\gamma$) consists of a single point, say p (resp. q). Let t be an inhomogeneous coordinate function on $C_\gamma \approx \boldsymbol{P}^1$ such that $t(p) = 0$, $t(q) = \infty$. The function $z(t) = (z \circ \pi|_{C_\gamma})(t)$ takes zeros only for $t = 0$ and poles only for $t = \infty$. Therefore, $z(t) = a \cdot t^\nu$, where $a \in \boldsymbol{C}^* = \boldsymbol{C} - \{0\}$, $\nu \in \boldsymbol{N} - \{0\}$. Thus, $(C_\alpha \cdot C_\gamma) = (C_\beta \cdot C_\gamma) = \nu$.

(b) As we have seen in (a), C_α and C_β $((C_\alpha^2) > 0, (C_\beta^2) > 0)$ must intersect each other, at only one point P. Blow up this point of intersection P and denote the images of S, C_α, C_β, P etc. by S', C'_α, C'_β, P' etc. respectively. We have $(C'^2_\alpha) = (C'^2_\beta) = 0$, $C'_\alpha \cap C'_\beta = \varnothing$. Since any singular point of C is an ordinary double point, there is no component of C' intersecting P' other than C'_α, C'_β. Therefore, applying the result of (a), we see that $\Gamma(C')$ looks like Figure 4 with $m = -1$, so that $\Gamma(C)$ looks like Figure 3.

LEMMA 7. *Let C be a simply connected curve composed of three non-singular rational irreducible curves C_α, C_β, C_γ on a non-singular projective algebraic surface S, such that $(C_\alpha^2) = (C_\beta^2) = 0$, $C_\alpha \cap C_\beta = \varnothing$, $(C_\alpha \cdot C_\gamma) = (C_\beta \cdot C_\gamma) = \nu \geqq 1$. Suppose further that* rank $H_2(S; \boldsymbol{R}) = 2$ *and that $S - C$ is biholomorphic to $\boldsymbol{C} \times \boldsymbol{C}^*$. Then, $S = \boldsymbol{F}_n$ $(n \geqq 0)$ and $\nu = 1$. ($\Gamma(C)$ looks like Figure 4.)*

PROOF. Let $\pi: S \to \boldsymbol{P}^1$, z, t be the same as in Proof (a) of Lemma 6. Recall that $C_\alpha = \pi^{-1}(0)$, $C_\beta = \pi^{-1}(\infty)$, $z(t) = (z \circ \pi|_{C_\gamma})(t) = a \cdot t^\nu$ $(a \neq 0)$. Since rank $H_2(S; \boldsymbol{R}) = 2$, we have $S = \boldsymbol{F}_n$ by Lemma 4 $(r = 0)$; the projection $\pi: S \to \boldsymbol{P}^1$ has no singular fiber. Put $\boldsymbol{C}^* = \boldsymbol{P}^1 - \{0, \infty\}$. Since

$C_\gamma^* = C_\gamma \cap \pi^{-1}(C^*)$ has no branching point as a covering over C^* with respect to the projection $\pi|_{C_\gamma^*}$, $S - C$ is therefore topologically a fiber bundle over C^* with fiber $F \approx P^1 - \{\nu \text{ points}\}$ and with projection $\pi|_{S-C}$: $S - C \to C^*$. Consider the exact sequence of homotopy groups of this fiber bundle:

$$\cdots \underset{\substack{\| \\ 0}}{\pi_2(C^*)} \xrightarrow{\partial} \pi_1(F) \to \pi_1(S - C) \to \pi_1(C^*) \xrightarrow{\partial} \underset{\substack{\| \\ 0}}{\pi_0(F)} \cdots .$$

Since $\pi_1(C^*) \cong Z$, $\pi_1(S - C) \cong \pi_1(C \times C^*) \approx Z$, it follows that $\pi_1(F) = 0$, which implies $\nu = 1$.

