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CLASSICAL SOLUTIONS OF THE STEFAN PROBLEM

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Introduction. The purpose of the present paper is to prove the local (in-time) existence of the classical solutions for the initial value problem of the one-phase multidimensional Stefan problem, by using Nash's implicit function theorem.

The Stefan problem is a mathematical model for melting of a body of ice in contact with water. The initial value problem for the one-phase Stefan problem is formulated as follows:

(S-1) The unknowns are the thermal distribution in water and the shape of ice. The initial data are prescribed.

(S-2) The temperature of ice is maintained at 0°C. (The problem in which one considers the thermal distribution in ice is called the two-phase problem, which is not discussed in this paper.)

(S-3) The thermal distribution u satisfies the heat equation $(\partial_t - \Delta)u = 0$, where t is the time variable and Δ stands for the Laplacian with respect to the space variables $x = (x_1, \cdots, x_n)$.

(S-4) The body of ice melts, at each point of the interface, with velocity in proportion to the normal gradient of u. The locus of the interface in the (x, t)-space is the free boundary to be determined.

(S-5) The region occupied by water has possibly another connected component of the boundary. This component is fixed as t varies, and the heat may be supplied through it. The temperature is always non-negative.

This is a naive and typical free boundary problem, posed by Stefan [36]-[39].

In the one dimensional case, this problem (and also the two-phase problem) has been extensively studied. The problem of existence and uniqueness of the classical solution was settled by Rubinstein [27], [28]. It has also been proved that the classical solutions exist globally in time, for the initial and boundary data in various classes of function spaces (Rubinstein [29]; Friedman [10]; Cannon and Hill [5]; Cannon, Hill and Primicerio [6]; Cannon and Primicerio [7]). An excellent historical survey for the result before 1967 is provided by a monograph by Rubinstein [30]. See also Nogi [25] and Yamaguti and Nogi [40].

On the other hand, in the multidimensional case, as the ice melts, it may possibly break into two or more pieces in a finite time. This means that the classical solutions may not exist for all time in general, even if the given data are sufficiently smooth.

Let us briefly refer to studies in the multidimensional one-phase Stefan problem. The classical solutions are not expected to exist for all time. This fact motivates the study of the solutions in a generalized sense, i.e., the weak solutions. In [17], Kamenomostskaja introduced the notion of the weak solution of this problem, and proved its global existense and uniqueness. Her work was generalized by Oleinik [26] and Friedman [11]. The formulation of the problem as a parabolic variational inequality was initiated by Duvaut [9]. This method was developed by Friedman and Kinderlehrer [12], Caffarelli [1]-[3], Caffarelli and Friedman [4] and Kinderlehrer and Nirenberg [18], [19]. In [1]-[4], the Lipschitz continuity of the free boundary and the continuity of the thermal distribution up to the free boundary have been proved. Since the free boundary of the classical solution should be of C^1-class and the thermal distribution u of the classical solution should have the derivatives $\partial_{x_i}u$ which are continuous up to the free boundary, the type of such conclusions as in [1]-[4] is slightly weaker than the required one. In [12] and [19], a case in which we can obtain the C^∞ solution is posed. To formulate the problem as a variational inequality, one needs the positivity of the initial and boundary data. Further, in order to obtain the smoothness result in [12] and [19], they need a restrictive geometrical assumption on the initial and boundary data which assures that the melting is rapid and free from the breaking (see [12] and [19]). With assumptions of such kind, one may get around the difficulty, explained later in this introduction, in the multidimensional Stefan problem. In the case in [12] and [19], however, the smoothness up to the initial time was not proved.

What we do in the present paper is to construct the classical solutions in a sufficiently small time interval in general. Our proof has an advantage in revealing the character of the difficulty in the multidimensional problem.

In order to state our result, we introduce the following notations. Let Ω_0 be a bounded domain in $\boldsymbol{R}^n$, $n \geqq 2$, with C^∞ boundary. The domain Ω_0 is regarded as a region occupied by water. Suppose the boundary $\partial\Omega_0$ has two connected components Γ_0 and J_0, where the exterior boundary Γ_0 is in contact with ice, and the heat is supplied through the interior boundary J_0. Given $0 < T_0 < \infty$, we set, for $0 < T < T_0$, $\Omega_T = \Omega_0 \times [0, T]$,

$J_T = J_0 \times [0, T]$, $\Gamma_T = \Gamma_0 \times [0, T]$. As the ice melts, the interface Γ_0 varies and forms a free boundary, which will be diffeomorphic to Γ_T, as long as T is small enough. We shall parametrize this free boundary by the distance function ρ from Γ_0 (in $\boldsymbol{R}^n$), and denote it by $\Gamma_{\rho,T}$. The corresponding space-time domain in $\boldsymbol{R}^n \times \boldsymbol{R}$, which will be diffeomorphic to Ω_T, is denoted by $\Omega_{\rho,T}$. The one-phase Stefan problem is a problem determining the free boundary $\Gamma_{\rho,T}$ and the thermal distribution u in the region $\Omega_{\rho,T}$ occupied by water. By the preceding formulation (S-1)-(S-5), we are led to the following equations for (ρ, u):

$$(\partial_t - \Delta)u = 0 \quad \text{in} \quad \Omega_{\rho,T} . \tag{1_u}$$

$$u|_{t=0} = a_0 . \tag{2_u}$$

$$u = b_0 \quad \text{on} \quad J_T . \tag{3_u}$$

$$u = 0 \quad \text{on} \quad \Gamma_{\rho,T} . \tag{4_u}$$

$$\partial_t \Phi_\rho - c_0\langle \operatorname{grad} \Phi_\rho, \operatorname{grad} u\rangle = 0 \quad \text{on} \quad \Gamma_{\rho,T} . \tag{5_u}$$

Here a_0 (resp. b_0) is a nonnegative function on Ω_0 (resp. J_{T_0}), Φ_ρ is a defining function of $\Gamma_{\rho,T}$ and c_0 is a positive (because of its physical meaning) constant. The equation (5_u) is the so-called Stefan condition which makes the problem complicated. Now our result is as follows (for the precise statement, see Section 1).

THEOREM. *If a_0 and b_0 are sufficiently smooth and satisfy some compatibility conditions, then, for a sufficiently small T, there exists a classical solution (ρ, u) of (1_u)-(5_u).*

The above theorem is obtained by using Nash's implicit function theorem. Recently, several articles have appeared on the applications of Nash's implicit function theorem (Guillemin [13], in differential geometry; Hamilton [15], in complex analysis; Hörmander [16] and Schaeffer [31]-[33], in free boundary problems; Klainerman [20], in the theory of nonlinear wave equations; Zehnder [41], in Hamiltonian mechanics). However, Nash's theorem and its character are not so popular yet. Hence, before explaining why and how this theorem is used in our proof, let us sketch the idea of this theorem.

The classical implicit function theorem or the inverse function theorem in the finite dimensional Euclidean space asserts the existence of the local inverse of a smooth mapping with the nonzero Jacobian determinant. This theorem extends to a Fréchet differentiable, i.e., a linearizable, mapping between infinine dimensional Banach spaces, provided the Fréchet derivative, i.e., the linearized operator, has the bounded inverse, namely, Banach's implicit function theorem. However, it is sometimes possible that the linearized operator has merely an unbounded inverse. In the case that the Banach spaces considered are certain function spaces, e.g., the set of functions of C^k-class, and that the given mapping corresponds to a nonlinear differential equation, such a phenomenon happens when the operator solving the linearized problem does not gain the differentiability of the same order as that of the equation. In such a case, the original problem is said to have a derivative loss. Nash encountered such a problem with derivative loss in his work [23] on the isometric embedding problem for Riemannian manifolds and overcame this difficulty by establishing a new method, Nash's implicit function theorem.

The statement of Nash's implicit function theorem is somewhat complicated. The following is the simplified version due to Moser [22]. Consider two finite scales of Banach spaces $E_0 \supset E_1 \supset \cdots \supset E_k$ and $F_1 \supset F_2 \supset \cdots \supset F_k$, e.g., $E_i = C^{mi+p}$, $F_i = C^{mi+q}$ with $p > q$ and $m > 0$. Let $\mathscr{F}$ be a nonlinear operator defined in a neighborhood V of 0 of E_1 into F_1 such that $\mathscr{F}(\rho) \in F_i$ for $\rho \in V \cap E_i$. Nash's theorem asserts that the equation $\mathscr{F}(\rho) = 0$ has a solution $\rho \in V$, provided the following assumptions are satisfied (for the exact statement, see Section 5):

(N-1) An operator $\mathscr{S}_\theta\colon E_0 \to E_k$ with parameter $\theta \geqq 1$ is defined so that for $0 \leqq i \leqq j \leqq k$,

$$|\mathscr{S}_\theta f|_{E_j} \leqq C\theta^{j-i}|f|_{E_i}\,, \qquad |f - \mathscr{S}_\theta f|_{E_i} \leqq C\theta^{-(j-i)}|f|_{E_j}\,.$$

This $\mathscr{S}_\theta$ is called a smoothing operator.

(N-2) The operator $\mathscr{F}$ is Fréchet differentiable, i.e., $\mathscr{F}$ is linearizable.

(N-3) The linearized equation $D\mathscr{F}(\rho)\delta\rho = \delta G$ can be solved possibly with derivative loss for each $\rho \in V$, i.e., there exists a linear operator $\mathscr{J}(\rho)\colon F_i \to E_{i-1}$ for $i \geqq 1$ and $\rho \in V \cap E_i$, which is a right inverse to $D\mathscr{F}(\rho)$.

(N-4) For $i \geqq 1$ and $\rho \in V \cap E_i$,

$$|\mathscr{J}(\rho)\mathscr{F}(\rho)|_{E_{i-1}} \leqq C(1 + |\rho|_{E_i})\,.$$

(N-5) The value $|\mathscr{F}(0)|_{F_1}$ is sufficiently small.

The assumptions (N-2, 3, 5) will be in no need of explanation. In contrast to these, it seems that the assumptions (N-1) and (N-4) are artificial and indistinct, though it is obvious from the proof of Nash's theorem ([16]; [22]; [23]; Schwartz [34]; Sergeraert [35]) that they are essential and indispensable. Later in this introduction, we will give a short account for (N-1) and (N-4). Further, we add that the present paper has two mathematical cores, which are to solve the linearized Stefan problem and to introduce a logical frame of a general character in which the assumptions (N-1) and (N-4) can be verified.

