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ALMOST PERIODIC SOLUTIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE RETARDATION, II

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

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In [8], we have discussed the existence theorems for almost periodic solutions of functional differential equations with infinite retardation by introducing new concepts of stabilities. Furthermore, the author [9] has considered linear almost periodic systems with bounded solutions which are uniformly stable and discussed the existence of almost periodic solutions. Recently, Sawano [10] has considered a linear almost periodic system with a bounded solution which is uniformly asymptotically stable and discussed the existence of a unique almost periodic solution by utilizing the properties of a Liapunov functional.

For functional differential equations with finite delay, Halanay [2], Hale [4] and Yoshizawa [11] have discussed the existence of a unique almost periodic solution of a linear perturbed system whose perturbed term satisfies a Lipschitz condition, by assuming uniformly asymptotic stability of the null solution of a unperturbed system. In studying these book and papers, it seems meaningful to consider the following problem: Can we extend existence theorems to the case where unperturbed systems are not necessarily linear and perturbed terms do not necessarily satisfy a Lipschitz condition?

In this paper, we shall consider this problem for functional differential equations with infinite retardation and present a partial result.

First, we shall give the space B discussed by Hale [5] (also, refer to [6, 9, 10]). Let $|x|$ be any norm of x in R^n. Let B be a real linear vector space of functions mapping $(-\infty, 0]$ into R^n with a semi-norm $|\cdot|_B$. For any elements ϕ and ψ in B, $\phi = \psi$ means $\phi(t) = \psi(t)$ for all $t \in (-\infty, 0]$. For a $\beta \geqq 0$ and a $\phi \in B$, let ϕ^β denote the restriction of ϕ to the interval $(-\infty, -\beta]$. We shall denote by B^β the space of such functions ϕ^β. For any $\eta \in B^\beta$, we define the semi-norm $|\cdot|_\beta$ by

$$|\eta|_\beta = \inf_{\psi \in B} \{|\psi|_B : \psi^\beta = \eta\} .$$

If x is a function defined on $(-\infty, a)$, then for each t in $(-\infty, a)$ we define the function x_t by the relation $x_t(s) = x(t+s)$, $-\infty < s \leqq 0$. For a number $a > 0$, we denote by A^a the class of functions x mapping $(-\infty, a)$ into R^n such that x is a continuous function on $[0, a)$ and $x_0 \in B$. The space B is assumed to have the following properties:

(I) If x is in A^a, then x_t is in B for all t in $[0, a)$ and x_t is a continuous function of t, where $0 < a \leqq \infty$.

(II) There is a $K > 0$ such that $|\phi|_B \leqq K(\sup_{-\beta \leqq \theta \leqq 0} |\phi(\theta)| + |\phi^\beta|_\beta)$ for any $\phi \in B$ and any β, $\beta \geqq 0$.

(III) If a sequence $\{\phi^k\}$, $\phi^k \in B$, is uniformly bounded on $(-\infty, 0]$ with respect to $|\cdot|$ and converges to ϕ uniformly on any compact subset of $(-\infty, 0]$, then $\phi \in B$ and $|\phi^k - \phi|_B \to 0$ as $k \to \infty$.

(IV) There is a positive continuous function $M(\beta)$, $M(\beta) \to 0$ as $\beta \to \infty$, such that $|\tau^\beta \phi|_\beta \leqq M(\beta)|\phi|_B$ for any $\phi \in B$ and $\beta \geqq 0$, where τ^β is a linear operator from B into B^β defined by $\tau^\beta \phi(\theta) = \phi(\beta + \theta)$, $\theta \in (-\infty, -\beta]$.

REMARK 1. In our previous papers [7, 8], the phase space is given in a little different manner. The previous setting involves some vagueness and our present setting based on the work in [6] gives a precise reconstruction. However, in our present context, there is no difference between the two.

REMARK 2. As was stated in [6], Properties (I) $\sim$ (IV) imply that all bounded continuous functions ϕ mapping $(-\infty, 0]$ into R^n are in B, and it will not be difficult to see that $|\phi|_B \leqq K \sup_{s \leqq 0} |\phi(s)|$. Hence, for any bounded continuous function ϕ defined on R, we have $\sup_{t \in R} |\phi_t|_B \leqq K|\phi|^\infty$, where $|\phi|^\infty = \sup_{t \in R} |\phi(t)|$.

