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NOTES ON THE CANCELLATION OF RIEMANNIAN MANIFOLDS

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Introduction. Let M, N and B be Riemannian manifolds. Then, we have a question "Is M isometric to N, if $M\times B$ is isometric to $N\times B$?" Uesu [1] proved that the answer is affirmative if M and N are complete and if B is compact locally symmetric. In the present note, we shall show that the answer is affirmative also if the last condition on B is replaced by one of the following (1) and (2):

(1) B is simply connected and complete.

(2) B is complete and B has the irreducible restricted holonomy group.

The last assertion is stated as Theorems A and B in the next section.

We assume, in the present note, that all Riemannian manifolds are connected and C^∞.

1. Proof of the theorems. First, we give some lemmas. Let $\Omega'=\{1, \cdots, r\}$ and $\Omega''=\{r+1, \cdots, n\}$. For a subset $\Omega\subset\Omega'\cup\Omega''$, we denote by $S(\Omega)$ the symmetric group of Ω. And, by S_n and S_r, we denote $S(\Omega'\cup\Omega'')$ and $S(\Omega')$, respectively. Let G be a subgroup of S_r and $\Omega'_0=\{i\in\Omega'|\tau(i)=i$ for all $\tau\in G\}$, $\Omega'_1=\{i\in\Omega'\,|\,\tau(i)\neq i$ for some $\tau\in G\}$.

LEMMA 1. *Let σ be an element of S_n.*

(i) *If $\omega\in S(\sigma(\Omega'_0\cup\Omega''))$, then $(\omega\sigma)\tau(\omega\sigma)^{-1}=\sigma\tau\sigma^{-1}$ for all $\tau\in G$.*

(ii) *If $\sigma G\sigma^{-1}\subset S_r$, then $\sigma(\Omega'_1)\subset\Omega'$.*

PROOF. (i) We note $\tau(\Omega'_1)=\Omega'_1$ for all $\tau\in G$. If $i\in\Omega'_1$, then $\omega\sigma(i)=\sigma(i)$, and hence $\omega\sigma\tau\sigma^{-1}\omega^{-1}(\omega\sigma(i))=\omega\sigma(\tau(i))=\sigma\tau(i)$, $\sigma\tau\sigma^{-1}(\omega\sigma(i))=\sigma\tau\sigma^{-1}(\sigma(i))=\sigma\tau(i)$. If $i\in\Omega'_0\cup\Omega''$, then $\omega\sigma\tau\sigma^{-1}\omega^{-1}(\omega\sigma(i))=\omega\sigma\tau(i)=\omega\sigma(i)$ and $(\sigma\tau\sigma^{-1})(\omega\sigma(i))=\omega\sigma(i)$, as $\omega\sigma(i)=\sigma(j)$ for some $j\in\Omega'_0\cup\Omega''$. (ii) Assume $i\in\Omega'_1$ and $\sigma(i)\in\Omega''$. Since $\sigma G\sigma^{-1}\subset S_r$, we have $\sigma\tau\sigma^{-1}(\sigma(i))=\sigma(i)$ and hence $\tau(i)=i$ for all $\tau\in G$, a contradiction. q.e.d.

Throughout the present note, $I(M)$ denotes the group of all isometries of a simply connected and complete Riemannian manifold M.

Let $M_1=M_2=\cdots=M_n$ be a simply connected and complete Riemannian manifold whose homogeneous holonomy group is irreducible. Let M be the direct product Riemannian manifold $M_1\times M_2\times\cdots\times M_n$. For

each $\sigma \in S_n$, $\lambda(\sigma): M \to M$ is defined by

$$\lambda(\sigma)(x_1, \cdots, x_n) = (x_{\sigma^{-1}(1)}, \cdots, x_{\sigma^{-1}(n)}), \quad x_i \in M_i .$$

Then, $\lambda(\sigma) \in I(M)$ and $\lambda: S_n \to I(M)$ is an isomorphism. Briefly we denote $\lambda(\sigma)$ by σ. Then S_n is a subgroup of $I(M)$.