Returning to our Stefan problem, let us sketch briefly how the problem is rewritten as a nonlinear operator equation $\mathscr{F}(\rho) = 0$ and the essential feature of solving the linearized Stefan problem $D\mathscr{F}(\rho)\delta\rho = \delta G$. Regarding the thermal distribution u as an auxiliary unknown determined from the distance function ρ with (1_u)-(4_u) and introducing an operator $\mathscr{F}$ which transforms ρ to the pull back of the function $\partial_t\Phi_\rho - c_0\langle \mathrm{grad}\, \Phi_\rho$, $\mathrm{grad}\, u\rangle$ on the free boundary $\Gamma_{\rho,T}$ to the function on the flattened boundary Γ_T, we are led to a nonlinear equation $\mathscr{F}(\rho) = 0$ which is equivalent to (1_u)-(5_u). This equation can be linearized and the concrete form of the linearized problem $D\mathscr{F}(\rho)\delta\rho = \delta G$ consists of two parts, as follows. The first part, corresponding to (1_u)-(4_u), is an initial boundary value problem of the Dirichlet type for a linear second order parabolic equation in Ω_T, where the unknown is an auxiliary one corresponding to the formal Fréchet derivative of u, and given data are ρ and $\delta\rho$. The second part, corresponding to the Stefan condition (5_u), is a linear first order equation of hyperbolic type for $\delta\rho$ on the flattend boundary Γ_T, with the data containing the normal derivative of the unknown of the first part. To solve the above linearized problem, we eliminate $\delta\rho$ from the system, by substituting the solution $\delta\rho$ of the latter hyperbolic equation for the data $\delta\rho$ in the former initial boundary value problem of parabolic type. Then, introducing a new unknown δX for convenience, we find that the essential point in solving the above system without $\delta\rho$ is to solve a linear parabolic initial boundary value problem for δX in Ω_T whose boundary condition on Γ_T is given by a linear hyperbolic first order equation. We extend this first order operator to one in the domain Ω_T. Then, to invert this first order operator, we need the nonnegativity of the coefficients of the normal derivation in this operator on the boundary Γ_T. In fact, this condition assures that the characteristic curves starting from points on the domain Ω_0 at the initial time cover the cylinder Ω_T. In our case, by using the maximum principle of the heat equation, we can prove the required nonnegativity because the temperature on Ω_0 at the initial time and on the fixed boundary J_T are nonnegative (namely, this physical requirement has also a natural mathematical meaning). Then, after a technical deformation, the above problem for δX is solved by decomposing it to a parabolic initial boundary value problem of Dirichlet type and an initial value problem for the above first order operator in Ω_T. Consequently, the solution $\delta\rho$ of the linearized Stefan problem $D\mathscr{F}(\rho)\delta\rho = \delta G$ is obtained as a linear combination containing the normal derivative of δX.

Since, except in the one-dimensional case, the operator $\delta G \mapsto \delta X$ gains only the same regularity in the weighted Hölder spaces, standard in the theory of parabolic equations, the linearized Stefan problem can be solved, but actually with an essential derivative loss. This is the reason why we need Nash's implicit function theorem.

From the above consideration on solving the linearized Stefan problem, we find that the reason why derivative loss occurs in the multidimensional Stefan problem consists in the following fact: Since the mapping solving the linear first order equation of hyperbolic type which corresponds to the Stefan condition (5_u) gains the regularity only along the characteristic curves, it cannot cover the bad influence of the diffusion effect, i.e., the normal derivative of δX. This seems to be a new observation showing why the multidimensional Stefan problem is difficult.

In the one-dimensional case, since direction of the diffusion can be covered by the characteristic curve, we can solve the linearized problem without derivative loss. Therefore, the local existence theorem for classical solutions is obtained by usual Banach's implicit function theorem, because, in an appropriate setting, the norm $|\mathscr{F}(0)|$ can be taken small if T is sufficiently small.

Now, we account for the remaining assumptions (N-1) and (N-4). The general principle to verify the assumption (N-4), i.e., the estimate $|\mathscr{I}(\rho)\mathscr{F}(\rho)|_{i-1} \leqq C(1+|\rho|_i)$, is as follows: This estimate is automatically obtained if we prove that each of all mathematical operations constructing $\mathscr{F}$ and $\mathscr{I}$ is a "balanced" operator. Here the meaning of the term "balanced" will be illustrated by the following examples. In the weighted Hölder spaces, addition, multiplication, division, composition and to take the solution u of a parabolic initial boundary value problem

$$(\partial_t - \sum A_{ij}\partial_{x_i}\partial_{x_j} - \sum A_i\partial_{x_i} - A_0)u = f \quad \text{in} \quad \Omega_T\,, \quad u|_{t=0} = a\,, \quad u|_{\partial\Omega_T} = b\,,$$

are balanced operations, because we have estimates of the following type:

$$|f+g|_{(r)} \leqq |f|_{(r)} + |g|_{(r)}\,, \qquad |fg|_{(r)} \leqq C(|f|_{(r)}|g|_{(0)} + |f|_{(0)}|g|_{(r)})\,,$$

$$|f/g|_{(r)} \leqq C[|f|_{(r)} + (1+|g|_{(r)})|f|_{(0)}]\,, \quad \text{if} \quad \inf g \geqq B > 0\,,$$

$$|f\circ g|_{(r)} \leqq C(|f|_{(r)} + |f|_{(1)}|g|_{(r)})\,, \quad \text{if} \quad r \geqq 1 \quad \text{and} \quad |g|_{(1)} \leqq B\,,$$

$$\begin{aligned} |u|_{(r+2)} \leqq C[&(\sum |A_{ij}|_{(r)} + \sum |A_i|_{(r)} + |A_0|_{(r)})(|f|_{(\varepsilon)} + |a|_{(\varepsilon+2)} + |b|_{(\varepsilon+2)}) \\ &+ |f|_{(r)} + |a|_{(r+2)} + |b|_{(r+2)}]\,, \end{aligned}$$

if $r \geqq \varepsilon > 0$, r and ε are not integers and $|A_{ij}|_{(\varepsilon)}$, $|A_i|_{(\varepsilon)}$, $|A_0|_{(\varepsilon)} \leqq B$, where $|\ \ |_{(r)}$ is the weighted Hölder norm of r-th order. Now the preceding "general principle" is an ovbious fact. This "principle" is consciously used, e.g., in [16] and [31]-[33]. Further, e.g., in [16], in place of the parabolic boundary value problem and the weighted Hölder norm, with an elliptic boundary value problem and the usual Hölder norm, $\mathscr{F}$ and $\mathscr{I}$ are constructed only by operations listed in the above examples. Then, with the fact that to construct smoothing orerators on a usual Hölder space is not so difficult, the above consideration is sufficient to apply Nash's theorem to the problem in [16]. Moreover, it seems that the above consideration is also sufficient for the other applications of Nash's theorem listed above in this introduction.

Our new trouble on the verification of (N-4) in the Stefan problem lies in the fact that, to invert the linearized problem, we had to take the Neumann series to solve linear integral equations of Volterra type several times. If an operator norm is less than 1 and we can complete the proof by taking Neumann series only once, then to prove (N-4) is not so difficult (see [33, Lemma 8.1]); however, both cannot be expected in our proof. The method adopted in the present paper is as follows. Instead of the usual weighted Hölder spaces $C^{(r)}$, (r denotes the order of regularity), we can use $C_{\#}^{(r)}$ spaces each of which consists of all $C^{(r)}$ functions whose derivatives up to r-th order are vanishing at the initial time, because the considered problem is a linearized one, i.e., a variational one. In $C_{\#}^{(r)}$ spaces, the integral operator $\int_0^t$ commutes with the norm, i.e., $\left|\int_0^t f\right|_{(r),T} \leqq \int_0^T |f|_{(r),t}\,dt$, where the subscripts T and t denote the width of the time intervals. It should be noted that this commutativity does not hold in $C^{(r)}$ spaces. Now, in $C_{\#}^{(r)}$ spaces, by using the iterated estimate $\left|\int_0^t \cdots \int_0^t f\right|_{(r)} \leqq (t^n/n!)|f|_{(r)}$, it is obvious from the standard argument that the operator solving a linear integral equation of Volterra type is a balanced one.

Then, we account for the assumption (N-1) on the existence of smoothing operators. Since we are working in the setting of $C_{\#}^{(r)}$ spaces, we have to construct smoothing operators on $C_{\#}^{(r)}$ spaces. It is known that for C^m, m is an integer, Sobolev and usual Hölder spaces, smoothing operators can be constructed as integral operators whose kernels are defined by using Fourier transform (see, e.g., [16] and [23]). In the same method, we can construct smoothing operators on $C^{(r)}$ spaces; however, these operators do not preserve the $C_{\#}^{(r)}$ property, i.e., the image of a $C_{\#}^{(r)}$ function is not necessarily a $C_{\#}^{(r)}$ function. Therefore the above method cannot be used in our setting. On the other hand, in his work on the isometric embedding problem for analytic Riemannian manifolds [24], in order to get smoothing operators which do not shrink the radii of convergence of real analytic functions at each point in the domain, Nash introduced a new method to construct smoothing operators. In [24], smoothing operators are constructed as integral operators, but their kernels are defined not by using Fourier transform, but by taking a linear combination of functions each of which is obtained from the heat kernel with an appropriate coordinate transform. (See also Gromov [14, Section 3] whose exposition may be more precise than that in [24].) Then, though the appearance of the problem is different from that in [24] and we must take a $C_{\#}^{(\infty)}$ function instead of the heat kernel, we can construct smoothing operators on $C_{\#}^{(r)}$ spaces by using the method in [24].

The finite scale with $C_{\#}^{(r)}$ spaces and the smoothing operators constructed in the above method constitute the logical frame, announced in this introduction, in which the operator solving a linear integral equation of Volterra type becomes a balanced one. This frame seems to be natural and useful in applying Nash's theorem to initial value problems.

