Tôhoku Math. Journ.
32 (1980), 63–70.

FUBINI PRODUCTS OF C^*-ALGEBRAS

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(Received February 19, 1979)

1. Introduction. Let C and D be C^*-algebras and let $C \otimes D$ denote their minimal (or spatial) C^*-tensor product. For each $g \in C^*$ there is a unique bounded linear map R_g of $C \otimes D$ to D satisfying $R_g(c \otimes d) = \langle g, c\rangle d$. Similarly, for each $h \in D^*$ there is a unique bounded linear map L_h of $C \otimes D$ to C satisfying $L_h(c \otimes d) = \langle h, d\rangle c$. Let A and B be C^*-subalgebras of C and D, respectively. We define the Fubini product of A and B with respect to $C \otimes D$ to be

$$F(A, B, C \otimes D) = \{x \in C \otimes D\colon R_g(x) \in B,\ L_h(x) \in A \text{ for every } g \in C^*,\ h \in D^*\}$$

(see [10]). If C_1, C_2 and A are C^*-algebras such that $C_1 \supseteqq C_2 \supseteqq A$, and if D_1, D_2 and B are C^*-algebras such that $D_1 \supseteqq D_2 \supseteqq B$, then $F(A, B, C_1 \otimes D_1)$ contains $F(A, B, C_2 \otimes D_2)$. In this paper we show that there is the largest Fubini product of A and B, denoted by $A \otimes_F B$. We also consider a condition for a C^*-algebra to have property S [13]. Aided by [15], we give several Fubini products $A \otimes_F B$ strictly containing $A \otimes B$.

The author would like to thank Professor J. Tomiyama for his useful suggestions. He would also like to thank Professor S. Wassermann for sending him a copy of the preprint [15].

2. Some properties of Fubini products. In this section we study certain elementary properties of Fubini products. The following result is known [12, Proposition 4.1] and is easy to check.

LEMMA 1. *Let C and D be C^*-algebras with C^*-subalgebras A and B, respectively. Let $\bar{C}$ and $\bar{D}$ be the enveloping W^*-algebras of C and D. Under the canonical embedding of $C \otimes D$ into the W^*-tensor product $\bar{C} \mathbin{\bar{\otimes}} \bar{D}$, let $\bar{A} \mathbin{\bar{\otimes}} \bar{B}$ denote the weak closure of $A \otimes B$. Then $F(A, B, C \otimes D)$ is just $(C \otimes D) \cap (\bar{A} \mathbin{\bar{\otimes}} \bar{B})$ and is a C^*-subalgebra of $C \otimes D$.*

LEMMA 2. *Let A, C_1 and C_2 be C^*-algebras such that $C_1 \supseteqq A$ and $C_2 \supseteqq A$, and let B, D_1 and D_2 be C^*-algebras such that $D_1 \supseteqq B$ and $D_2 \supseteqq B$. Suppose that there are four contractive and completely positive maps*:

$$\phi_1\colon C_1 \to C_2,\quad \phi_2\colon C_2 \to C_1,\quad \phi_i(a) = a \qquad (i = 1, 2,\quad a \in A)\ ,$$
$$\psi_1\colon D_1 \to D_2,\ \psi_2\colon D_2 \to D_1,\ \psi_i(b) = b \qquad (i = 1, 2,\quad b \in B)\ .$$

*Then there is a *-isomorphism* Ψ of $F(A, B, C_1 \otimes D_1)$ *onto* $F(A, B, C_2 \otimes D_2)$ *such that* $\Psi(x) = x$ *for all* $x \in A \otimes B$.