LEMMA 2. (i) *$I(M)$ is generated by S_n and $I(M_1) \times \cdots \times I(M_n)$.* (ii) *If $\sigma \in S_n$ and $(f_1, \cdots, f_n) \in I(M_1) \times \cdots \times I(M_n)$, then*

$$\sigma(f_1, \cdots, f_n)\sigma^{-1} = (f_{\sigma^{-1}(1)}, \cdots, f_{\sigma^{-1}(n)}),$$

where $(f_{\sigma^{-1}(1)}, \cdots, f_{\sigma^{-1}(n)}) \in I(M_1) \times \cdots \times I(M_n)$. In particular, $I(M_1) \times \cdots \times I(M_n)$ is a normal subgroup of $I(M)$.

PROOF. (i) is easily seen by the uniqueness of de Rham's decomposition (cf. Uesu [1] and Wolf [2]). (ii) $\sigma(f_1, \cdots, f_n)\sigma^{-1}(x_1, \cdots, x_n) = \sigma(f_1, \cdots, f_n)(x_{\sigma(1)}, \cdots, x_{\sigma(n)}) = \sigma(f_1(x_{\sigma(1)}), \cdots, f_n(x_{\sigma(n)})) = (f_{\sigma^{-1}(1)}(x_1), \cdots, f_{\sigma^{-1}(n)}(x_n)) = (f_{\sigma^{-1}(1)}, \cdots, f_{\sigma^{-1}(n)})(x_1, \cdots, x_n)$. q.e.d.

By the above lemma, S_n is isomorphic to the quotient group $I(M)/I(M_1) \times \cdots \times I(M_n)$. Let μ be the natural projection of $I(M)$ onto $I(M)/I(M_1) \times \cdots \times I(M_n)$. Then, the image $\mu(\Gamma)$ of a subgroup Γ of $I(M)$ is considered as a subgroup of S_n.

Let us decompose M in Lemma 2 into M' and M'', where $M' = M_1 \times \cdots \times M_r$ and $M'' = M_{r+1} \times \cdots \times M_n$. Then $M = M' \times M''$.

LEMMA 3. *Let Γ be a subgroup of $I(M')$ and $f \in I(M)$. If $f\Gamma f^{-1} \subset I(M')$, then there exists $f' \in I(M')$ satisfying $f'hf'^{-1} = fhf^{-1}$ for all $h \in \Gamma$.*

PROOF. Let G be the subgroup $\mu(\Gamma)$ which is a subgroup of S_r. Then, we may apply Lemma 1 with the other notations used. f is written as $f = \sigma(f_1, \cdots, f_n)$ by Lemma 2, where $\sigma \in S_n$ and $(f_1, \cdots, f_n) \in I(M_1) \times \cdots \times I(M_n)$. Let h be an element of Γ. Then h is written as $h = \tau(h_1, \cdots, h_r, h_{r+1}, \cdots, h_n)$, where $\tau \in S_r$, $(h_1, \cdots, h_r, h_{r+1}, \cdots, h_n) \in I(M_1) \times \cdots \times I(M_r) \times I(M_{r+1}) \times \cdots \times I(M_n)$ and $h_{r+1} = \cdots = h_n = 1$. Then we have

$$(*) \qquad fhf^{-1} = \sigma\tau\sigma^{-1}(f_{\tau\sigma^{-1}(1)}h_{\sigma^{-1}(1)}f^{-1}_{\sigma^{-1}(1)}, \cdots, f_{\tau\sigma^{-1}(n)}h_{\sigma^{-1}(n)}f^{-1}_{\sigma^{-1}(n)}),$$