The outline of this paper is as follows. In Section 1, we give the exact statement of our result and define the weighted Hölder spaces with which our theorem is formulated. In Section 2, we add a technical assumption to our theorem to simplify the account of the proof and restate our theorem by using some notations which are also introduced in this section. The elimination of this technical assumption will be carried out in Section 10. In Section 3, we collect together some fundamental lemmas which assure that multiplication, division, composition and to take the solution of a parabolic initial boundary value problem are balanced operators, in the weighted Hölder spaces. In Section 4, we introduce a nonlinear operator $\mathscr{F}$ by which Theorem′ in Section 2 is restated as the equation $\mathscr{F}(\rho) = 0$. We apply Nash's implicit function theorem to this operator equation. In Section 5, we state Nash's theorem and pose two finite scales of Banach spaces on which Nash's theorem is applied. In Section 6, smoothing operators in our setting are constructed; in other words, the assumption (N-1) is verified. In Section 7, the Fréchet differentiability of $\mathscr{F}$, i.e., the fact that the problem is linearizable, is proved; in other words, the assumption (N-2) is verified. In Section 8, we again collect together some technical lemmas, which include two key lemmas to our proof. One of the two lemmas assures that in $C_{\#}^{(r)}$ spaces, the operator solving a linear integral equation of Volterra type is a balanced one (see Lemma 8.A.4). The other lemma asserts that we can solve a linear parabolic initial boundary value problem whose boundary condition is given by a first order equation such that the coefficient of the normal derivative is nonnegative. This lemma also asserts that the operator solving the problem is a balanced one (see Lemma 8.B.9). In Section 9, by using the lemmas in Section 8, we solve the linearized problem $D\mathscr{F}(\rho)\delta\rho = \delta G$ and observe that the solution $\delta\rho$ satisfies the assumptions (N-3) and (N-4). Section 9 contains the key fact which enables us to prove the result of this paper, that is, the fact that the coefficient corresponding to the one in the lemma in Section 8 on the solvability of a parabolic initial boundary value problem is, in fact, nonnegative (see (9.16)). At the end of Section 9, the assumption (N-5) is verified, so that the proof of Theorem′ in Section 2 is completed. In Section 10, we sketch how our proof is modified when the technical assumption in Section 2 is replaced by the general compatibility condition in Theorem in Section 1.

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1. **The result.** In this section, we state our result. At the beginning, we prepare some notations.

(A) Let n be an integer with $n \geqq 2$. Let Ω_0 be a bounded domain in $\boldsymbol{R}^n$ whose boundary consists of the outside component Γ_0 and the inside one J_0. Suppose Γ_0 and J_0 are C^∞. Let T_0 be a positive constant. For $T \in (0, T_0]$, let $\Omega_T = \Omega_0 \times [0, T]$, $\Gamma_T = \Gamma_0 \times [0, T]$, and $J_T = J_0 \times [0, T]$.

(B) We define the Hölder spaces.

DEFINITION 1.B.1. Let d be a positive integer. Let D be a domain with C^∞ boundary in $\boldsymbol{R}^d$ or $D = \boldsymbol{R}^d$. Let $r \geqq 0$. Then $C^r(\bar{D})$ is the set of real-valued functions f on $\bar{D}$ such that:

(i) The derivatives $\partial_x^\alpha f$ with $|\alpha| \leqq [r]$ are continuously extended to $\bar{D}$, where $\alpha = (\alpha_1, \cdots, \alpha_n)$ denotes a d-tuple of non-negative integers, $\partial_x = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, $|\alpha| = \sum_{i=1}^d \alpha_i$, and $[r]$ is the greatest integer not greater than r.

(ii) The norm $|f|_r$ is finite.

Here $|f|_r$ is defined by:

(i) $|f|_r = \sum_{|\alpha| \leqq r} \sup_{x \in \bar{D}} |\partial_x^\alpha f(x)|$ if r is an integer.

(ii) $|f|_r = \sum_{|\alpha| \leqq [r]} \sup_{x \in \bar{D}} |\partial_x^\alpha f(x)| + \sum_{|\alpha| = [r]} \sup_{x, y \in \bar{D}} |\partial_x^\alpha f(x) - \partial_x^\alpha f(y)| / |x - y|^{r-[r]}$ if r is not an integer where $|x - y|$ denotes $(\sum_{i=1}^d |x_i - y_i|^2)^{1/2}$.

DEFINITION 1.B.2. Let d be a positive integer. Let I be an interval in $\boldsymbol{R}$. Let D be a domain with piecewise-C^1 boundary in $\boldsymbol{R}^d \times I$. Let $r \geqq 0$. Then $C^{(r)}(\bar{D})$ is the set of real-valued functions f on $\bar{D}$ such that:

(i) The derivatives $\partial_x^\alpha \partial_t^a f(x, t)$ with $|\alpha| + 2a \leqq [r]$ are continuously extended to $\bar{D}$.

(ii) The norm $|f|_{(r)}$ is finite.

The norm $|f|_{(r)}$ is defined by:

(i) $|f|_{(r)} = \sum_{i=0}^{r} \langle f \rangle_{(i)}$ if r is an integer.

(ii) $|f|_{(r)} = \sum_{i=0}^{[r]} \langle f \rangle_{(i)} + \langle f \rangle_{(r)}$ if r is not an integer.

Here $\langle f \rangle_{(i)}$ and $\langle f \rangle_{(r)}$ are defined by:

(i) $\langle f \rangle_{(i)} = \sum_{|\alpha|+2a=i} \sup_{(x,t)\in\bar{D}} |\partial_x^\alpha \partial_t^a f(x, t)| + \sum_{|\alpha|+2a=i-1} \sup_{(x,t),(x,s)\in\bar{D}} |\partial_x^\alpha \partial_t^a f(x, t) - \partial_x^\alpha \partial_t^a f(x, s)|/|t-s|^{1/2}$ if i is an integer.

(ii) $\langle f \rangle_{(r)} = \sum_{|\alpha|+2a=[r]} \sup_{(x,t),(y,t)\in\bar{D}} |\partial_x^\alpha \partial_t^a f(x, t) - \partial_x^\alpha \partial_t^a f(y, t)|/|x-y|^{r-[r]} + \sum_{|\alpha|+2a=[r]} \sup_{(x,t),(x,s)\in\bar{D}} |\partial_x^\alpha \partial_t^a f(x, t) - \partial_x^\alpha \partial_t^a f(x, s)|/|t-s|^{(r-[r])/2} + \sum_{|\alpha|+2a=[r]-1} \sup_{(x,t),(x,s)\in\bar{D}} |\partial_x^\alpha \partial_t^a f(x, t) - \partial_x^\alpha \partial_t^a f(x, s)|/|t-s|^{(r-[r]+1)/2}$ if r is not an integer.

DEFINITION 1.B.3. Let M be a compact C^∞ manifold (with or without boundary). Let I be an interval in $\boldsymbol{R}$. Let $r \geqq 0$. Then the set $C^{(r)}(M \times I)$ and the norm $|\ |_{(r)}$ in it are defined by using a finite covering of M by coordinate neighborhoods and a C^∞ partition of unity subordinate to it.

REMARK 1.B.4. All the normed spaces in Definitions 1.B.1–1.B.3 are Banach spaces.

(C) Let $\boldsymbol{n}_\omega$ be the outward unit normal at $\omega \in \Gamma_0$. Let γ_0 be a positive constant so small that a mapping $x: \Gamma_0 \times [-\gamma_0, \gamma_0] \to \boldsymbol{R}^n$ defined by $(\omega, \lambda) \mapsto \omega + \lambda \boldsymbol{n}_\omega$ is regular and one-to-one. Let

$$N_0 = \{x(\omega, \lambda); (\omega, \lambda) \in \Gamma_0 \times [-\gamma_0, \gamma_0]\}$$

and

$$N_0^- = \{x(\omega, \lambda); (\omega, \lambda) \in \Gamma_0 \times [-\gamma_0, 0]\} .$$

We denote the inverse mapping of x on N_0 onto $\Gamma_0 \times [-\gamma_0, \gamma_0]$ by $x \mapsto (\omega(x), \lambda(x))$. Clearly the mappings $x(\omega, \lambda)$, $\omega(x)$ and $\lambda(x)$ are C^∞. We often use $(\omega_1(x), \cdots, \omega_{n-1}(x))$ as local coordinates of $x \in N_0$, where ω_i is the i-th component of ω with respect to local coordinates in Γ_0.

(D) For $\rho \in C^0(\Gamma_T)$ with $|\rho|_0 < \gamma_0$, let

$$\Gamma_{\rho,T} = \{(x(\omega, \rho(\omega, t)), t); (\omega, t) \in \Gamma_T\}$$

and let $\Omega_{\rho,T}$ be the domain in $R^n \times [0, T]$ bounded by $\Gamma_{\rho,T}$ and J_T. For $\rho \in C^0(\Gamma_T)$ with $|\rho|_0 < \gamma_0$ and for $(x, t) \in N_0 \times [0, T]$, let

$$\Phi_\rho(x, t) = \lambda(x) - \rho(\omega(x), t) . \tag{1.1}$$

Note that

$$\Gamma_{\rho,T} = \{(x, t) \in N_0 \times [0, T]; \Phi_\rho(x, t) = 0\} .$$

(E) Now our result can be stated as follows.

THEOREM. *Let $r_0 = n_0 + \varepsilon_0$ where n_0 is an integer with $n_0 \geqq 7$ and $0 < \varepsilon_0 < 1$. Suppose that:*

(A.1) $a_0 \in C^{r_0+43}(\bar{\Omega}_0)$ *and* $b_0 \in C^{(r_0+39)}(J_{T_0})$.