3. Rational compactification of $(C^*)^2$. Let (S, C) be a minimal normal rational compactification of $(C^*)^2$. We know by Ueda ([13] n° 4) and Simha [12] that

(i) the first Betti number $b_1(C)$ is equal to 1, and

(ii) there exists an exact sequence of homology groups (with real coefficients) as follows:

$$0 \to R^2 \to H_2(C; R) \xrightarrow{i_*} H_2(S; R) \to 0 ,$$

where i_* is the homomorphism induced by the inclusion map $C \hookrightarrow S$. Let $C_1, C_2, \cdots, C_b$ be the irreducible components of C. Since $b^+(S) = 1$ (S is rational), it follows from this exact sequence that

(iii) the intersection matrix $((C_i \cdot C_j))$ is degenerated, but *not* negative semi-definit.

Further, since rank $H_2(S, R) \geqq 1$, it follows from (ii) that $b \geqq 3$. Now, by Lemmas 2 and 1, C must be one of the curves listed in Table II. Since $b_1(C) = 1$ (by (i)), C is of type $\beta(n_1, n_2, \cdots, n_b)$ with $\Gamma(C)$ as in Figure 8, where we have $\max\{n_1, n_2, \cdots, n_b\} \geqq 0$, because of (iii) and Remark 1 (to Lemma 1). We may assume $(C_i^2) = n_i$ $(i = 1, 2, \cdots, b)$.

Suppose first $(C_1^2) = n_1 = 0$. Then, according to Preliminary 3°, S has a structure of rational ruled surface $S \xrightarrow{\pi} P^1$ such that C_1 is its regular fiber, and $S = Q_{P_r} Q_{P_{r-1}} \cdots Q_{P_1}(F_n)$, where $r = \operatorname{rank} H_2(S) - 2 = b - 4$ by Lemma 4 and (ii), so that $b \geqq 4$. On the other hand, since the components $C_3, C_4, \cdots, C_{b-1}$ are contained in a fiber of π, say $\pi^{-1}(z_0)$, we have

$$b - 4 \leqq \alpha(z_0) \leqq \sum_{z \in P^1} \alpha(z) = r , \qquad \text{(Lemma 5)} .$$

Hence, $\alpha(z_0) = b - 4$ and the fiber $\pi^{-1}(z_0)$ is composed of $C_3, C_4, \cdots, C_{b-1}$. Therefore, by Nagata's theorem cited in Preliminary 3°, if $b \geqq 5$, there is at least one component C_i $(3 \leqq i \leqq b - 1)$ with $(C_i^2) = -1$, which is contradictory to the assumption of the minimality of C. Thus, we have $b=4$ and $r=0$, so that $S=F_n$. Since C_2 and C_4 are two disjoint holomorphic sections of π, we can take them as 0-section and ∞-section, while C_1 and C_3 are its regular fibers. This is the case (2) of Theorem 1.

Hence, we assume from now on that $n_i \neq 0$ for all i, and $n_1 > 0$. Blow up n times successively, the point of intersection corresponding to the line between the vertices with weights n_1 and n_b of $\Gamma(C)$ (Figure 8) to obtain (S', C') with $\Gamma(C')$ which looks like Figure 14. Since $n_i \neq -1$

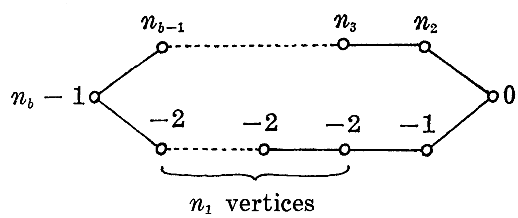

FIGURE 14

for all i and $n_b - 1 \neq -1$, the above argument asserts that $S' = F_n$ and $\Gamma(C')$ looks like Figure 2. Here $n=1$, for C' contains a component with the self-intersection number -1. Blowing down this component, we thus obtain: $S = \boldsymbol{P}^2$ (projective plane) and $\Gamma(C)$ as in Figure 1. C is therefore composed of three lines in $S = \boldsymbol{P}^2$ in general position. This completes the proof of our Theorem 1 (see Introduction).