where $(f_{\tau\sigma^{-1}(1)}h_{\sigma^{-1}(1)}f^{-1}_{\sigma^{-1}(1)}, \cdots, f_{\tau\sigma^{-1}(n)}h_{\sigma^{-1}(n)}f^{-1}_{\sigma^{-1}(n)}) \in I(M_1) \times \cdots \times I(M_n)$. Since $fhf^{-1} \in I(M')$, we have $\sigma\tau\sigma^{-1} \in S_r$ and $f_{\tau\sigma^{-1}(i)}h_{\sigma^{-1}(i)}f^{-1}_{\sigma^{-1}(i)} = 1$ if $i \in \Omega''$. On the other hand, by (ii) of Lemma 1, if $i \in \Omega''$, then $\sigma^{-1}(i) \in \Omega'_0 \cup \Omega''$. Hence $\tau\sigma^{-1}(i) = \sigma^{-1}(i)$ and $h_{\sigma^{-1}(i)} = 1$. Thus, if $j \in \sigma^{-1}(\Omega'') \cup \Omega''$, then $\tau(j) = j$ and $h_j = 1$. We may assume for brevity that $\sigma^{-1}(\Omega'') \cap \Omega' = \{1, \cdots, s\}$, $s \leqq r$. Then $\Gamma \subset I(M_{s+1} \times \cdots \times M_r)$.

Now, we define $\omega \in S(\{\sigma(j_1), \cdots, \sigma(j_s), \sigma(1), \cdots, \sigma(s)\})$ by $\omega(\sigma(j_k)) = \sigma(k)$ and $\omega(\sigma(k)) = \sigma(j_k)$, $k = 1, \cdots, s$, where $\sigma^{-1}(\Omega') \cap \Omega'' = \{j_1, \cdots, j_s\}$, $r + 1 \leqq j_1 < \cdots < j_s \leqq n$. And we define $f' \in I(M')$ by $f' = \omega\sigma(1, \cdots, 1, f_{s+1}, \cdots, f_r, 1, \cdots, 1)$. Then, f' is the desired one. Indeed, as $\Gamma \subset I(M_{s+1} \times \cdots \times M_r)$, $\sigma(1, \cdots, 1, f_{s+1}, \cdots, f_r, 1, \cdots, 1)h(1, \cdots, 1, f_{s+1}, \cdots, f_r, 1, \cdots, 1)^{-1}\sigma^{-1} = fhf^{-1}$ for any $h \in \Gamma$. For any $h = \tau(h_1, \cdots, h_n) \in \Gamma$, $\sigma\tau\sigma^{-1} \in S(\{\sigma(s + 1), \cdots, \sigma(r)\})$, as $\tau \in S(\{s + 1, \cdots, r\})$. On the other hand, $f_{\tau\sigma^{-1}(j)}h_{\sigma^{-1}(j)}f^{-1}_{\sigma^{-1}(j)} = 1$ for any $j \in \{\sigma(1), \cdots, \sigma(s), \sigma(r + 1), \cdots, \sigma(n)\}$. Then, by (*), $\sigma(1, \cdots, 1, f_{s+1}, \cdots, f_r, 1, \cdots, 1)\Gamma(1, \cdots, 1, f_{s+1}, \cdots, f_r, 1, \cdots, 1)^{-1}\sigma^{-1} \subset I(M_{\sigma(s+1)} \times \cdots \times M_{\sigma(r)})$. But, as $\omega \in S(\{\sigma(j_1), \cdots, \sigma(j_s), \sigma(1), \cdots, \sigma(s)\})$, we have $f'hf'^{-1} = fhf^{-1}$.

Let E^n be an n-dimensional Euclidean space. Let $E(n) = I(E^n)$. Then $E(n)$ is the semi-direct product group $O(n) \dotplus \boldsymbol{R}^n$, where $O(n)$ is the orthogonal group of the n-dimensional Euclidean vector space $\boldsymbol{R}^n$ and, if (A, a), $(B, b) \in E(n)$, the $(A, a)(B, b) = (AB, Ab + a)$.