(A.2) *The pair $\{a_0, b_0\}$ satisfies the compatibility condition up to order $[(r_0 + 39)/2]$ of the Dirichlet problem for the heat equation, on J_0.*

(A.3) $a_0 \geqq 0$ *on* $\bar{\Omega}_0$, $b_0 \geqq 0$ *on* J_{T_0} *and c_0 is a positive constant.*

(A.4) *The function a_0 satisfies the compatibility condition up to order $[(r_0 + 39)/2]$ of the Stefan problem (see Remark* 1.E.4 *in the following).*

Then, for a sufficiently small $T \in (0, T_0]$, there exist $\rho \in C^{(r_0)}(\Gamma_T)$ with $\rho|_{t=0} = 0$ and $|\rho|_0 < \gamma_0$ and $u \in C^{(r_0)}(\bar{\Omega}_{\rho,T})$ which satisfy:

$$(1_u) \qquad (\partial_t - \Delta)u = 0 \ \text{ in } \Omega_{\rho,T} \, .$$

$$(2_u) \qquad u|_{t=0} = a_0 \, .$$

$$(3_u) \qquad u = b_0 \quad \text{on} \quad J_T \, .$$

$$(4_u) \qquad u = 0 \quad \text{on} \quad \Gamma_{\rho,T} \, .$$

$$(5_u) \qquad \partial_t \Phi_\rho - c_0 \langle \operatorname{grad} \Phi_\rho, \operatorname{grad} u \rangle = 0 \quad \text{on} \quad \Gamma_{\rho,T} \, .$$

Here $\Delta = \sum_{i=1}^n \partial_{x_i}^2$, $\operatorname{grad} = (\partial_{x_1}, \cdots, \partial_{x_n})$ *and* $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$ *for* $x, y \in \boldsymbol{R}^n$.

REMARK 1.E.1. The numbers 43 and 39 appearing in our theorem have no specific meaning. They come out because we employ Moser's version of Nash's implicit function theorem [22], which is the most popular one. Nash's original version [23] or Hörmander's version [16] provides smaller numbers, but such a refinement may not be essential.

REMARK 1.E.2. Consider the mixed problem of the heat equation $(\partial_t - \Delta)u = 0$ with the initial condition $u = a$ and the boundary condition $u = b$ (resp. $\partial_{\boldsymbol{n}} u = b$ where $\boldsymbol{n}$ is the outward unit normal of the boundary). The compatibility condition up to order i of the above problem requires that $\partial_t^j b = \Delta^j a$ (resp. $\partial_t^j b = \partial_{\boldsymbol{n}} \Delta^j a$) on the boundary at $t = 0$ for $j = 0, \cdots, i$. For other mixed problems of parabolic type, analogous definitions are adopted. (For details, see, e.g., Ladyzenskaja, Solonnikov and Uralceva [21, p. 319-320].)

REMARK 1.E.3. The assumption (A.3), which is natural in the physical sense, enables us to solve the linearized Stefan problem (see (9.16)).

REMARK 1.E.4. Let i be a nonnegative integer and let (ρ, u) be a $C^{(2i)}$ solution of the above system (1_u)-(5_u) with $\rho|_{t=0} = 0$. Because of (4_u), on Γ_0 at $t = 0$, the time variable derivatives up to order i of u

along $\Gamma_{\rho,T}$ vanish. Therefore, we have algebraic relations of u, $\partial_t u, \cdots,$ $\partial_t^i u$ and $\partial_t \rho, \cdots, \partial_t^i \rho$ on Γ_0 at $t=0$. On the other hand, in view of (1_u), (2_u) and (5_u), all of the above derivatives are determined only by a_0. Then, we have algebraic relations of the derivatives $\partial_x^\alpha a_0$ with $|\alpha| \leqq 2i$ on Γ_0. These relations constitute the compatibility condition up to order i of the Stefan problem.

REMARK 1.E.5. As is mentioned in Introduction, the uniqueness of the solution has been established even in the class of weak solutions.

REMARK 1.E.6. The author does not know whether we can take $r_0=\infty$ in the above theorem. However, when the melting is rapid, our solution (ρ, u) is of C^∞-class at least for $0<t<T$, according to the regularity theorem for classical solutions by Kinderlehrer and Nirenberg [18], [19].

REMARK 1.E.7. The width T of the existence interval is determined only by $|a_0|_{r_0+43}$, $|b_0|_{(r_0+39)}$ and the shape of Γ_0.

2. **Technical assumption.** We shall first prove our theorem, assuming the technical assumption

$$\text{(T)} \qquad \partial_x^\alpha a_0 = 0 \quad \text{on} \quad \Gamma_0 \quad \text{if} \quad |\alpha| \leqq [r_0+39]\,.$$

The elimination of this assumption will be carried out in Section 10. In this section, assuming (T) and introducing a few new notations, we restate our result in somewhat different form from that in Section 1.

(A) We introduce a new class of Hölder spaces.

DEFINITION 2.A.1. Let d be a positive integer. Let $r \geqq 0$. For $T \in (0, T_0]$ and for a domain D with piecewise-C^1 boundary in $R^d \times [0, T]$, we set

$$C_\#^{(r)}(\bar{D}) = \{f \in C^{(r)}(\bar{D});\, (\partial_t^a f)|_{t=0} = 0 \text{ for } a=0, \cdots, [r/2]\}\,.$$

Further, for $T \in (0, T_0]$ and for a compact C^∞ manifold M (with or without boundary), $C_\#^{(r)}(M \times [0, T])$ is similarly defined. In these cases, when we emphasize that the interval of the "time variable" t is $[0, T]$, we denote the norm $|f|_{(r)}$ by $|f|_{(r),T}$.

REMARK 2.A.2. The above normed spaces are Banach spaces.

(B) For $T \in (0, T_0]$, we set

$$V_T = \{\rho \in C_\#^{(r_0)}(\Gamma_T);\, |\rho|_{(r_0)} < \delta_0\}\,,$$

where δ_0 is a positive constant so small that:

(i) $4\delta_0 \leqq \gamma_0$.

(ii) There is a positive constant σ_0 such that

$$\sigma_0^{-1}\sum_{i=1}^{n}\xi_i^2 \leqq \sum_{i,j=1}^{n} A_{\rho,ij}(x, t)\xi_i\xi_j \leqq \sigma_0\sum_{i=1}^{n}\xi_i^2$$

for $\rho \in V_T$, $(x, t) \in \Omega_T$, and $\xi \in \boldsymbol{R}^n$. Here $A_{\rho,ij}$ are the coefficients of the second order derivations in the operator $\mathscr{L}_\rho$ defined in (D) in this section.

(iii) There is a positive constant B_0 such that

$$S_\rho(\omega, t) \geqq B_0^{-1}$$

for $\rho \in V_T$ and $(\omega, t) \in \Gamma_T$. Here S_ρ is the function defined in (D) in this section.

This constant δ_0 is determined only by the shape of Γ_0.

(C) Choose a function $\chi_0 \in C_0^\infty(R)$ so that:

(i) $\chi_0(\lambda) = 1$ if $|\lambda| \leqq \delta_0$.

(ii) $\chi_0(\lambda) = 0$ if $|\lambda| \geqq 3\delta_0$.

(iii) $|\partial_\lambda \chi_0(\lambda)| \leqq 3/4\delta_0$ for $\lambda \in \boldsymbol{R}$.

For $\rho \in V_T$, define a diffeomorphism e_ρ: $\boldsymbol{R}^n \times [0, T] \to \boldsymbol{R}^n \times [0, T]$ by:

$$\text{(2.1.i)} \qquad e_\rho(x(\omega, \lambda), t) = (x(\omega, \lambda + \chi_0(\lambda)\rho(\omega, t)), t) \quad \text{for} \quad (x, t) = (x(\omega, \lambda), t) \in N_0 \times [0, T] .$$

$$\text{(2.1.ii)} \qquad e_\rho(x, t) = (x, t) \quad \text{for} \quad (x, t) \in (\boldsymbol{R}^n - N_0) \times [0, T] .$$

Note that $e_\rho(\Omega_T) = \Omega_{\rho,T}$, $e_\rho(\Gamma_T) = \Gamma_{\rho,T}$, and $e_\rho|_{t=0}$ is the identity mapping.

Define a function η: $\boldsymbol{R} \times [-\delta_0, \delta_0] \to \boldsymbol{R}$ by $\eta(\lambda, \mu) + \chi_0(\eta(\lambda, \mu))\mu = \lambda$ for $(\lambda, \mu) \in \boldsymbol{R} \times [-\delta_0, \delta_0]$. Since $|\partial_\lambda \chi_0(\lambda)| \leqq 3/(4\delta_0)$, the function η is well-defined and C^∞. We easily observe that:

$$\text{(2.2.i)} \qquad e_\rho^{-1}(x(\omega, \lambda), t) = (x(\omega, \eta(\lambda, \rho(\omega, t))), t) \quad \text{for} \quad (x, t) = (x(\omega, \lambda), t) \in N_0 \times [0, T] .$$

$$\text{(2.2.ii)} \qquad e_\rho^{-1}(x, t) = (x, t) \quad \text{for} \quad (x, t) \in (\boldsymbol{R}^n - N_0) \times [0, T] .$$

(D) We define the operator $\mathscr{L}_\rho$ and the function S_ρ by:

$$\text{(2.3.i)} \qquad \mathscr{L}_\rho V = [(\partial_t - \Delta)(V \circ e_\rho^{-1})] \circ e_\rho \quad \text{for} \quad V \in C^{(2)}(\bar{\Omega}_T) .$$

$$\text{(2.3.ii)} \qquad [\langle \operatorname{grad} \Phi_\rho, \operatorname{grad}(V \circ e_\rho^{-1})\rangle] \circ e_\rho = (\partial_\lambda V) S_\rho \quad \text{on} \quad \Gamma_T \quad \text{for} \quad V \in C^1(\bar{\Omega}_T) \quad \text{with} \quad V|_{\Gamma_T} = 0 .$$

For the explicit expressions, see Section 4.

(E) Now our restricted and modified theorem is as follows.

THEOREM$'$. *Assume the assumptions* (A.1)-(A.3) *and* (T). *Then, there exist* $\rho \in V_T$ *and* $u \in C^{(r_0)}(\bar{\Omega}_{\rho,T})$ *which satisfy* (1_u)-(5_u).

REMARK 2.E.1. When we assume (T), we can replace the assumption

$a_0 \in C^{r_0+43}(\bar{\Omega}_0)$ by a weaker one $a_0 \in C^{r_0+39}(\bar{\Omega}_0)$.

3. Technical lemmas I. We collect together some lemmas which are used in the following sections. Throughout this paper, "X is bounded with Y" means that X is bounded when Y is.

(A) We begin with the following fact. By the term "a manifold", we mean both one with boundary and one without boundary.

LEMMA 3.A.1. *Let M be a compact C^∞ manifold. Let $\gamma < \delta$ and $0 \leqq p \leqq q \leqq r$. Let f belong to $C^{(r)}(M \times [\gamma, \delta])$. Then*

$$|f|_{(q)} \leqq C|f|_{(p)}^{(r-q)/(r-p)}|f|_{(r)}^{(q-p)/(r-p)} .$$

Here C is a constant bounded with r and $(\delta-\gamma)^{-1}$.