4. **Compactifications $\boldsymbol{C} \times \boldsymbol{C}^*$.** Let (S, C) be a minimal normal compactification of $\boldsymbol{C} \times \boldsymbol{C}^*$. With Ueda [13], let us consider the exact sequence of homology groups with real coefficients for the pair (S, C):

$$0 \to H_3(S) \to H_3(S, C) \xrightarrow{\partial} H_2(C) \xrightarrow{i_*} H_2(S) \to$$
$$H_2(S, C) \xrightarrow{\partial} H_1(C) \to H_1(S) \to 0 .$$

Since $S - C \approx \boldsymbol{C} \times \boldsymbol{C}^*$, we have $H_3(S, C) \cong \boldsymbol{R}$, $H_2(S, C) = 0$ and $H_1(C) \cong H_1(S) \cong \boldsymbol{R}$ or 0.

Let us denote the k-th Betti number of $*$ by $b_k(*)$, and suppose $b_1(C)=1$. We have then $b_1(S)=1$. By Theorems 3 and 25 of Kodaira [6],

(a) $b^+(S) = 2p_g$ and (b) S is not algebraic .

On the other hand, by the Poincaré duality, $b_3(S) = b_1(S) = 1$, so that

(c) $i_*\colon H_2(C) \to H_2(S)$ is an isomorphism .

Now, applying Lemmas 2 and 1, we see that C is of type $\beta(n_1, n_2, \cdots, n_b)$

($b \geqq 1$) in Table II. If $b = 1$, we must have $b_2(S) = b_2(C) = 1$, so that $b^+(S) = 0$ by (a). This implies that $(C^2) < 0$ and that C is exceptional, which is absurd. Thus $b \geqq 2$. Now, $\det((C_i \cdot C_j)) \neq 0$ by (c). According to Remark 1 to Lemma 1, we have then, $\max\{n_1, n_2, \cdots, n_b\} \geqq 0$ and by Preliminary 3°, S must be a rational surface, contradicting (b).

Thus, we have

(i) $b_1(C) = 0$.

Therefore, $b_3(S) = b_1(S) = b_1(C) = 0$. Letting $C_1, C_2, \cdots, C_n$ to be the irreducible components of C, we have the following two conditions:

(ii) $n = b_2(C) = b_2(S) + 1$,

(iii) the intersection matrix $((C_i \cdot C_j))$ is degenerated.

Further, since $b_1(C) = 0$ (even), we have, by Kodaira [6, Theorem 3], $b^+(S) = 2p_g + 1 \geqq 1$. Since $i_*: H_2(C) \to H_2(S)$ is surjective, we get the condition:

(iv) $((C_i \cdot C_j))$ is *not* negative semi-definite.

Now, by Lemmas 2 and 1, C belongs to Table II. The above conditions (i), (iii), (iv) and Remarks 1-3 to Lemma 1 imply that C is of one of the following three types:

1. $\gamma(n_1, n_2, \cdots, n_b)$ (Figure 9) with $b \geqq 2$ and $\max\{n_1 + 1, n_2, \cdots, n_{b-1}, n_b + 1\} \geqq 0$,
2. $\gamma'(n_1, n_2, \cdots, n_b)$ (Figure 10) with $b \geqq 2$ and $\max\{n_1 + 1, n_2, \cdots, n_b\} \geqq 0$,
3. $\varepsilon(n_1, n_2, \cdots, n_b)$ (Figure 12) with $b \geqq 2$ and $\max\{n_1, n_2, \cdots, n_b\} \geqq 0$.

(By $b^+(S) \geqq 1$, we have $n = b_2(C) = b_2(S) + 1 \geqq 2$.)

We denote by $d\Gamma(C)$ the determinant $\det(-(C_i \cdot C_j))$ of the intersection matrix with opposite sign $(-(C_i \cdot C_j))$. By (iii), $d\Gamma(C) = 0$. We put:

1. $w_1 = n_1 + 1$, $w_2 = n_2, \cdots, w_{b-1} = n_{b-1}$, $w_b = n_b + 1$, if C is of type γ,
2. $w_1 = n_1 + 1$, $w_2 = n_2, \cdots, w_b = n_b$, if C is of type γ',
3. $w_i = n_i$ for all i $(=1, 2, \cdots, b)$, if C is of type ε.