LEMMA 4. *Let G be a subgroup of $O(n)$, $A \in O(n)$ and $\bar{G} = AGA^{-1}$. Let $V = \{v \in \boldsymbol{R}^n \mid Xv = v$ for all $X \in G\}$ and $W = \{w \in \boldsymbol{R}^n \mid Yw = w$ for all $Y \in \bar{G}\}$. Then $A(V) = W$ and hence $A(V^\perp) = W^\perp$, where $V^\perp$ and $W^\perp$ are orthogonal complements in $\boldsymbol{R}^n$ of V and W, respectively.*

Let us consider E^n as the direct product Riemannian manifold $E^r \times E^{n-r}$ of the Euclidean spaces E^r and E^{n-r}.

LEMMA 5. *Let Γ be a subgroup of $E(r) = I(E^r)$ and $f \in E(n)$. If $f\Gamma f^{-1} \subset E(r)$, then there exists $f' \in E(r)$ satisfying $f'hf'^{-1} = fhf^{-1}$ for all $h \in \Gamma$.*

PROOF. Let $\bar{\Gamma} = f\Gamma f^{-1}$, $V_0 = \{v \in \boldsymbol{R}^r \mid \mu(h)v = v$ for all $h \in \Gamma\}$ and $W_0 = \{w \in \boldsymbol{R}^r \mid \mu(\bar{h})w = w$ for all $\bar{h} \in \bar{\Gamma}\}$, where μ is the projection $E(n) \to O(n)$. Let $V = V_0 \oplus \boldsymbol{R}^{n-r}$ and $W = W_0 \oplus \boldsymbol{R}^{n-r}$. Then $\boldsymbol{R}^r = V^\perp \oplus V_0 = W^\perp \oplus W_0$ and f is considered as a mapping f: $V^\perp \oplus V \to W^\perp \oplus W$, where $V^\perp$ and $W^\perp$ are orthogonal complements in $\boldsymbol{R}^n$ of V and W, respectively. Then, by Lemma 4, $\mu(f)(V^\perp) = W^\perp$ and $\mu(f)(V) = W$. Let $h = (X, x) \in \Gamma$ and $f = (A, a)$. Then $X|_V = 1$ and $x \in \boldsymbol{R}^r$. On the other hand, we have

$$(**) \quad fhf^{-1} = (A, a)(X, x)(A^{-1}, -A^{-1}a) = (AXA^{-1}, -AXA^{-1}a + Ax + a) .$$

Since $fhf^{-1} \in E(r)$, we have $AXA^{-1} \in O(r)$, $-AXA^{-1}a + Ax + a \in \boldsymbol{R}^r$ and $AXA^{-1}|_W = 1$. Here, a is written as $a = a' + a''$, where $a' \in W^\perp$ and $a'' \in W$. Then $-AXA^{-1}a + a = -AXA^{-1}a' + a' \in W^\perp \subset \boldsymbol{R}^r$. Thus $Ax \in \boldsymbol{R}^r$, as $-AXA^{-1}a + Ax + a \in \boldsymbol{R}^r$.

Now, let $U = \{v \in V_0 \mid Av \in \boldsymbol{R}^r\}$. Then $V_0 = U \oplus U^\perp$, where $U^\perp$ is the orthogonal complement of U in V_0. Let A' be an element of $O(r)$

satisfying $A'|_{V^\perp\otimes U} = A|_{V^\perp\otimes U}$, $A'|_{\boldsymbol{R}^{n-r}} = 1$ and $A'(U^\perp) = A(U)^\perp$, where $A(U)^\perp$ is the orthogonal complement of $A(U)$ in W_0. Then $f' = (A', a') \in E(r)$ is the desired one. In fact, let $h = (X, x) \in \Gamma$. If $v \in V^\perp$, then $AXA^{-1}(Av) = AX(v) = A'XA'^{-1}(Av)$ as $Xv \in V^\perp$. If $v \in U$, then $AXA^{-1}(Av) = Av = A'XA'^{-1}(Av)$ as $Xv = v$. If $v \in \boldsymbol{R}^{n-r}$, then $AXA^{-1}(Av) = Av = A'XA'^{-1}(Av)$ as $Av \in W$, $A'(V_0) = W_0$ and $Xv = v$. If $v \in U^\perp$, then $AXA^{-1}(Av) = Av = A'XA'^{-1}(Av)$ as $Xv = v$ and $Av \in \boldsymbol{R}^{n-r}$. Then we have $AXA^{-1} = A'XA'^{-1}$. Moreover, $Ax = A'x$ as $x \in V^\perp \oplus U$. Then $-AXA^{-1}a + Ax + a = -A'XA'^{-1}a' + A'x + a'$. Thus, by $(**)$, we have $f'hf'^{-1} = fhf^{-1}$. q.e.d.