Lemma 3.A.1 can be proved in the same way as [16, Theorem A.5] or [32, Corollary 1.3] (with [21, Lemma 3.2, p. 80]), which are analogues for C^r spaces.

COROLLARY 3.A.2. *Let g and h belong to $C^{(q+r)}(M \times [\gamma, \delta])$. Then*

$$|g|_{(q)}|h|_{(r)} \leqq C(|g|_{(q+r)}|h|_0 + |g|_0|h|_{(q+r)}) .$$

Here C is a constant bounded with $q+r$ and $(\delta-\gamma)^{-1}$.

PROOF. From Lemma 3.A.1 and the obvious inequality $a^{1-\mu}b^\mu \leqq (1-\mu)a + \mu b$ for $a, b \geqq 0$ and $\mu \in [0, 1]$, the corollary follows immediately.

COROLLARY 3.A.3. *Let g and h belong to $C^{(r)}(M \times [\gamma, \delta])$. Then their product fg belongs to $C^{(r)}(M \times [\gamma, \delta])$ and satisfies*

$$|fg|_{(r)} \leqq C(|f|_{(r)}|g|_0 + |f|_0|g|_{(r)}) .$$

Here C is a constant bounded with r and $(\delta-\gamma)^{-1}$.

PROOF. From Definitions 1.B.2, 1.B.3, Leibnitz's formula and Corollary 3.A.2, the corollary follows immediately.

REMARK 3.A.4. Corollary 3.A.3 implies that $C^{(r)}(M \times [\gamma, \delta])$ is a ring and $C_{\#}^{(r)}(M \times [0, T])$ is an ideal in $C^{(r)}(M \times [0, T])$.

We introduce a special class of functions.

DEFINITION 3.A.5. Let M be a compact C^∞ manifold. Let I be a subset of nonnegative real numbers having the maximal element. Let $G \geqq 1$. Then, for $T \in (0, T_0]$, $E_G^I(M \times [0, T])$ is the set of functions $f \in C^{(\max I)}(M \times [0, T])$ having an extension $\tilde{f} \in C^{(\max I)}(M \times [-T_0, T])$ such that $|\tilde{f}|_{(q)} \leqq G|f|_{(q)}$ for $q \in I$. We call the above $\tilde{f}$ an E_G^I-extension of f.

LEMMA 3.A.6. *Let M be a compact C^∞ manifold. Let $r \geqq 0$. Then* *we have*:

(i) $C_{\#}^{(r)}(M \times [0, T]) \subset E_1^{[0,r]}(M \times [0, T])$.

(ii) *If* $r < 2$, *then* $C^{(r)}(M \times [0, T]) = E_1^{[0,r]}(M \times [0, T])$.

(iii) *If* $r \geqq 2$, *then*

$$\{f \in C^{(r)}(M \times [0, T]); \partial_t f \in C_{\#}^{(r-2)}(M \times [0, T])\} \subset E_1^{[0,r]}(M \times [0, T]) .$$

PROOF. In the above cases, we can construct an $E_1^{[0,r]}$-extension $\tilde{f}$ of f by putting $\tilde{f}(x, t) = f(x, 0)$ for $(x, t) \in M \times [-T_0, 0]$. This proves the lemma.

LEMMA 3.A.7. *Let M be a compact C^∞ manifold. Let $0 \leqq p, q \leqq r$ and $G \geqq 1$. Let α and β be n-tuples of nonnegative integers. Suppose that*:

(i) $f \in E_G^{\{p+|\alpha|+a, r+|\alpha|+a\}}(M \times [0, T])$.

(ii) $g \in E_G^{\{q+|\beta|+b, r+|\beta|+b\}}(M \times [0, T])$.

Then the product $(\partial_x^\alpha \partial_t^a f)(\partial_x^\beta \partial_t^b g)$ belongs to $C^{(r)}(M \times [0, T])$ and satisfies

$$|(\partial_x^\alpha \partial_t^a f)(\partial_x^\beta \partial_t^b g)|_{(r)} \leqq C(|f|_{(r+|\alpha|+a)}|g|_{(q+|\beta|+b)} + |f|_{(p+|\alpha|+a)}|g|_{(r+|\beta|+b)}) .$$

Here C is a constant bounded with r, $|\alpha| + a$, $|\beta| + b$, and G.

PROOF. From Corollary 3.A.3 and Definition 3.A.5, the lemma follows immediately.

REMARK 3.A.8. In Lemma 3.A.7, the constant C is independent of $T \in (0, T_0]$, when G is.

LEMMA 3.A.9. *Let d_1 and d_2 be positive integers. Let D_1 (resp. D_2) be a bounded domain with C^∞ boundary in $\boldsymbol{R}^{d_1}$ (resp. $\boldsymbol{R}^{d_2}$). Let $r \geqq 1$ and $G \geqq 1$. Suppose that*:

(i) $f \in E_G^{\{1,r\}}(\bar{D}_1 \times [0, T])$.

(ii) *The components $g_1, \cdots, g_{d_1}$ of a mapping $g: \bar{D}_2 \times [0, T] \to \bar{D}_1$ belong to $E_G^{\{1,r\}}(\bar{D}_2 \times [0, T])$.*

(iii) $\sum_{i=1}^{d_1} |g_i|_{(1)} \leqq B$.

Then the composed function $(x, t) \mapsto f(g(x, t), t)$ belongs to $C^{(r)}(\bar{D}_2 \times [0, T])$ and satisfies

$$|f(g(x, t), t)|_{(r)} \leqq C\left(|f|_{(r)} + |f|_{(1)} \sum_{i=1}^{d_1} |g_i|_{(r)}\right) .$$

Here C is a constant bounded with r, G, and B.

Lemma 3.A.9 can be proved in the same way as [16, Theorem A.8] or [32, Lemma 1.6], which are analogues for C^r spaces.

Finally we state the following fact, which follows immediately from Definitions 1.B.2 and 1.B.3.

LEMMA 3.A.10. *Let M be a compact C^∞ manifold. Let $\gamma < \delta$ and $r \geqq 0$. Let $B > 0$. Suppose that:*

(i) $f \in C^{(r)}(M \times [\gamma, \delta])$.

(ii) $|f(x, t)| \geqq B^{-1}$ *for* $(x, t) \in M \times [\gamma, \delta]$.

Then the function $(x, t) \mapsto 1/f(x, t)$ *belongs to* $C^{(r)}(M \times [\gamma, \delta])$ *and satisfies*

$$|1/f|_{(r)} \leqq C(1 + |f|_{(r)}) .$$

Here C is a constant bounded with r and B.

(B) Throughout (B), let $\mathscr{L}$ be a differential operator on $\bar{\Omega}_T$ of the form

$$\mathscr{L} = \partial_t - \sum_{i,j=1}^n A_{ij}(x, t)\partial_{x_i}\partial_{x_j} - \sum_{i=1}^n A_i(x, t)\partial_{x_i} - A_0(x, t) \quad \text{for} \quad (x, t) \in \bar{\Omega}_T .$$

DEFINITION 3.B.1. Let $0 \leqq k \leqq r$. Let σ and B be positive constants. We say that $\mathscr{L}$ is a (k, r, σ, B)-$\#$-parabolic operator if:

(i) $A_{ij}, A_i, A_0 \in C^{(r)}(\bar{\Omega}_T)$ for $i, j = 1, \cdots, n$.

(ii) If $r \geqq 2$, then $\partial_t A_{ij}, \partial_t A_i, \partial_t A_0 \in C_\#^{(r-2)}(\bar{\Omega}_T)$ for $i, j = 1, \cdots, n$.

(iii) $\sum_{i,j=1}^n |A_{ij}|_{(k)} + \sum_{i=1}^n |A_i|_{(k)} + |A_0|_{(k)} \leqq B$.

(iv) $\sum_{i,j=1}^n |(A_{ij}|_{t=0})|_r + \sum_{i=1}^n |(A_i|_{t=0})|_r + |(A_0|_{t=0})|_r \leqq B$.

(v) $\sigma^{-1} \sum_{i=1}^n \xi_i^2 \leqq \sum_{i,j=1}^n A_{ij}(x, t)\xi_i\xi_j \leqq \sigma \sum_{i=1}^n \xi_i^2$ for $(x, t) \in \bar{\Omega}_T$ and $\xi \in \boldsymbol{R}^n$.

LEMMA 3.B.2. *Let i be a nonnegative integer. Let σ and B be positive constants. Suppose that:*

(i) *The operator $\mathscr{L}$ is $(\varepsilon_0, \varepsilon_0 + i, \sigma, B)$-$\#$-parabolic.*

(ii) $f \in C_\#^{(\varepsilon_0+i)}(\bar{\Omega}_T)$.

(iii) $u \in C^{(\varepsilon_0+i+2)}(\bar{\Omega}_T)$.

(iv) $\mathscr{L}u = f$ *in* Ω_T.

Then u belongs to $E_G^{\{0, \varepsilon_0+2, \varepsilon_0+3, \cdots, \varepsilon_0+i+2\}}(\bar{\Omega}_T)$. Here G is a constant bounded with i, σ, and B.

PROOF. Let $\tilde{A}_{gh}$, $\tilde{A}_g$, and $\tilde{A}_0$ be extensions of $A_{gh}|_{t=0}$, $A_g|_{t=0}$, and $A_0|_{t=0}$ to $\boldsymbol{R}^n$, respectively, such that:

(i) $\tilde{A}_{gh}, \tilde{A}_g, \tilde{A}_0 \in C^{\varepsilon_0+i}(\boldsymbol{R}^n)$.

(ii) $|\tilde{A}_{gh}|_{\varepsilon_0+j} \leqq C_1|(A_{gh}|_{t=0})|_{\varepsilon_0+j}$, $|\tilde{A}_g|_{\varepsilon_0+j} \leqq C_1|(A_g|_{t=0})|_{\varepsilon_0+j}$, $|\tilde{A}_0|_{\varepsilon_0+j} \leqq C_1|(A_0|_{t=0})|_{\varepsilon_0+j}$, for $j = 0, \cdots, i$.