Consider the systems

$$\dot{x}(t) = A(t, x_t) \tag{1}$$

and

$$\dot{x}(t) = A(t, x_t) + \eta F(t, x_t)\,, \tag{2}$$

where $A(t, \phi)$ and $F(t, \phi)$ are continuous in $(t, \phi) \in R \times B$ and almost periodic in t uniformly for $\phi \in B$, and $\eta \geqq 0$ is a parameter. In addition, we shall assume that $A(t, \phi)$ and $F(t, \phi)$ satisfy the following conditions, respectively:

(A) For any $\alpha > 0$, there exists a positive, continuous and increasing function $M_A(\alpha)$ such that $|A(t, \phi)| \leqq M_A(\alpha)$ on $R \times \bar{B}_\alpha$, where $\bar{B}_\alpha = \{\phi \in B: |\phi|_B \leqq \alpha\}$.

(F) For any $r > 0$ and $N > 0$, there exists an $L_F > 0$ such that for any ϕ, $\psi \in R^-_{r,N}$ and $t \in R$, $|F(t, \phi) - F(t, \psi)| \leqq L_F |\phi - \psi|_B$, where $R^-_{r,N} = \{\phi \in C((-\infty, 0], R^n): |\phi(t)| \leqq r$ for $t \in (-\infty, 0]$ and $|\phi(t_1) - \phi(t_2)| \leqq N|t_1 - t_2|$,

$t_1, t_2 \in (-\infty, 0]\}$, which is a subset of B by Remark 2.

Condition (F) is weaker than a Lipschitz condition. In fact, the following example presents a function which does not satisfy a Lipschitz condition but satisfies Condition (F).

EXAMPLE. Let $\mathscr{C}$ be the space which consists of all continuous functions mapping $(-\infty, 0]$ into R^n such that $\phi(\theta)e^{\gamma\theta} \to 0$ as $\theta \to -\infty$ with norm $|\phi|_{\mathscr{C}} = \sup_{-\infty<\theta\leqq 0} |\phi(\theta)|e^{\gamma\theta}$, where $\gamma > 0$ is a fixed constant. This space satisfies all the conditions given for the space B (cf. [6, 7]). Consider a function $F(t, \phi) = \phi(-|\phi(0)|)$. Then it is known that $F(t, \phi)$ defined on $R \times \mathscr{C}$ does not satisfy a Lipschitz condition but satisfies Condition (F) (refer to [3]).

Define AP by

$$\mathrm{AP} = \{\phi \in C(R, R^n)\colon \phi(t) \text{ is almost periodic in } t\}\,.$$

For $r > 0$ and $N > 0$, define $R_{r,N}$ and $\mathrm{AP}_{r,N}$ by

$$R_{r,N} = \{\phi \in C(R, R^n)\colon |\phi|^\infty \leqq r \text{ and } |\phi(t_1) - \phi(t_2)| \leqq N|t_1 - t_2| \text{ for } t_1, t_2 \in R\}$$

and $\mathrm{AP}_{r,N} = \mathrm{AP} \cap R_{r,N}$, respectively.

LEMMA. *Let $r > 0$ and $N > 0$. Then $\mathrm{AP}_{r,N}$ is a closed subset of the Banach space $C_0(R, R^n)$ with norm $|\cdot|^\infty$, where $C_0(R, R^n)$ consists of all bounded continuous functions mapping R into R^n. Furthermore, if $\phi \in \mathrm{AP}_{r,N}$ and $t \in R$, then $F(t, \phi_t) \in \mathrm{AP}$ and it is bounded uniformly for $\phi \in \mathrm{AP}_{r,N}$ and $t \in R$.*

PROOF. Since AP is the Banach space with norm $|\cdot|^\infty$ (cf. [1]), we can easily show that $\mathrm{AP}_{r,N}$ is a closed subset of the Banach space $C_0(R, R^n)$ with norm $|\cdot|^\infty$. It is well known that if a continuous function $f(t, x)$ is almost periodic in t uniformly for $x \in R^n$ and if $x(t)$ is almost periodic in t and takes its value in some compact set S in R^n, then $f(t, x(t))$ is almost periodic in t (cf. Theorem 2.7 in [12]) and $f(t, x)$ is bounded on $R \times S$ (cf. Theorem 2.1 in [12]). Hence, we have the second assertion, because for any $\phi \in \mathrm{AP}_{r,N}$ and $t \in R$, $\phi_t \in R^-_{r,N}$ and $R^-_{r,N}$ is compact in B.

Now we shall give our theorem.