LEMMA 6. *Let M, N and B be complete Riemannian manifolds. If $M\times B$ is isometric to $N\times B$, then $\tilde{M}$ is isometric to $\tilde{N}$, where $\tilde{M}$ and $\tilde{N}$ are universal Riemannian covering manifolds of M and N, respectively.*

PROOF. Let $p: \tilde{M} \to M$, $p': \tilde{N} \to N$ and $q: \tilde{B} \to B$ be the universal Riemannian coverings. And let $\phi: M\times B \to N\times B$ be an isometry. Then, the covering $\phi\circ(p, q): \tilde{M}\times\tilde{B} \to N\times B$ has a lift $\tilde{\phi}: \tilde{M}\times\tilde{B} \to \tilde{N}\times\tilde{B}$, since $\tilde{M}\times\tilde{B}$ and $\tilde{N}\times\tilde{B}$ are simply connected. Then $\tilde{\phi}$ is a covering and a local isometry as $(p', q)\circ\tilde{\phi} = \phi\circ(p, q)$. Hence, $\tilde{\phi}$ is an isometry. Thus $\tilde{M}$ is isometric to $\tilde{N}$ by de Rham's decomposition theorem. q.e.d.

LEMMA 7. *Let $\tilde{M}$ be a simply connected and complete Riemannian manifold. Let Γ and $\bar{\Gamma}$ be subgroups of $I(\tilde{M})$ acting freely and properly discontinuously on $\tilde{M}$. Then the quotient $\tilde{M}/\Gamma$ is isometric to the quotient $\tilde{M}/\bar{\Gamma}$ if and only if there exists an element $f \in I(\tilde{M})$ satisfying $f\Gamma f^{-1} = \bar{\Gamma}$.*

PROOF. See Wolf [2].

REMARK. Let $\tilde{M}$ and $\tilde{N}$ be simply connected and complete Riemannian manifolds. Let $\tilde{\phi}: \tilde{M} \to \tilde{N}$ be an isometry. Let Γ and $\bar{\Gamma}$ be subgroups of $I(\tilde{M})$ acting freely and properly discontinuosly on $\tilde{M}$. Then, $\Delta = \tilde{\phi}\Gamma\tilde{\phi}^{-1}$ and $\bar{\Delta} = \tilde{\phi}\bar{\Gamma}\tilde{\phi}^{-1}$ are subgroups of $I(\tilde{N})$ acting freely and properly discontinuously on $\tilde{N}$. And $\tilde{\phi}$ induces natural isometries $\phi: \tilde{M}/\Gamma \to \tilde{N}/\Delta$ and $\bar{\phi}: \tilde{M}/\bar{\Gamma} \to \tilde{N}/\bar{\Delta}$. If there exists $f \in I(\tilde{M})$ satisfying $\bar{\Gamma} = f\Gamma f^{-1}$, then $\bar{\Delta} = (\tilde{\phi}f\tilde{\phi}^{-1})\Delta(\tilde{\phi}f\tilde{\phi}^{-1})^{-1}$. Conversely, if there exists $g \in I(\tilde{N})$ satisfying $\bar{\Delta} = g\Delta g^{-1}$, then $\bar{\Gamma} = (\tilde{\phi}^{-1}g\tilde{\phi})\Gamma(\tilde{\phi}^{-1}g\tilde{\phi})^{-1}$. By Lemma 7, $\tilde{M}/\Gamma$ is isometric to $\tilde{M}/\bar{\Gamma}$ if and only if $\tilde{N}/\Delta$ is isometric to $\tilde{N}/\bar{\Delta}$.