PROOF. For convenience, if $i = 1, 2$, let $F_i = F(A, B, C_i \otimes D_i)$. By Lemma 1, each F_i is a C^*-subalgebra of $C_i \otimes D_i$. Let $x \in F_1$. If $g \in C_2^*$, we have $R_g(\phi_1 \otimes \psi_1(x)) = \psi_1(R_{(g\circ\phi_1)}(x)) \in B$. Similarly, if $h \in D_2^*$, we have $L_h(\phi_1 \otimes \psi_1(x)) \in A$. Let $y \in F_1$. If $g \in C_1^*$ and $h \in D_1^*$, then

$$\langle g \otimes h, (\phi_2 \otimes \psi_2) \circ (\phi_1 \otimes \psi_1)(y)\rangle = \langle g, (\phi_2 \circ \phi_1)(L_{(h\circ\psi_2\circ\psi_1)}(y))\rangle$$
$$= \langle g, L_{(h\circ\psi_2\circ\psi_1)}(y)\rangle = \langle h, (\psi_2\circ\psi_1)(R_g(y))\rangle = \langle h, R_g(y)\rangle = \langle g \otimes h, y\rangle .$$

Hence $(\phi_2 \otimes \psi_2) \circ (\phi_1 \otimes \psi_1)(y) = y$ for all $y \in F_1$. Similarly, we have $\phi_2 \otimes \psi_2(F_2) \subseteqq F_1$ and $(\phi_1 \otimes \psi_1) \circ (\phi_2 \otimes \psi_2)(y) = y$ for all $y \in F_2$.

Since $\phi_1 \otimes \psi_1$ and $\phi_2 \otimes \psi_2$ are contractive and completely positive, by [4, Lemma 3.9] we may assume that F_1 and F_2 are unital, and that both $\phi_1 \otimes \psi_1 | F_1$ and $\phi_2 \otimes \psi_2 | F_2$ are unital and completely positive. Let $\Psi = \phi_1 \otimes \psi_1 | F_1$. Then $\Psi = (\phi_2 \otimes \psi_2 | F_2)^{-1}$, so that Ψ is isometric. It follows from [3, Corollary 3.2] and [9, Theorem 7] that Ψ is a *-isomorphism (cf. the proof of [6, Theorem 3.1]). It is easy to see that $\Psi(x) = x$ for all $x \in A \otimes B$. This completes the proof.

A unital C^*-algebra A is said to be injective [5] if for any unital C^*-algebras B and C such that $C \supseteqq B$ with the common unit, any completely positive map of B into A extends to a completely positive map of C into A. Arveson proved in [2, Theorem 1.2.3] that for any Hilbert space H, the C^*-algebra $L(H)$ is injective.

LEMMA 3. *Let* B, C_1 *and* C_2 *be* C^**-algebras such that* $C_1 \supseteqq B$ *and* $C_2 \supseteqq B$. *Suppose that* C_2 *is injective. Then there is a contractive and completely positive map* Ψ *of* C_1 *into* C_2 *such that* $\Psi(b) = b$ *for all* $b \in B$.

PROOF. For $i = 1, 2$ let $\tilde{C}_i$ be the unital extension of C_i by the complex number field, let e_i be the unit of $\tilde{C}_i$, and let B_i be the C^*-subalgebra of $\tilde{C}_i$ generated by B and e_i. It follows from [4, Lemma 3.9] that the map $\psi_0 \colon b + ce_1 \to b + ce_2$ is a completely positive map of B_1 onto B_2. Define a completely positive map ψ_1 of B_1 into C_2 *by* $\psi_1(x) = e\psi_0(x)e$, where e is the unit of C_2. Since C_2 is injective, the map ψ_1 extends to a completely positive map ψ_2 of $\tilde{C}_1$ into C_2. Put $\Psi = \psi_2 | C_1$, which has the desired properties.

The following result is a generalization of [11], and is an immediate consequence of Lemmas 2 and 3.

THEOREM 4. *Let* A *and* B *be* C^**-algebras. Let* C_1 *and* C_2 *be injective* C^**-algebras such that* $C_1 \supseteqq A$ *and* $C_2 \supseteqq A$, *and let* D_1 *and* D_2 *be injective*

C^*-algebras such that $D_1 \supseteqq B$ and $D_2 \supseteqq B$. Then the following two statements hold.