THEOREM. *Suppose that there exists a Liapunov functional $V(t, \phi, \psi)$ defined on $I \times B \times B$, $I = [0, \infty)$, which has the following properties:*

(V.1) $M_V|\phi(0) - \psi(0)| \leqq V(t, \phi, \psi) \leqq b(|\phi - \psi|_B)$, *where M_V is a positive constant and $b(r)$ is a continuous and increasing function on I with $b(0) = 0$.*

(V.2) $|V(t, \phi_1, \psi_1) - V(t, \phi_2, \psi_2)| \leqq L_V|(\phi_1 - \phi_2) - (\psi_1 - \psi_2)|_B$, *where* L_V *is a positive constant.*

(V.3) $\dot{V}_{(1)^*}(t, \phi, \psi) = \limsup_{\delta \to 0+} [V(t+\delta, x_{t+\delta}, y_{t+\delta}) - V(t, x_t, y_t)]/\delta \leqq -cV(t, \phi, \psi)$, *where* (x, y) *is a solution of the product system*

$$(1)^* \qquad \dot{x}(t) = A(t, x_t), \qquad \dot{y}(t) = A(t, y_t)$$

with initial data (t, ϕ, ψ) *and* c *is a positive constant. Moreover, we assume that* (1) *has a solution* $\xi(t)$ *such that* $|\xi(t)| \leqq \beta$ *for* $t \in I$ *and some positive constant* β. *Then for any* $r > \beta$ *and* $N > M_A(K\beta)$, *there is an* $\eta_0 > 0$ *such that if* $0 \leqq \eta < \eta_0$, *then the system* (2) *has a unique solution in* $\mathrm{AP}_{r,N}$.

(Throughout this paper we shall denote by * the product system associated with an equation considered.)

Let $u(t)$ and $v(t)$ be solutions of $\dot{u}(t) = A(t, u_t) + f(t)$ and $\dot{v}(t) = A(t, v_t) + g(t)$, respectively. Define $\dot{V}(t, u_t, v_t)$ by

$$\dot{V}(t, u_t, v_t) = \limsup_{\delta \to 0+} [V(t+\delta, u_{t+\delta}, v_{t+\delta}) - V(t, u_t, v_t)]/\delta .$$

Then we shall note that

$$(3) \qquad \dot{V}(t, u_t, v_t) \leqq KL_V|f(t) - g(t)| - cV(t, u_t, v_t)$$

by Properties (II), (V.2) and (V.3).

Proof of Theorem. Let $r > \beta$ and let $N > M_A(K\beta)$. First, we shall show that there is an $\eta_1 > 0$ such that if $0 \leqq \eta < \eta_1$, then for any $\phi \in \mathrm{AP}_{r,N}$ the system

$$(4) \qquad \dot{x}(t) = A(t, x_t) + \eta F(t, \phi_t)$$

has a unique solution in $\mathrm{AP}_{r,N}$. Let $C_1 = \sup\{|F(t, \phi_t)| : t \in R, \phi \in \mathrm{AP}_{r,N}\}$. Then $C_1 < \infty$ by Lemma. By choosing $\{\tau_k\}$, $\tau_k \to \infty$ as $k \to \infty$, suitably, we see that $\xi(t+\tau_k)$ converges to a solution $\zeta(t)$ of (1) uniformly on any compact set in R as $k \to \infty$. Clearly, $|\zeta(t)| \leqq \beta$ for all $t \in R$. Let $\phi \in \mathrm{AP}_{r,N}$ and let $x(t)$ be a solution of (4) with $x_0 = \zeta_0$. By the relation (3), we have $\dot{V}(t, \zeta_t, x_t) \leqq L_V K\eta|F(t, \phi_t)| - cV(t, \zeta_t, x_t) \leqq L_V K\eta C_1 - cV(t, \zeta_t, x_t)$, as long as x_t exists, which implies $M_V|\zeta(t) - x(t)| \leqq V(t, \zeta_t, x_t) \leqq e^{-ct}V(0, \zeta_0, x_0) + L_V KC_1\eta/c \leqq L_V KC_1\eta/c$ by (V.1). Hence we have

$$(5) \qquad |x(t)| \leqq L_V KC_1\eta/(cM_V) + |\zeta(t)| \leqq L_V KC_1\eta/(cM_V) + \beta .$$

It follows from (5) and Remark 2 that

$$(6) \qquad |x_t|_B \leqq K\{L_V KC_1\eta/(cM_V) + \beta\}$$

for all $t \in R$, because $|x(t)| \leqq \beta$ for $t \leqq 0$. Therefore, since the right hand side of (4) is completely continuous by Property (A), x_t exists for all $t \in R$.

We shall show that $x(t)$ is an asymptotically almost periodic solution of (4). It is known that if the closure of $\{x_t: t \geqq 0\}$ is compact, then the existence of a Liapunov functional $V(t, \phi, \psi)$ which has Properties (V. 1), (V. 2) and (V. 3) implies that $x(t)$ is asymptotically almost periodic (see [10]). By (6), we have

$$(7) \qquad |\dot{x}(t)| \leqq |A(t, x_t)| + \eta|F(t, \phi_t)| \leqq M_A(K^2 L_V C_1 \eta/(cM_V) + K\beta) + \eta C_1$$

for $t \in I$, which implies the closure of $\{x_t: t \geqq 0\}$ is compact (cf. see Remark 1 in [7]). Hence $x(t)$ is asymptotically almost periodic.