THEOREM A. *Let M and N be complete Riemannian manifolds. Let $\tilde{B}$ be a simply connected and complete Riemannian manifolds. If $M\times\tilde{B}$ is isometric to $N\times\tilde{B}$, then M is isometric to N.*

PROOF. By Lemma 6, we may assume that M and N are isometric

to $\tilde{M}/\Gamma$ and $\tilde{M}/\bar{\Gamma}$, respectively, where Γ and $\bar{\Gamma}$ are subgroups of $I(\tilde{M})$ acting freely and properly discontinuously on $\tilde{M}$. Since $\tilde{M}/\Gamma\times\tilde{B}$ is isometric to $\tilde{M}/\bar{\Gamma}\times\tilde{B}$,, there exists $f\in I(\tilde{M}\times\tilde{B})$ satisfying $f\Gamma f^{-1}=\bar{\Gamma}$ by Lemma 7. It is sufficient to prove that there exists $f'\in I(\tilde{M})$ satisfying $f'hf'^{-1}=fhf^{-1}$ for all $h\in\Gamma$.

Now, by de Rham's decomposition theorem, we may assume that $\tilde{M}$ and $\tilde{B}$ are isometric to the direct product Riemannian manifolds $N_0\times N_1\times\cdots\times N_m\times N^*$ and $B_0\times B_1\times\cdots\times B_m\times B^*$, respectively, which have the following properties (1)$\sim$(4):

(1) $N_0, \cdots, N_m, N^*, B_0, \cdots, B_m$ and B^* are all simply connected and complete.

(2) $N_0\times B_0$ is a Euclidean space.

(3) For each $i\in\{1, \cdots, m\}$, $N_i\times B_i$ is a product of some Riemannian manifolds which are all isometric to one simply connected and complete Riemannian manifold M_i whose homogeneous holonomy group is irreducible. And if $i\neq j$, then M_i is not isometric to M_j.

(4) Any component of de Rham's decomposition of $N^*\times B^*$ has the irreducible homogeneous holonomy group. And any component of N^* is not isometric to any of B^*.

By the above remark, we may suppose $\tilde{M}=N_0\times N_1\times\cdots\times N_m\times N^*$ and $\tilde{B}=B_0\times B_1\times\cdots\times B_m\times B^*$. Moreover, we have a natural isometry ϕ: $\tilde{M}\times\tilde{B}\to P=(N_0\times B_0)\times\cdots\times(N_m\times B_m)\times N^*\times B^*$. By the uniqueness of de Rham's decomposition, we have $I(P)=I(N_0\times B_0)\times\cdots\times I(N_m\times B_m)\times I(N^*)\times I(B^*)$, (cf. Uesu [1]). Since Γ and $\bar{\Gamma}$ are contained in $I(\tilde{M})$, $\phi\Gamma\phi^{-1}$ and $\phi\bar{\Gamma}\phi^{-1}$ are contained in $I(N_0)\times\cdots\times I(N_m)\times I(N^*)\times\{1\}$, where $I(N_i)$ is interpreted as $I(N_i)\subset I(N_i\times B_i)$ for each $i\in\{0, 1, \cdots, m\}$. Again, by the remark, we may consider $\phi\Gamma\phi^{-1}$ and $\phi\bar{\Gamma}\phi^{-1}$ as Γ and $\bar{\Gamma}$, respectively. Then, it is sufficient to prove the following: Let Γ and $\bar{\Gamma}$ be subgroups of $I(N_0)\times\cdots\times I(N_m)\times I(N^*)\times\{1\}$. If there exists $f\in I(P)$ satisfying $f\Gamma f^{-1}=\bar{\Gamma}$, then there exists $f'\in I(N_0)\times\cdots\times I(N_m)\times I(N^*)\times\{1\}$ satisfying $f'hf'^{-1}=fhf^{-1}$ for all $h\in\Gamma$.