Here $g, h = 1, \cdots, n$ and C_1 is a constant bounded with i. (This is possible by the Hestenes-Whitney technique. See, e.g., [21, p. 296-297].) Let Ω_1 be a domain with C^∞ boundary in $\boldsymbol{R}^n$ such that:

(i) $\bar{\Omega}_0 \subset \Omega_1$.

(ii) $(2\sigma)^{-1} \sum_{g=1}^n \xi_g^2 \leqq \sum_{g,h=1}^n \tilde{A}_{gh}(x)\xi_g\xi_h \leqq 2\sigma \sum_{g=1}^n \xi_g^2$ for $x \in \bar{\Omega}_1$ and $\xi \in \boldsymbol{R}^n$.

The domain Ω_1 is determined only by Ω_0, σ, and B. Let $\tilde{\alpha}$ be an extension of $u|_{t=0}$ to $\bar{\Omega}_1$ such that:

(i) $\operatorname{supp} \tilde{a} \subset \Omega_1$.

(ii) $\tilde{a} \in C^{\varepsilon_0+i+2}(\bar{\Omega}_1)$.

(iii) $|\tilde{a}|_{\varepsilon_0+j+2} \leqq C_2|(u|_{t=0})|_{\varepsilon_0+j+2}$ for $j = 0, \cdots, i$.

Here C_2 is a constant bounded with i, σ, and B. Let $\partial\Omega_1$ be the boundary of Ω_1. Let w be the solution of the mixed problem:

$$(1) \qquad \left(\partial_t - \sum_{i,j=1}^{n} \tilde{A}_{ij}(x)\partial_{x_i}\partial_{x_j} - \sum_{i=1}^{n} \tilde{A}_i(x)\partial_{x_i} - \tilde{A}_0(x)\right)w = 0 \quad \text{in} \quad \Omega_1 \times [0, T_0] .$$

$$(2) \qquad w|_{t=0} = \tilde{a} .$$

$$(3) \qquad w = 0 \quad \text{on} \quad \partial\Omega_1 \times [0, T_0] .$$

It is known that w exists in $C^{(\varepsilon_0+i+2)}(\bar{\Omega}_1 \times [0, T_0])$ and satisfies $|w|_{(q)} \leqq C_3|\tilde{a}|_q$ for $q = 0, \varepsilon_0 + 2, \varepsilon_0 + 3, \cdots, \varepsilon_0 + i + 2$. Here C_3 is a constant bounded with i. (See [21, Theorem 2.3, p. 16–17] and [21, Theorem 5.2, p. 320].) We easily observe that:

(i) $(\partial_t^j w)|_{t=0} = (\partial_t^j u)|_{t=0}$ on $\bar{\Omega}_0$ for $j = 0, \cdots, [(i + 2)/2]$.

(ii) $|w|_{(q),T_0} \leqq C_2C_3|u|_{(q),T}$ for $q = 0, \varepsilon_0 + 2, \varepsilon_0 + 3, \cdots, \varepsilon_0 + i + 2$.

Let $\tilde{w}$ be the Hestenes-Whitney extension of w to $\bar{\Omega}_1 \times [-T_0, T_0]$. Now we can construct an $E_G^{\{0,\varepsilon_0+2,\varepsilon_0+3,\cdots,\varepsilon_0+i+2\}}$-extension $\tilde{u}$ of u by setting $\tilde{u}(x, t) = \tilde{w}(x, t)$ for $(x, t) \in \bar{\Omega}_0 \times [-T_0, 0]$. This proves the lemma.

LEMMA 3.B.3. *Let i be a nonnegative integer. Let σ and B be positive constants. Suppose that:*

(i) *The operator $\mathscr{L}$ is $(\varepsilon_0, \varepsilon_0 + i, \sigma, B)$-#-parabolic.*

(ii) $f \in C_{\#}^{(\varepsilon_0+i)}(\bar{\Omega}_T)$.

(iii) $a \in C^{\varepsilon_0+i+2}(\bar{\Omega}_0)$.

(iv) $b_1 \in C^{(\varepsilon_0+i+2)}(J_T)$.

(v) $b_2 \in C^{(\varepsilon_0+i+2)}(\Gamma_T)$ *(resp.* $b_2 \in C^{(\varepsilon_0+i+1)}(\Gamma_T)$*).*

Consider the following mixed problem:

$$(1) \qquad \mathscr{L}u = f \quad in \quad \Omega_T .$$

$$(2) \qquad u|_{t=0} = a .$$

$$(3) \qquad u = b_1 \quad on \quad J_T .$$

$$(4) \qquad u = b_2 \quad (resp.\ \partial_\lambda u = b_2) \quad on \quad \Gamma_T .$$

(Here ∂_λ is the partial differentiation with respect to λ in the $(\omega_1, \cdots, \omega_{n-1}, \lambda)$ coordinates.) Suppose that:

(i) *The set $\{a, b_1\}$ satisfies the compatibility condition of order $[(i + 2)/2]$ of the Dirichlet problem for the equation $\mathscr{L}u = f$ on J_0 at $t = 0$.*

(ii) *The set* $\{a, b_2\}$ *satisfies the compactibility condition of order* $[(i+2)/2]$ (*resp.* $[(i+1)/2]$) *of the Dirichlet* (*resp. Neumann*) *problem for the equation* $\mathscr{L}u = f$ *on* Γ_0 *at* $t=0$.
Then the problem (1)-(4) *has a unique solution* u *in* $C^{(\varepsilon_0+i+2)}(\bar{\Omega}_T)$ *which satisfies*

$$|u|_{(\varepsilon_0+i+2)} \leqq C\Big[\Big(\sum_{g,h=1}^{n} |A_{gh}|_{(\varepsilon_0+i)} + \sum_{g=1}^{n} |A_g|_{(\varepsilon_0+i)} + |A_0|_{(\varepsilon_0+i)}\Big) \times (|f|_{(\varepsilon_0)} + |a|_{\varepsilon_0+2} + |b_1|_{(\varepsilon_0+2)} + |b_2|_{(\varepsilon_0+2)}) + (|f|_{(\varepsilon_0+i)} + |a|_{\varepsilon_0+i+2} + |b_1|_{(\varepsilon_0+i+2)} + |b_2|_{(\varepsilon_0+i+2)})\Big]$$

(*resp.* $|u|_{(\varepsilon_0+i+2)} \leqq C[(\sum_{g,h=1}^{n} |A_{gh}|_{(\varepsilon_0+i)} + \sum_{g=1}^{n} |A_g|_{(\varepsilon_0+i)} + |A_0|_{(\varepsilon_0+i)})(|f|_{(\varepsilon_0)} + |a|_{\varepsilon_0+2} + |b_1|_{(\varepsilon_0+2)} + |b_2|_{(\varepsilon_0+1)}) + (|f|_{(\varepsilon_0+i)} + |a|_{\varepsilon_0+i+2} + |b_1|_{(\varepsilon_0+i+2)} + |b_2|_{(\varepsilon_0+i+1)})]$). *Here* C *is a constant bounded with* i, σ, *and* B.

For the proof, refer to [16, Theorem A.14] which is an analogue for the elliptic boundary value problem. With the aid of [21, Theorems 5.2, 5.3, p. 320-321], [21, Theorem 2.3, p. 16-17], Lemmas 3.A.6, 3.A.7 and Lemma 3.B.2, we can prove Lemma 3.B.3 in a manner similar to the proof of [16, Theorem A.14].

4. **Reformulation of the problem.** In order to apply Nash's implicit function theorem, we reformulate Theorem′ (in Section 2).

(A) For $\rho \in V_T$, by using the notations $\mathscr{L}_\rho$ and S_ρ introduced in Section 2 and by setting $U = u \circ e_\rho$, we can reformulate (1_u)-(5_u) as follows:

$$(1_U) \qquad \mathscr{L}_\rho U = 0 \quad \text{in} \quad \Omega_T .$$

$$(2_U) \qquad U|_{t=0} = a_0 .$$

$$(3_U) \qquad U = b_0 \quad \text{on} \quad J_T .$$

$$(4_U) \qquad U = 0 \quad \text{on} \quad \Gamma_T .$$

$$(5_U) \qquad \partial_t\rho + c_0(\partial_\lambda U)S_\rho = 0 \quad \text{on} \quad \Gamma_T .$$

Let us express $\mathscr{L}_\rho$ as

$$\mathscr{L}_\rho = \partial_t - \sum_{i,j=1}^{n} A_{\rho,ij}(x,t)\partial_{x_i}\partial_{x_j} - \sum_{i=1}^{n} A_{\rho,i}(x,t)\partial_{x_i}$$

for $(x,t) \in \bar{\Omega}_T$ with $A_{\rho,ij} = A_{\rho,ji}$. By (2.1)-(2.3), a routine calculation gives us:

(4.1.i) $A_{\rho,ij}(x,t) = A_{ij}(x, \rho(\omega(x),t), \partial_{\omega_1}\rho(\omega(x),t), \cdots, \partial_{\omega_{n-1}}\rho(\omega(x),t))$

for $i, j = 1, \cdots, n$ and $(x,t) \in N_0^- \times [0, T]$, where $(\omega_1, \cdots, \omega_{n-1})$ are local coordinates in Γ_0.

(4.1.ii)
$$A_{\rho,ij}(x, t) = \delta_{ij}$$
for $i, j = 1, \cdots, n$ and $(x, t) \in (\bar{\Omega}_0 - N_0^-) \times [0, T]$, where δ_{ij} is Kronecker's delta.

(4.1.iii)
$$A_{\rho,i}(x, t) = A_i(x, \rho(\omega(x), t), \partial_{\omega_1}\rho(\omega(x), t), \cdots, \partial_{\omega_{n-1}}\rho(\omega(x), t), \partial_t\rho(\omega(x), t), \partial^2_{\omega_1}\rho(\omega(x), t), \partial_{\omega_1}\partial_{\omega_2}\rho(\omega(x), t), \cdots, \partial^2_{\omega_{n-1}}\rho(\omega(x), t))$$
for $i = 1, \cdots, n$ and $(x, t) \in N_0^- \times [0, T]$.

(4.1.iv) $A_{\rho,i}(x, t) = 0$ for $i = 1, \cdots, n$ and $(x, t) \in (\bar{\Omega}_0 - N_0^-) \times [0, T]$.