By the standard arguments (cf. Theorem 1 in [8]), it is easy to show that $x(t + \tau_k)$ converges to an almost periodic solution $p(t)$ of (4) for a suitable sequence $\{\tau_k\}$, $\tau_k \to \infty$ as $k \to \infty$. Clearly, $p(t)$ and $\dot{p}(t)$ are bounded on R and their bounds are given by the right hand sides of (5) and (7), respectively. Since $\dot{V}_{(4)*}(t, \psi, \chi) \leqq -cV(t, \psi, \chi)$ by the relation (3), $p(t)$ is a unique almost periodic solution of (4). Hence we can choose a desirable η_1, because $r > \beta$, $N > M_A(K\beta)$ and $M_A(\alpha)$ is continuous and increasing.

For a unique solution $p(t) \in \mathrm{AP}_{r,N}$ of (4), put $T\phi(t) = p(t)$. Then T is a mapping from $\mathrm{AP}_{r,N}$ into $\mathrm{AP}_{r,N}$. Let $\phi, \psi \in \mathrm{AP}_{r,N}$ and $t \geqq 0$. Define a scalar function $w(t)$ by $w(t) = V(t, (T\phi)_t, (T\psi)_t)$. Then it holds that $\dot{w}(t) \leqq -cw(t) + L_V K\eta|F(t, \phi_t) - F(t, \psi_t)|$ by the relation (3). Hence we have $\dot{w}(t) \leqq -cw(t) + L_V K\eta L_F|\phi_t - \psi_t|_B \leqq -cw(t) + L_V K^2\eta L_F|\phi - \psi|^\infty$ by Condition (F) and Remark 2. It follows from (V. 1) that $M_V|T\phi(t) - T\psi(t)| \leqq V(t, (T\phi)_t, (T\psi)_t) \leqq w(t) \leqq e^{-ct}b(|(T\phi)_0 - (T\psi)_0|_B) + L_V K^2\eta L_F|\phi - \psi|^\infty/c$, which implies

$$(8) \qquad |T\phi(t) - T\psi(t)| \leqq e^{-ct}b(|(T\phi)_0 - (T\psi)_0|_B)/M_V + C_2\eta|\phi - \psi|^\infty$$

for all $t \geqq 0$, where $C_2 = L_V K^2 L_F/(M_V c)$. It is possible to choose a sequence $\{t_k\}$, $t_k \to \infty$ as $k \to \infty$, so that $T\phi(t + t_k) - T\psi(t + t_k) \to T\phi(t) - T\psi(t)$ as $k \to \infty$ uniformly on R. Therefore, by replacing t with $t + t_k$ in (8) and by setting $k \to \infty$, we have $|T\phi(t) - T\psi(t)| \leqq C_2\eta|\phi - \psi|^\infty$ for all $t \in R$. Thus if we take $\eta_0 = \min\{\eta_1, 1/C_2\}$, then for $0 \leqq \eta < \eta_0$ we see that T is a contraction mapping and T has a unique fixed point in $\mathrm{AP}_{r,N}$, because $\mathrm{AP}_{r,N}$ is a closed subset of a Banach space $C_0(R, R^n)$ with norm $|\cdot|^\infty$ by Lemma. This completes the proof.

In addition, we suppose that the space B has the following property:

(V) $|\phi(0)| \leqq M_1|\phi|_B$ for an $M_1 > 0$.

We can find a Liapunov functional $V(t, \phi, \psi)$ which has Properties (V. 1), (V. 2) and (V. 3), when $A(t, \phi)$ is linear in ϕ and the null solution of (1) is uniformly asymptotically stable (see [10]). (In this case, we can take

$M_V = M_1$ and $b(r) = L_V r$.) Hence we have the following:

COROLLARY. *Suppose that the space B has Properties* (I) $\sim$ (V). *Assume that $A(t, \phi)$ is linear in ϕ and the null solution of* (1) *is uniformly asymptotically stable. Let $r > 0$ and $N > 0$. Then there is an $\eta_0 > 0$ such that if $0 < \eta < \eta_0$, then the system* (2) *has a unique solution in* $\mathrm{AP}_{r,N}$.

REMARK. We note that $A(t, \phi)$ satisfies Condition (A) automatically, if it is linear in ϕ and almost periodic in t uniformly for $\phi \in B$ (cf. [10]).

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