(1) *There is a *-isomorphism Ψ_1 of $F(A, B, C_1 \otimes D_1)$ onto $F(A, B, C_2 \otimes D_2)$ such that $\Psi_1(x) = x$ for all $x \in A \otimes B$.*

(2) *There is a *-isomorphism Ψ_2 of $F(A, B, A \otimes D_1)$ onto $F(A, B, A \otimes D_2)$ such that $\Psi_2(x) = x$ for all $x \in A \otimes B$.*

DEFINITION. By $A \otimes_F B$ we denote any one of the *-isomorphic Fubini products of A and B of Theorem 4 (1). Then $A \otimes_F B$ is independent of the choice of injective C^*-algebras C and D such that $C \supseteqq A$ and $D \supseteqq B$, and is the largest of all Fubini products of A and B. In fact, if A_1 and B_1 are C^*-algebras such that $A_1 \supseteqq A$ and $B_1 \supseteqq B$, there are injective C^*-algebras C and D such that $C \supseteqq A_1$ and $D \supseteqq B_1$, hence $F(A, B, A_1 \otimes B_1) \subseteqq F(A, B, C \otimes D) = A \otimes_F B$. Similarly, any one of the *-isomorphic Fubini products of A and B of Theorem 4 (2) is independent of the choice of injective C^*-algebra D such that $D \supseteqq B$, and is the largest of Fubini products of A and B with respect to $A \otimes D$ with D taken over all C^*-algebras such that $D \supseteqq B$.

We now consider a condition for a C^*-algebra to have property S. A C^*-algebra A is said to have property S [13] if $F(A, B, A \otimes C) = A \otimes B$ for any C^*-algebras B and C such that $C \supseteqq B$.

LEMMA 5. *Let C and D be C^*-algebras with C^*-subalgebras A and B, respectively. Suppose that $F(A, B, C \otimes D) \supsetneqq A \otimes B$. Then there are separable C^*-subalgebras A_0, B_0, C_0 and D_0 of A, B, C and D, respectively, such that $C_0 \supseteqq A_0$, $D_0 \supseteqq B_0$,*

(1) *$F(A, B_0, C \otimes D_0) \supsetneqq A \otimes B_0$ and*

(2) *$F(A_0, B_0, C_0 \otimes D_0) \supsetneqq A_0 \otimes B_0$.*

PROOF. Let $z \in F(A, B, C \otimes D)$ with $z \notin A \otimes B$. Then there is a sequence (z_n) such that

$$z_n = \sum_{i=1}^{m_n} x_i^{(n)} \otimes y_i^{(n)} \quad \text{and} \quad \lim_n z_n = z ,$$

where each $x_i^{(n)} \in C$ and $y_i^{(n)} \in D$. Let A_0 be the C^*-subalgebra of A generated by $\{L_h(z) \colon h \in D^*\}$ and let C_0 be the C^*-subalgebra of C generated by $\{x_i^{(n)} \colon i = 1, \cdots, m_n, n = 1, 2, \cdots\}$. Since $L_h(z_n) \in C_0$ $(h \in D^*)$, we have $C_0 \supseteqq A_0$. Since C_0 is separable, so is A_0. Similarly, let B_0 be the C^*-subalgebra of B generated by $\{R_g(z) \colon g \in C^*\}$ and let D_0 be the C^*-subalgebra of D generated by $\{y_i^{(n)} \colon i = 1, \cdots, m_n, n = 1, 2, \cdots\}$. Then D_0 is a separable C^*-algebra containing B_0. It is easy to see that $z \in F(A_0, B_0, C_0 \otimes D_0) \subseteqq F(A, B_0, C \otimes D_0)$. If $z \in A \otimes B_0$, we have $z \in A \otimes B$, a contradiction to the assumption. Thus $F(A, B_0, C \otimes D_0) \supsetneqq A \otimes B_0$ and $F(A_0, B_0, C_0 \otimes D_0) \supsetneqq A_0 \otimes B_0$.

THEOREM 6. *Let A be C^*-algebra and let H be a separable infinite dimensional Hilbert space. Then the following two statements are equivalent.*

(1) *A has property* S.

(2) *$F(A, B, A \otimes L(H)) = A \otimes B$ for every separable C^*-subalgebra B of $L(H)$.*

PROOF. Since every separable C^*-algebra can be regarded as a C^*-subalgebra of $L(H)$, and $L(H)$ is injective [2, Theorem 1.2.3], this theorem is an immediate consequence of the second remark in the definition following Theorem 4 and Lemma 5 (1).