Here A_{ij} (resp. A_i) are C^∞ functions on $N_0^- \times [-\delta_0, \delta_0] \times \boldsymbol{R}^{n-1}$ (resp. $N_0^- \times [-\delta_0, \delta_0] \times \boldsymbol{R}^{(n^2+n)/2}$). We easily observe that if $\rho \in V_T \cap C_\#^{(r)}(\Gamma_T)$ for $r \geqq r_0$, then $\mathscr{L}_\rho$ is an $(r_0 - 2, r - 2, \sigma_0, B)$-#-parabolic operator (see (B) in Section 2). Here B is a positive constant.

On the other hand, express S_ρ as

(4.2)
$$S_\rho(\omega, t) = S(\omega, \rho(\omega, t), \partial_{\omega_1}\rho(\omega, t), \cdots, \partial_{\omega_{n-1}}\rho(\omega, t))$$
for $(\omega, t) \in \Gamma_T$. By (1.1), a routine calculation gives us
$$S(\omega, \rho(\omega, t), \partial_{\omega_1}\rho(\omega, t), \cdots, \partial_{\omega_{n-1}}\rho(\omega, t)) = \sum_{i=1}^n \left\{\partial_{x_i}\lambda(x(\omega, \rho(\omega, t))) - \sum_{j=1}^{n-1} [\partial_{\omega_j}\rho(\omega, t)][\partial_{x_i}\omega_j(x(\omega, \rho(\omega, t)))]\right\}^2$$
for $(\omega, t) \in \Gamma_T$. As is assumed in Section 1, the function S_ρ has a positive constant B_0^{-1} as a lower bound. Note that $\operatorname{grad}\lambda \neq 0$ on Γ_T because Γ_0 is embedded in $\boldsymbol{R}^n$ without critical points. We easily observe that if $\rho \in V_T \cap C_\#^{(r)}(\Gamma_T)$ for $r \geqq r_0$, then $\partial_t(S_\rho) \in C_\#^{(r-3)}(\Gamma_T)$.

(B) Define a mapping $\mathscr{F}: V_T \to C_\#^{(r_0-2)}(\Gamma_T)$ by

(4.3)
$$\mathscr{F}(\rho) = \partial_t\rho + c_0[(\partial_\lambda U)|_{\lambda=0}]S_\rho \quad \text{for} \quad \rho \in V_T .$$
Here U_ρ is the solution U of (1_U)-(4_U). This is possible by Lemma 3.B.3 and the assumptions (A.2) and (T) in theorem.

Clearly Theorem′ (in Section 2) is reformulated as:

THEOREM′ (reformulated form). *Under the same assumptions as in Theorem′ in Section 2, for a sufficiently small* $T \in (0, T_0]$, *there exists* $\rho \in V_T$ *with* $\mathscr{F}(\rho) = 0$.

Here we suppose ρ is the only unknown of the problem because u is obtained once ρ is determined.

(C) By Lemmas 3.A.9 and 3.B.3, we have

(4.4)
$$|U_\rho|_{(r_0+i)} \leqq C(1 + |\rho|_{(r_0+i)}) \quad \text{for} \quad i = 0, \cdots, 39 \quad \text{and} \quad \rho \in V_T \cap C_\#^{(r_0+i)}(\Gamma_T) .$$

Here C is a constant bounded with $|a_0|_{r_0+39}$ and $|b_0|_{(r_0+39)}$. By (4.3), (4.4) and Lemma 3.A.7, we have:

(4.5.i) $\mathscr{F}(V_T \cap C_{\#}^{(r_0+i)}(\Gamma_T)) \subseteqq C_{\#}^{(r_0+i-2)}(\Gamma_T)$ for $i = 0, \cdots, 40$.

(4.5.ii) $|\mathscr{F}(\rho)|_{(r_0+i-2)} \leqq C(1 + |\rho|_{(r_0+i)})$ for $i = 0, \cdots, 40$ and $\rho \in V_T \cap C_{\#}^{(r_0+i)}(\Gamma_T)$.

Here C is a constant bounded with $|a_0|_{r_0+39}$ and $|b_0|_{(r_0+39)}$.

5. Nash's implicit function theorem. (A) We recall the well-known version of Nash's implicit function theorem due to Moser [22]. For the proof, refer to [22], [34, Chapter 2] and [35].

NASH'S IMPLICIT FUNCTION THEOREM. *Let $E_0, E_1, \cdots,$ and E_{11} (resp. $F_1, F_2, \cdots,$ and F_{11}) be real Banach spaces such that:*

(i) $E_0 \supset E_1 \supset \cdots \supset E_{11}$ *(resp.* $F_1 \supset F_2 \supset \cdots \supset F_{11}$*)*

(ii) $|x|_i \leqq |x|_{i+1}$ *for* $i = 0, \cdots, 11$ *and* $x \in E_{i+1}$ *(resp.* $|y|_i \leqq |y|_{i+1}$ *for* $i = 1, \cdots, 11$ *and* $y \in F_{i+1}$*).*

Here $|\ |_i$ *denotes the norm in* E_i *(resp.* F_i*). Let* δ *be a positive constant and let* $V = \{x \in E_1; |x_1| < \delta\}$. *Let* $\mathscr{F}: V \to F_1$ *be a mapping such that* $\mathscr{F}(V \cap E_i) \subseteqq F_i$ *for* $i = 1, \cdots, 11$.

Suppose that:

(I) *There exists a "smoothing" linear operator* $\mathscr{S}_\theta: E_0 \to E_{11}$ *for* $\theta \geqq 1$ *which satisfies:*

(I.i) $|\mathscr{S}_\theta x|_j \leqq C_1\theta^{j-i}|x|_i$ *for* i, j *with* $0 \leqq i \leqq j \leqq 11$ *and* $x \in E_i$.

(I.ii) $|x - \mathscr{S}_\theta x|_i \leqq C_1\theta^{-(j-i)}|x|_j$ *for* i, j *with* $0 \leqq i \leqq j \leqq 11$ *and* $x \in E_j$.

(II) *There exists a "Fréchet derivative" linear operator* $D\mathscr{F}(x)$: $E_1 \to F_1$ *for* $x \in V$ *which satisfies:*

(II.i) $|D\mathscr{F}(x)h|_1 \leqq C_2|h|_1$ *for* $h \in E_1$.

(II.ii) $|\mathscr{F}(x + h) - \mathscr{F}(x) - D\mathscr{F}(x)h|_1 \leqq C_2|h|_1^2$ *for* $h \in E_1$ *with* $x + h \in V$.

(III) *There exists a "right inverse" linear operator* $\mathscr{J}(x): F_1 \to E_0$ *for* $x \in V$ *which satisfies:*

(III.i) $\mathscr{J}(x)F_i \subseteqq E_{i-1}$ *for* $i = 1, \cdots, 11$ *and* $x \in V \cap E_i$.

(III.ii) $D\mathscr{F}(x)\mathscr{J}(x)y = y$ *for* $x \in V \cap E_2$ *and* $y \in F_2$.

(III.iii) $|\mathscr{J}(x)y|_0 \leqq C_3|y|_1$ *for* $y \in F_1$.

(III.iv) $|\mathscr{J}(x)\mathscr{F}(x)|_{i-1} \leqq C_3(1 + |x|_i)$ *for* $i = 1, \cdots, 11$ *and* $x \in V \cap E_i$.

Here C_1, C_2, and C_3 are constants.

Then there is a positive constant ε determined by C_1, C_2, C_3 and δ such that: If $|\mathscr{F}(0)|_1 < \varepsilon$, then there exists $x \in V$ with $\mathscr{F}(x) = 0$.

(B) We prove Theorem′ (in Section 4) by using Nash's implicit function theorem. The setting in which we apply Nash's implicit function theorem is:

(i) $E_i = C_\#^{(r_0-4+4i)}(\Gamma_T)$ for $i = 0, \cdots, 11$.

(ii) $F_i = C_\#^{(r_0-6-(\varepsilon_0/2)+4i)}(\Gamma_T)$ for $i = 1, \cdots, 11$.

(iii) $\delta = \delta_0$ (so that $V = V_T$).

(iv) The mapping $\mathscr{F}$ is the one defined by (4.3) (see (4.5.i)).

In the following sections, we show that the mapping $\mathscr{F}$ and the spaces E_i, F_i together with δ satisfy the conditions in Nash's implicit function theorem.

6. Smoothing operators. We verify the condition (I) in Nash's theorem. Choose a function ζ_0 on $\boldsymbol{R}^{n-1} \times \boldsymbol{R}$ so that:

(i) $\zeta_0 \in C_0^\infty(\boldsymbol{R}^{n-1} \times \boldsymbol{R})$.

(ii) $\zeta_0(x, t) = 0$ for $x \in \boldsymbol{R}^{n-1}$ and $t \leqq 0$.

(iii) $\int_{\boldsymbol{R}^{n-1}\times\boldsymbol{R}} \zeta_0(x, t)dxdt = 1$, where $dx = \prod_{i=1}^{n-1} dx_i$.

Define a sequence of functions $\{\zeta_0, \zeta_1, \cdots\}$ by

$$\zeta_i(x, t) = (1 - 2^i)^{-1}[\zeta_{i-1}(x, t) - 2^{i+n+1}\zeta_{i-1}(2x, 4t)]$$

for $i = 1, 2, \cdots$ and $(x, t) \in \boldsymbol{R}^{n-1} \times \boldsymbol{R}$. For $i = 0, 1, \cdots$, we can easily verify that:

(i) $\zeta_i \in C_0^\infty(\boldsymbol{R}^{n-1} \times \boldsymbol{R})$.

(ii) $\zeta_i(x, t) = 0$ for $x \in \boldsymbol{R}^{n-1}$ and $t \leqq 0$.

(iii) $\int_{\boldsymbol{R}^{n-1}\times\boldsymbol{R}} \zeta_i(x, t)dxdt = 1$.

(iv) $\int_{\boldsymbol{R}^{n-1}\times\boldsymbol{R}} x^\alpha t^a \zeta_i(x, t)dxdt = 0$ if $0 < |\alpha| + 2a \leqq i$.