Indeed, f is written as $f=(g_0, g_1, \cdots, g_m, g^*, g^{**})$, where $g_i\in I(N_i\times B_i)$, $g^*\in I(N^*)$ and $g^{**}\in I(B^*)$. h is written as $h=(k_0, k_1, \cdots, k_m, k^*, 1)$, where $k_i\in I(N_i)$ and $k^*\in I(N^*)$. Then $fhf^{-1}=(g_0k_0g_0^{-1}, g_1k_1g_1^{-1}, \cdots, g_mk_mg_m^{-1}, g^*k^*g^{*-1}, 1)$. Now, the assertion is clear by Lemmas 3 and 5. q.e.d.

THEOREM B. *Let B be a complete Riemannian manifold whose restricted homogeneous holonomy group is irreducible. And let M and N be complete Riemannian manifolds. If $M\times B$ is isometric to $N\times B$, then M is isometric to N.*

PROOF. Let $\tilde{B}$ and $\tilde{M}$ be universal Riemannian covering manifolds of B and M, respectively. Then, by Lemma 6, M, N and B are isometric to the quotients $\tilde{M}/\Gamma$, $\tilde{M}/\bar{\Gamma}$ and $\tilde{B}/\Delta$, respectively. Then, by Lemma 7, it is sufficient to prove: If there exists $f \in I(\tilde{M}\times\tilde{B})$ satisfying $f(\Gamma\times\Delta)f^{-1} = \bar{\Gamma}\times\Delta$, then there exists $f' \in I(\tilde{M})$ satisfying $f'\Gamma f'^{-1} = \bar{\Gamma}$.

Let $M_1\times\cdots\times M_{n-1}\times M_n$ be de Rham's decomposition of $\tilde{M}\times\tilde{B}$, where $\tilde{M} = M_1\times\cdots\times M_{n-1}$ and $\tilde{B} = M_n$. Then, by the uniqueness of de Rham's decomposition, $I(\tilde{M}\times\tilde{B})$ is generated by $I(M_1), \cdots, I(M_n)$ and by all permutations of M_i's which are isometric to each other, where we identify M_i with M_j by an isometry if M_i is isometric to M_j (cf. Uesu [1]). Moreover, we have a statement similar to (ii) of Lemma 2. Then f is written as $f = \sigma(f_1, \cdots, f_{n-1}, f_n)$, where $\sigma \in S_n$ and $(f_1, \cdots, f_{n-1}, f_n) \in I(M_1)\times\cdots\times I(M_{n-1})\times I(M_n)$. Let $r = \sigma(n)$ and $s = \sigma^{-1}(n)$. Suppose $r \leqq n-1$ and hence $s \leqq n-1$. Then M_r, M_s and M_n are isometric to each other. We shall prove

$$\bar{\Gamma} = g_r\Delta_r g_r^{-1}\times(f\Gamma f^{-1}\cap\bar{\Gamma}) \qquad \text{(the direct product group)},$$

where $g_r = (1, \cdots, f_n, 1, \cdots, 1, 1) \in I(M_r)$, $f\Gamma f^{-1}\cap\bar{\Gamma}\subset I(M_1\times\cdots\times M_{r-1}\times M_{r+1}\times\cdots\times M_{n-1})$ and Δ_r is the group Δ considered as a subgroup of $I(M_r)$. Let $h = \tau(h_1, \cdots, h_{n-1}, h_n) \in \Gamma\times\Delta$, where $\tau \in S_{n-1}$ and $(h_1, \cdots, h_{n-1}, h_n) \in I(M_1)\times\cdots\times I(M_{n-1})\times I(M_n)$. Then

$$fhf^{-1} = \sigma\tau\sigma^{-1}(f_{\tau\sigma^{-1}(1)}h_{\sigma^{-1}(1)}f_{\sigma^{-1}(1)}^{-1}, \cdots, f_{\tau\sigma^{-1}(n)}h_{\sigma^{-1}(n)}f_{\sigma^{-1}(n)}^{-1}),$$