We have $\max\{w_1, w_2, \cdots, w_b\} \geqq 0$.

1° Suppose first that there is only one $w_i \geqq 0$. Then, $(w) = (w_1, w_2, \cdots, w_n)$ is of the form:

$$(-p_1, -p_2, \cdots, -p_r, m, -q_s, -q_{s-1}, \cdots, -q_1),$$

where $p_1 \geqq 1$, $p_i \geqq 2$ $(i = 2, 3, \cdots, b)$, $q_1 \geqq 1$, $q_j \geqq 2$ $(j = 2, 3, \cdots, s)$, $m \geqq 0$, $r \geqq 0$, $s \geqq 0$ and $r + s = b - 1 \geqq 1$. $d\Gamma(C)$ can be calculated as follows (cf. Ramanujam [11]): Let

(A) $a = 16, 4, 1$ according as C is of type γ, γ', ε respectively,

(B) $\alpha_1 = p_1,\ \alpha_i = p_i - \underline{1}|\overline{p_{i-1}} - \cdots - \underline{1}|\overline{p_1}$ for $2 \leqq i \leqq r$,
$\beta_1 = q_1,\ \beta_j = q_j - \underline{1}|\overline{q_{j-1}} - \cdots - \underline{1}|\overline{q_1}$ for $2 \leqq j \leqq s$.
(continued fractions)

We have then,

$$d\Gamma(C) = a \cdot \alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_r(-m - 1/\alpha_r - 1/\beta_r) \cdot \beta_s \cdot \beta_{s-1} \cdot \cdots \cdot \beta_1$$

(Omit the α's and $-1/\alpha_r$ if $r = 0$; the β's and $-1/\beta_s$ if $s = 0$.)

Since $\alpha_k \geqq 1$, $\beta_k \geqq 1$ for all $k \geqq 1$, $m \geqq 0$ and $r + s \geqq 1$, this leads to $d\Gamma(C) \neq 0$, a contradiction.

2° Next suppose that there is a pair (i_0, j_0) such that $w_{i_0} \geqq 0$, $w_{j_0} \geqq 0$ and $j_0 > i_0 + 1$. Blowing up the intersection points of C_i's suitably, we may assume $w_{i_0} = w_{j_0} = 0$. Let us denote by C' the total transform of C. The graph $\Gamma(C')$ looks like Figure 15. In case 15-a,

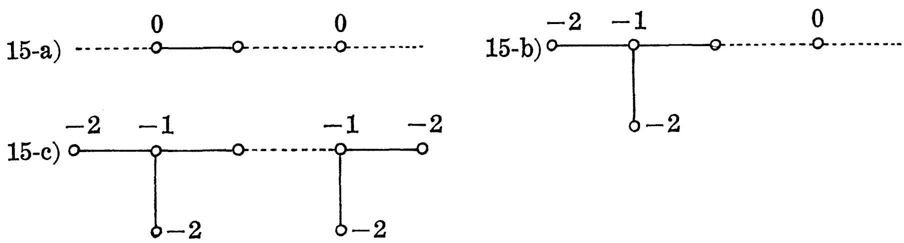

FIGURE 15

apply Lemma 6 to (S', C') to see that $S' = F_n$ and that $\Gamma(C')$ is of the form of Figure 4, which means that $\Gamma(C)$ was of the same form (Figure 4, $S = F_n$).

As for case 15-b (resp. 15-c), contracting the two (resp. four) curves corresponding to the vertices with weights -1 and -2 on the horizontal line of Figure 15-b (resp. 15-c), we get (S'', C'') with the graph $\Gamma(C'')$ which looks like Figure 16 (resp. 17). Applying Lemma 6 to this (S'', C''),

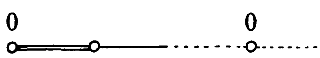

FIGURE 16

FIGURE 17

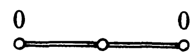

FIGURE 18

we see that the case 15-b cannot occur, and that in case 15-c, $\Gamma(C'')$ must be of the form of Figure 18. But note that $b_2(C'') - b_2(S'') = b_2(C') - b_2(S') = b_2(C) - b_2(S) = 1$, so that $b_2(S'') = 2$. By Lemma 7, also the case 15-c cannot occur.