REMARK. Let C and D be C^*-algebras. If $A \otimes_F B = A \otimes B$ for any separable C^*-subalgebras A and B of C and D, respectively, we have $C \otimes_F D = C \otimes D$ by the first remark in the definition following Theorem 4 and Lemma 5 (2).

3. **Examples.** In this section we need certain results and notation from [15]. Let H be a separable infinite dimensional Hilbert space, let $H = \bigoplus_{n=1}^{\infty} H_n$ denote a decomposition of H into subspaces of dimension n and write $M = \bigoplus_{n=1}^{\infty} L(H_n) = \{(x_n): x_n \in L(H_n), \sup_n \|x_n\| < \infty\}$. If U is a free ultrafilter on the positive integers and if tr_n is the trace on $L(H_n)$ so normalized that the unit has trace 1 ($n = 1, 2, \cdots$), then $I_U = \{(x_n): \lim_U \mathrm{tr}_n(x_n^* x_n) = 0\}$ is a maximal two-sided ideal in M and $N = M/I_U$ is a II_1 factor. If $x \in N$ is represented by the sequence $(x_n) \in M$, then $\mathrm{Tr}(x) = \lim_U \mathrm{tr}_n(x_n)$. For each n we identify $L(H_n)$ with the algebra of $n \times n$ complex matrices, and let $\sim$ denote the transposition of a matrix. For $x = (x_n) \in M$ let $\tilde{x} = (\tilde{x}_n)$. Then $(I_U)^\sim = I_U$ and an antiautomorphism of N is defined by $(x + I_U)^\sim = \tilde{x} + I_U$.

If K denotes the completion of N with respect to the canonical trace norm, K is a Hilbert space. A self-adjoint unitary operator J on K is defined by $Jx = \tilde{x}$ $(x \in N)$. N acts on K by left multiplication: if $L_x \in L(K)$ is given by $L_x a = xa$ $(x, a \in N)$, then the map $x \to L_x$ is a normal $*$-isomorphism of N into $L(K)$, the standard representation of N. If we identify N with its image in $L(K)$, the commutant N' is just the set of right multiplications by elements of N [7, I, §5, Théorème 1]. If $x, a \in N$, then $JxJa = J(x\tilde{a}) = a\tilde{x}$. Thus $JNJ \subseteq N'$ and the map $x \to JxJ$ is a $*$-isomorphism of N onto N'.

Let Φ be the quotient map of M onto N. By [14, Lemma 2.4] there is a representation σ of $M \otimes M$ on K such that

$$\sigma(a \otimes b) = \Phi(a)J\Phi(b)J \qquad (a, b \in M) .$$

Since $\sigma(M \otimes I) = N$ and $\sigma(I \otimes M) = JNJ = N'$, σ is irreducible.

Since $M \otimes M \subseteqq L(H) \otimes L(H)$, there are, by [8, 2.10.2], a Hilbert space K_0 with $K \subseteqq K_0$ and an irreducible representation π of $L(H) \otimes L(H)$ on K_0 such that $\pi(x) | K = \sigma(x)$ for $x \in M \otimes M$. Commuting factor representations π_1 and π_2 of $L(H)$ on K_0 are defined by

$$\pi_1(x) = \pi(x \otimes I),\ \pi_2(x) = \pi(I \otimes x) \qquad (x \in L(H)) .$$

Then $\ker \pi_1 = \ker \pi_2 = LC(H)$ by [15, Lemma 3].

In [15, Section 4] Wassermann showed that there is an isomorphism of the free group on two generators into the unitary group of M. Let C denote the C^*-subalgebra of M generated by its image. Anderson [1] showed that there is a projection p in M such that $\mathrm{Tr}\,(\Phi(p)) \geqq 1/2$ and $px \in LC(H)$ if $x \in C \cap I_C$. Let $C^*(C, p)$ denote the C^*-subalgebra of M generated by C and p.

From now on, we use $H, M, \pi, \pi_1, \pi_2, K_0$ and $C^*(C, p)$ in the above situation.