where $(f_{\tau\sigma^{-1}(1)}h_{\sigma^{-1}(1)}f_{\sigma^{-1}(1)}^{-1}, \cdots, f_{\tau\sigma^{-1}(n)}h_{\sigma^{-1}(n)}f_{\sigma^{-1}(n)}^{-1}) \in I(M_1)\times\cdots\times I(M_n)$. Since $\sigma\tau\sigma^{-1}(r) = r$, we have $\bar{\Gamma}\subset I(M_r)\times I(M_1\times\cdots\times M_{r-1}\times M_{r+1}\times\cdots\times M_{n-1})$. Moreover $g_r\Delta_r g_r^{-1} = f\Delta f^{-1}\subset\bar{\Gamma}\cap I(M_r)$. Next, let $\bar{h} \in \bar{\Gamma}$. Then $\bar{h}$ is written as $\bar{h} = fhf^{-1}$, where $h \in \Gamma\times\Delta$. But h is written as $h = h'h''$, where $h' \in \Gamma$, $h'' \in \Delta$. Then $fhf^{-1} = fh'f^{-1}fh''f^{-1}$. Since $fh''f^{-1} \in I(M_r)\cap\bar{\Gamma}$, we have $fh'f^{-1} \in \bar{\Gamma}$. Hence $fh'f^{-1} \in f\Gamma f^{-1}\cap\bar{\Gamma}$. By the above argument, it is evident that $fh'f^{-1} \in I(M_1\times\cdots\times M_{r-1}\times M_{r+1}\times\cdots\times M_{n-1})$.

Now, let ω be the transposition $(r, n) \in I(M_r\times M_n)$ and $f' = g_r\omega\sigma(f_1, \cdots f_{n-1}, 1) \in I(\tilde{M})$. Then f' is the desired one. In fact, let $h = \tau(h_1, \cdots, h_{n-1}, 1) \in \Gamma$. As $\tau \in S_{n-1}$, we have $fhf^{-1} = \sigma(f_1, \cdots, f_{n-1}, 1)h(f_1, \cdots, f_{n-1}, 1)^{-1}\sigma^{-1}$. On the other hand, as $fhf^{-1} \in \bar{\Gamma}\times\Delta$, we have $\sigma\tau\sigma^{-1}(n) = n$, that is, $\tau(s) = s$ and hence fhf^{-1} is written as $fhf^{-1} = (\bar{h}', f_s h_s f_s^{-1})$, where $(\bar{h}', 1) \in f\Gamma f^{-1}\cap\bar{\Gamma}\subset I(M_1\times\cdots\times M_{r-1}\times M_{r+1}\times\cdots\times M_{n-1})$ and $(1, \cdots, 1, f_s h_s f_s^{-1}) \in \Delta\subset I(M_n)$. Then we have $\omega\sigma(f_1, \cdots, f_{n-1}, 1)h(f_1, \cdots, f_{n-1}, 1)^{-1}\sigma^{-1}\omega^{-1} = (1, \cdots, 1, f_s h_s f_s^{-1}, 1, \cdots, 1, 1)(\bar{h}', 1) \in I(M_r)\times(f\Gamma f^{-1}\cap\bar{\Gamma})$. Hence $f'hf'^{-1} = g_r(1, \cdots, 1, f_s h_s f_s^{-1}, 1, \cdots, 1, 1)g_r^{-1}(\bar{h}', 1) \in (g_r\Delta_r g_r^{-1})\times(f\Gamma f^{-1}\cap\bar{\Gamma}) = \bar{\Gamma}$. Thus, $f'\Gamma f'^{-1}\subset\bar{\Gamma}$. By the above argument, it is evident that $f'\Gamma f'^{-1} = \bar{\Gamma}$. q.e.d.

REFERENCES

[1] K. UESU, On cancellation of compact Riemannian manifolds, to appear.
[2] J. A. WOLF, Spaces of Constant Curvature, McGraw-Hill, New York, 1967.

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