3° Suppose now that there is a pair (i_0, j_0) such that $w_{i_0} > 0$, $w_{j_0} > 0$ and $j_0 > i_0$. Blow up the point of intersection corresponding to the line between the vertices with weights n_{i_0} and n_{i_0+1} of $\Gamma(C)$ (Figures 9, 10 or 12) to get (S', C') with the graph $\Gamma(C')$ which looks like Figure 19.

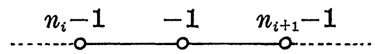

FIGURE 19

Apply the above result of 2° to this (S', C') to see that $\Gamma(C')$ is of the form of Figure 4 with $m = -1$ and that $S' = F_1$. This means that $\Gamma(C)$ was of the form of Figure 3 and $S = \boldsymbol{P}^2$.

4° Thus, we may assume from now on that $(w_1, w_2, \cdots, w_b)$ or $(w_b, w_{b-1}, \cdots, w_1)$ is of the form:

$$(-p_1, -p_2, \cdots, -p_r, m, 0, -q_s, -q_{s-1}, \cdots, -q_1)\ ,$$

where $p_1 \geqq 1$, $p_i \geqq 2$ $(i = 2, 3, \cdots, r)$, $q_1 \geqq 1$, $q_j \geqq 2$ $(j = 2, 3, \cdots, s)$, $r \geqq 0$, $s \geqq 0$ and $m \geqq 0$. We prove

PROPOSITION (P): *In this case,* $(S, \Gamma(C))$ *is of one of the types* (3), (4) *and* (5) *in Table* I (*see Introduction*).

Let a, α_i, β_j $(i = 1, 2, \cdots, r,\ j = 1, 2, \cdots, s)$ be the same as in 1° (Recall (A) and (B).) Then,

$$d\Gamma(C) = a \cdot \alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_r[(m + 1/\alpha_r)(0 + 1/\beta_s) - 1] \cdot \beta_s \cdot \beta_{s-1} \cdot \cdots \cdot \beta_1$$

(Omit the α's and $1/\alpha_r$ if $r = 0$; the β's and $1/\beta_s$ if $s = 0$) .

Since $d\Gamma(C) = 0$, we obtain,

$$(m + 1/\alpha_r)(0 + 1/\beta_s) = 1\ .$$

We deduce from this that ① $s \neq 0$ and $b \geqq 3$.

Suppose that $r = 0$ and $s \geqq 2$. Then, we must have $\beta_s = m$ (integer). This implies $q_s = m + 1$, $q_{s-1} = \cdots = q_2 = 2$ and $q_1 = 1$, so that $\Gamma(C)$ looks like Figure 20. Blow up the point of intersection corresponding

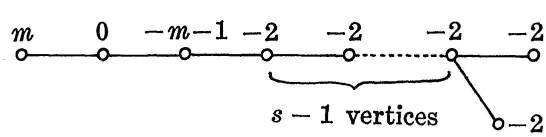

FIGURE 20

to the line between the vertices with weights m and 0, and contract the proper transform of the curve corresponding to the vertex with weight 0. Repeat this m times to get (S', C') with the graph $\Gamma(C')$ which looks like Figure 20 with $m = 0$. Contracting $s + 1$ curves with weights -1 and -2 on the horizontal line of $\Gamma(C')$, we get (S'', C'') with $\Gamma(C'')$ which looks like Figure 21. This contradicts Lemma 6. Therefore,

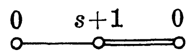

FIGURE 21

② if $r = 0$, we have $s = 1$ and $m = \beta_1 = q_1$, so that $\Gamma(C)$ is of type (3).

(Since $b_2(S) = 2$ in this case (recall $b_2(C) = b_2(S) + 1$), we see that $S = F_n$ by Lemma 4, and the existence of a holomorphic section with the self-intersection number $-m$ implies that $n = m$.)