LEMMA 7. *There exist no completely positive unital maps ρ_1 and ρ_2 of $L(K_0)$ to $\pi_2(C^*(C, p))'$ and $\pi_1(C^*(C, p))'$, respectively, such that*

$$\rho_1(axb) = a\rho_1(x)b \qquad (a, b \in \pi_1(C^*(C, p)), x \in L(K_0)) ,$$
$$\rho_2(axb) = a\rho_2(x)b \qquad (a, b \in \pi_2(C^*(C, p)), x \in L(K_0)) .$$

PROOF. It was shown in the proof of [15, Proposition 5] that such a ρ_1 cannot exist. It was also shown in the proof of [15, Theorem 8] that such a ρ_2 cannot exist.

LEMMA 8. *Let A and B be C^*-subalgebras of $L(H)$ both containing $C^*(C, p)$. Then*

(1) $F(A \cap LC(H), B, A \otimes B) \not\subseteqq \ker \pi$,

(2) $F(A, B \cap LC(H), A \otimes B) \not\subseteqq \ker \pi$.

PROOF. (1) Suppose that $F(A \cap LC(H), B, A \otimes B) \subseteqq \ker \pi$. As in the proof of [14, Proposition 2.5], the relation $\pi_1 \circ L_h = L_h \circ (\pi_1 \otimes I)$ $(h \in L(H)^*)$ shows that $\{x \in A \otimes B: \pi_1 \otimes I(x) = 0\} = F(A \cap LC(H), B, A \otimes B)$. Hence there is a representation $\bar{\pi}$ of $\pi_1(A) \otimes B$ such that $\bar{\pi}(a \otimes b) = a\pi_2(b)$ $(a \in \pi_1(A), b \in B)$. By [15, Lemma 1] there is a completely positive unital map ρ_1 of $L(K_0)$ to $\pi_2(B)'$ such that $\rho_1(axb) = a\rho_1(x)b$ $(a, b \in \pi_1(A), x \in L(K_0))$. Such a ρ_1 cannot exist by Lemma 7. Hence we obtain (1).

(2) This follows from an argument similar to (1).

THEOREM 9. *Let H be a separable infinite dimensional Hilbert space. Then $LC(H) \otimes_F LC(H)$ strictly contains $LC(H) \otimes LC(H)$.*

PROOF. Since $L(H)$ is injective, it is enough to show that $F(LC(H), LC(H), L(H) \otimes L(H)) \supsetneqq LC(H) \otimes LC(H)$. By Lemma 8 we have

$$\text{(1)} \qquad \begin{aligned} &F(LC(H), L(H), L(H) \otimes L(H)) \not\subseteq \ker \pi\ , \\ &F(L(H), LC(H), L(H) \otimes L(H)) \not\subseteq \ker \pi\ . \end{aligned}$$

Since $F(L(H), LC(H), L(H) \otimes L(H))$ is a closed two-sided ideal in $L(H) \otimes L(H)$ [10, Lemma 2.2], the restriction of π to $F(L(H), LC(H), L(H) \otimes L(H))$ is an irreducible representation. Let $\{u_\beta\}$ be an approximate identity for $F(L(H), LC(H), L(H) \otimes L(H))$. Then $\{\pi(u_\beta)\}$ converges strongly to the identity operator on K_0.

Suppose that $F(LC(H), LC(H), L(H) \otimes L(H)) \subseteq \ker \pi$. We note that $F(LC(H), LC(H), L(H) \otimes L(H)) = F(L(H), LC(H), L(H) \otimes L(H)) \cap F(LC(H), L(H), L(H) \otimes L(H))$. Since $F(LC(H), L(H), L(H) \otimes L(H))$ is a two-sided ideal [10, Lemma 2.2], it follows that if $x \in F(LC(H), L(H), L(H) \otimes L(H))$, we have $u_\beta x \in F(LC(H), LC(H), L(H) \otimes L(H))$, so that $\pi(x) = \lim_\beta \pi(u_\beta)\pi(x) = \lim_\beta \pi(u_\beta x) = 0$ (strongly). Hence we obtain $F(LC(H), L(H), L(H) \otimes L(H)) \subseteq \ker \pi$. This inclusion contradicts (1), and we have

$$\begin{aligned} F(LC(H), LC(H), L(H) \otimes L(H)) &\supsetneqq \ker(\pi \mid F(LC(H), LC(H)\ , \\ &\quad L(H) \otimes L(H))) \supseteq LC(H) \otimes LC(H)\ . \end{aligned}$$

This completes the proof.