Now, let us prove Proposition (P) by induction on b ($\geqq 3$). If $b = r + s + 2 = 3$, we have $r = 0$ by ①. Thus, $(S, \Gamma(C))$ is of type (3) in Table I (by ②). Next, assuming that Proposition (P) is true for $b \leqq b_0 - 1$, we consider the case $b = b_0$ ($\geqq 4$). Since $r = 0$ implies that $(S, \Gamma(C))$ is of type (3), we may assume $r \geqq 1$, $s \geqq 1$. We have $m + 1/\alpha_r = \beta_s$. Let $\alpha_r = l/q$, where q, l are coprime natural numbers such that $0 < q \leqq l$. Then,

$$\beta_s = (m + 1) - (l - q)/l , \qquad 0 \leqq (l - q)/l < 1 .$$

Hence, $q_s = m + 1$. $\Gamma(C)$ looks like Figure 22. Blow up the point of intersection corresponding to the line between the vertices with weights

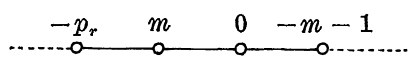

FIGURE 22

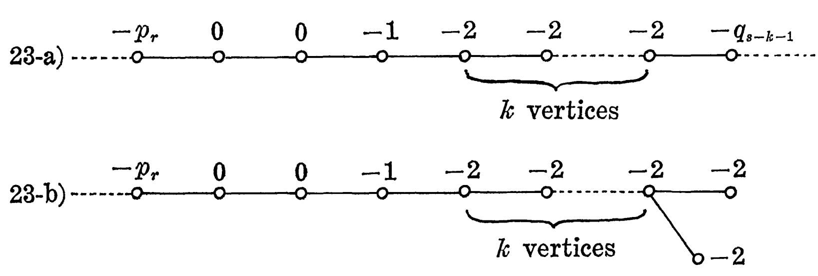

FIGURE 23

m and 0 in Figure 22, and contract the proper transform of the curve corresponding to the vertex of weight 0. Repeat this m times to get (S', C') with the graph $\Gamma(C')$ which looks like Figure 23 (a or b), where $k \geqq 0$ and, in case 23-a, either $q_{s-k-1} \geqq 3$ or $s = k + 1$ holds. The latter case 23-b does not occur, for if it did, contracting the $k + 2$ curves corresponding to the vertices with weights -1 and -2 on the horizontal line of Figure 23-b, we sould get a graph which looks like Figure 24-a,

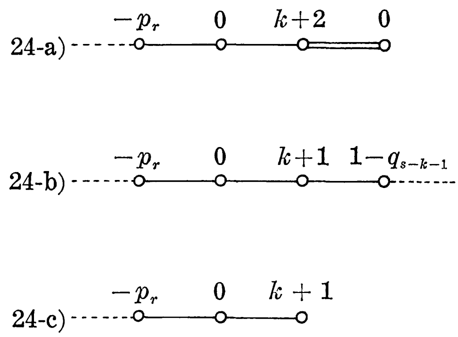

FIGURE 24

contradicting Lemma 6. Hence, $\Gamma(C')$ is of the form of Figure 23-a. Contracting the $k + 1$ curves corresponding to the vertices with weights -1 and -2 of Figure 23-a, we get (S'', C'') with $\Gamma(C'')$ which looks like Figure 24-b ($s - k \geqq 2$, $q_{s-k-1} \geqq 3$) or Figure 24-c ($s = k + 1$). Since the length of this graph $\Gamma(C'')$ is shorter than that of $\Gamma(C)$, the induction hypothesis asserts that $(S'', \Gamma(C''))$ is of one of the types (3), (4), (5) in Table I. Following the returning process: $(S'', C'') \to (S', C') \to (S, C)$, we see that $\Gamma(C)$ is of type (4) or (5). This completes the proof of Theorem 2. (Since the compactifications of types (4) and (5) are, as it is clear from the above proof, birational to that of type (3), we have shown at the same time that any compactification of $\boldsymbol{C} \times \boldsymbol{C}^*$ is a rational surface.)

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