THEOREM 10 (cf. [15, Theorem 8]). *Let K be an infinite dimensional Hilbert space and let A be a C^*-subalgebra of $L(K)$ such that $A \supseteq LC(K)$. Then $A \otimes_F LC(K)$ strictly contains $A \otimes LC(K)$.*

PROOF. As in the proof of Theorem 9, it is enough to show that $F(A, LC(K), L(K) \otimes L(K)) \supsetneqq A \otimes LC(K)$.

Suppose that

$$\text{(2)} \qquad F(A, LC(K), L(K) \otimes L(K)) = A \otimes LC(K)\ .$$

With H as in Theorem 9, we may assume that $H \subseteq K$. Then we have $LC(H) \subseteq LC(K) \subseteq A$, so that, by (2), $F(LC(H), LC(K), L(K) \otimes L(K)) \subseteq A \otimes LC(K)$. Thus $F(LC(H), LC(K), L(K) \otimes L(K)) = F(LC(H), LC(K), A \otimes LC(K))$. By [13, Theorem 22] we obtain $F(LC(H), LC(K), A \otimes LC(K)) = LC(H) \otimes LC(K)$, hence

$$\text{(3)} \quad F(LC(H), LC(H), L(K) \otimes L(K)) = F(LC(H), LC(H), LC(H) \otimes LC(K))\ .$$

Then a second application of [13, Theorem 22] shows that

(4) $\qquad F(LC(H), LC(H), LC(H) \otimes LC(K)) = LC(H) \otimes LC(H)$.

Since there is a projection of norm one from $L(K)$ onto $L(H)$, it follows from [10, Proposition 3.7] that

(5) $F(LC(H), LC(H), L(K) \otimes L(K)) = F(LC(H), LC(H), L(H) \otimes L(H))$.

Hence (3), (4) and (5) yield that $F(LC(H), LC(H), L(H) \otimes L(H)) = LC(H) \otimes LC(H)$. This contradicts Theorem 9, and we obtain the desired result.

Let C and D be C^*-algebras with C^*-subalgebras A and B, respectively. Tomiyama [10, Theorem 3.1] proved that if all the irreducible representations of A are finite dimensional of bounded dimension, then $F(A, B, C \otimes D) = A \otimes B$. However, if we remove the condition "of bounded dimension" from his theorem, we have the following situation.

EXAMPLE 11. *With H and M as in the beginning of this section, all the irreducible representations of $M \cap LC(H)$ are finite dimensional, and $(M \cap LC(H)) \otimes_F L(H)$ strictly contains $(M \cap LC(H)) \otimes L(H)$.*

PROOF. It is easy to see that $M \cap LC(H) = \{(x_n) \in M: \lim_n \|x_n\| = 0\}$. It follows from [8, 10.4.3 and 10.10.1] that all the irreducible representations of $M \cap LC(H)$ are finite dimensional. Since M is injective, we show that $F(M \cap LC(H), L(H), M \otimes L(H)) \supsetneqq (M \cap LC(H)) \otimes L(H)$. Applying Lemma 8 with $A = M$ and $B = L(H)$, we have

(6) $\qquad F(M \cap LC(H), L(H), M \otimes L(H)) \nsubseteq \ker \pi$.

Suppose that $F(M \cap LC(H), L(H), M \otimes L(H)) = (M \cap LC(H)) \otimes L(H)$. Then $F(M \cap LC(H), L(H), M \otimes L(H)) \subseteq \ker \pi$. This inclusion contradicts (6). It then follows that $F(M \cap LC(H), L(H), M \otimes L(H)) \supsetneqq (M \cap LC(H)) \otimes L(H)$. This completes the proof.

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