Let $i_0 = [r_0] + 40$. For $\bar\theta \geqq 1$, define a linear operator

$$\mathscr{S}_{\bar\theta}: C_\#^{(0)}(\boldsymbol{R}^{n-1} \times [0, T]) \to C_\#^{(i_0)}(\boldsymbol{R}^{n-1} \times [0, T])$$

by

$$[\mathscr{S}_{\bar\theta} f](x, t) = \int_0^T \left\{\int_{\boldsymbol{R}^{n-1}} [\bar\theta^{n+1}\zeta_{i_0}(\bar\theta(x - \xi), \bar\theta^2(t - \tau))f(\xi, \tau)]d\xi\right\}d\tau$$
$$\text{for } f \in C_\#^{(0)}(\boldsymbol{R}^{n-1} \times [0, T]) \text{ and } (x, t) \in \boldsymbol{R}^{n-1} \times [0, T] .$$

The fact that $\mathscr{S}_{\bar\theta} f$ is in $C_\#^{(i_0)}$ follows from the property (ii) of ζ_i. We can verify that:

(i) $|\mathscr{S}_{\bar\theta} f|_{(r)} \leqq C\bar\theta^{r-q}|f|_{(q)}$ for real numbers q, r with $0 \leqq q \leqq r \leqq r_0 + 40$ and for $f \in C_\#^{(q)}(\boldsymbol{R}^{n-1} \times [0, T])$.

(ii) $|f-\mathscr{S}_{\bar\theta}f|_{(q)} \leqq C\bar\theta^{-(r-q)}|f|_{(r)}$ for real numbers q, r with $0 \leqq q \leqq r \leqq r_0+40$ and for $f \in C_{\#}^{(r)}(\boldsymbol{R}^{n-1}\times[0, T])$.

Here C is a constant determined by ζ_{i_0}. For the proof, refer to [16, Theorem A.10] where analogous inequalities are proved for C^r spaces. The above inequalities (i) and (ii) can be proved similarly.

Set $\theta=\bar\theta^{1/4}$. Consider a finite covering of Γ_0 by coordinate neighborhoods and a C^∞ partition of unity subordinate to it. Then, for $\theta \geqq 1$, we can construct a linear operator $\mathscr{S}_\theta$: $C_{\#}^{(r_0-4)}(\Gamma_T) \to C_{\#}^{(r_0+40)}(\Gamma_T)$ which satisfies:

(i) $|\mathscr{S}_\theta f|_{(r_0-4+4j)} \leqq C\theta^{j-i}|f|_{(r_0-4+4i)}$ for i, j with $0 \leqq i \leqq j \leqq 11$ and $f \in C_{\#}^{(r_0-4+4i)}(\Gamma_T)$.

(ii) $|f-\mathscr{S}_\theta f|_{(r_0-4+4i)} \leqq C\theta^{-(j-i)}|f|_{(r_0-4+4j)}$ for i, j with $0 \leqq i \leqq j \leqq 11$ and $f \in C_{\#}^{(r_0-4+4j)}(\Gamma_T)$.

Here C is a constant. This proves (I).

7. Fréchet differentiability of $\mathscr{F}$. We verify the condition (II) in Nash's implicit function theorem.

(A) Let $\mathscr{E}_\rho=\partial_t-\mathscr{L}_\rho$. For $\rho \in V_T$ and $\delta\rho \in C_{\#}^{(r_0)}(\Gamma_T)$, let $\delta\mathscr{E}_\rho = \sum_{i,j=1}^n(\delta A_{\rho,ij})\partial_{x_i}\partial_{x_j}+\sum_{i=1}^n(\delta A_{\rho,i})\partial_{x_i}$ on Ω_T. Here $\delta A_{\rho,ij}$ and $\delta A_{\rho,i}$ are defined by:

$$(7.1.\text{i})\quad \delta A_{\rho,ij}=[(\partial/\partial\rho)A_{ij}]\delta\rho+\sum_{h=1}^n\{[\partial/\partial(\partial_{\omega_h}\rho)]A_{ij}\}\partial_{\omega_h}\delta\rho \quad \text{on} \quad N_0^-\times[0, T]\,.$$

$$(7.1.\text{ii})\quad \delta A_{\rho,ij}=0 \quad \text{on} \quad (\bar\Omega_0-N_0^-)\times[0, T]\,.$$

$$(7.1.\text{iii})\quad \delta A_{\rho,i}=[(\partial/\partial\rho)A_i]\delta\rho+\sum_{h=1}^{n-1}\{[\partial(\partial_{\omega_h}\rho)]A_i\}\partial_{\omega_h}\delta\rho+\{[\partial/\partial(\partial_t\rho)]A_i\}\partial_t\delta\rho$$
$$+\sum_{1\leqq g\leqq h\leqq n-1}\{[\partial/\partial(\partial_{\omega_g}\partial_{\omega_h}\rho)]A_i\}\partial_{\omega_g}\partial_{\omega_h}\delta\rho \quad \text{on} \quad N_0^-\times[0, T]\,.$$

$$(7.1.\text{iv})\quad \delta A_{\rho,i}=0 \quad \text{on} \quad N_0^-\times[0, T]\,.$$

(See (4.1).)

Let δU_ρ be the solution of the problem:

$$(1_{\delta U})\quad \mathscr{L}_\rho(\delta U_\rho)=(\delta\mathscr{E}_\rho)U_\rho \quad \text{in} \quad \Omega_T\,.$$

$$(2_{\delta U})\quad \delta U_\rho|_{t=0}=0\,.$$

$$(3_{\delta U})\quad \delta U_\rho=0 \quad \text{on} \quad J_T\,.$$

$$(4_{\delta U})\quad \delta U_\rho=0 \quad \text{on} \quad \Gamma_T\,.$$

Since $\delta\rho$ belongs to $C_{\#}^{(r_0)}(\Gamma_T)$, the compatibility condition of order $[r_0/2]$ of $(1_{\delta U})$-$(4_{\delta U})$ is satisfied. Then, by Lemmas 3.A.7, 3.B.2 and 3.B.3 and (4.4), the function δU_ρ is well-defined in $C_{\#}^{(r_0)}(\bar\Omega_T)$ and satisfies

$$(7.2)\quad |\delta U_\rho|_{(r_0)} \leqq C|\delta\rho|_{(r_0)}\,.$$

Here C is a constant bounded with $|a_0|_{r_0}$ and $|b_0|_{(r_0)}$.

Further we observe that if $\rho + \delta\rho$ belongs to V_T, then

$$(7.3) \qquad |U_{\rho+\delta\rho} - U_\rho - \delta U_\rho|_{(r_0)} \leqq C|\delta\rho|^2_{(r_0)} .$$

Here C is a constant bounded with $|a_0|_{r_0}$ and $|b_0|_{(r_0)}$. In fact we have:

$$(1) \qquad \mathscr{L}_{\rho+\delta\rho}(U_{\rho+\delta\rho} - U_\rho - \delta U_\rho) = (\mathscr{E}_{\rho+\delta\rho} - \mathscr{E}_\rho - \delta\mathscr{E}_\rho)U_\rho + (\mathscr{E}_{\rho+\delta\rho} - \mathscr{E}_\rho)\delta U_\rho \quad \text{on} \quad \Omega_T .$$

$$(2) \qquad (U_{\rho+\delta\rho} - U_\rho - \delta U_\rho)|_{t=0} = 0 .$$

$$(3) \qquad U_{\rho+\delta\rho} - U_\rho - \delta U_\rho = 0 \quad \text{on} \quad J_T .$$

$$(4) \qquad U_{\rho+\delta\rho} - U_\rho - \delta U_\rho = 0 \quad \text{on} \quad \Gamma_T .$$

By Lemmas 3.A.7, 3.B.2 and 3.B.3 and by (4.4), (7.1) and (7.2), we obtain (7.3).

(B) For $\rho \in V_T$, define a linear operator $D\mathscr{F}(\rho)$: $C_\#^{(r_0)}(\Gamma_T) \to C_\#^{(r_0-2)}(\Gamma_T)$ by

$$(7.4) \qquad D\mathscr{F}(\rho)\delta\rho = \partial_t\delta\rho + c_0[(\partial_\lambda U_\rho)|_{\lambda=0}]\delta S_\rho + c_0\{[\partial_\lambda(\delta U_\rho)]|_{\lambda=0}\}S_\rho$$

for $\delta\rho \in C_\#^{(r_0)}(\Gamma_T)$ (see (4.3)). Here

$$(7.5) \qquad \delta S_\rho = [(\partial/\partial\rho)S]\delta\rho + \sum_{i=1}^{n-1} \{[\partial/\partial(\partial_{\omega_i}\rho)S\}\partial_{\omega_i}\delta\rho .$$

(See (4.2).) By (4.4), (7.2), (7.3) and (7.4), we have

(i) $|D\mathscr{F}(\rho)\delta\rho|_{(r_0-2)} \leqq C|\delta\rho|_{(r_0)}$ for $\delta\rho \in C_\#^{(r_0)}(\Gamma_T)$.

(ii) $|\mathscr{F}(\rho + \delta\rho) - \mathscr{F}(\rho) - D\mathscr{F}(\rho)\delta\rho|_{(r_0-1)} \leqq C|\delta\rho|^2_{(r_0)}$ for $\delta\rho \in C_\#^{(r_0)}(\Gamma_T)$ with $\rho + \delta\rho \in V_T$.

Here C is a constant bounded with $|a_0|_{r_0}$ and $|b_0|_{(r_0)}$. This proves (II).

8. Technical lemmas II. We prove some lemmas which are used to verify (III). As in Section 3, by "a manifold", we mean both one with boundary and one without boundary.

(A) We begin with the following fact.

LEMMA 8.A.1. *Let M be a compact C^∞ manifold. Let $r \geqq 0$. Let f belong to $C_\#^{(r)}(M \times [0, T])$. Then the function $(x, t) \mapsto \int_0^t f(x, \tau)d\tau$ belongs to $C_\#^{(r)}(M \times [0, T])$ and satisfies*

$$\left|\int_0^t f(x, \tau)d\tau\right|_{(r),t} \leqq \int_0^t |f|_{(r),\tau}d\tau \quad \text{for} \quad t \in [0, T] .$$

PROOF. Extend f to $M \times (-\infty, T]$ by setting $f(x, t) = 0$ for $(x, t) \in M \times (-\infty, 0)$. Since $f \in C_\#^{(r)}$, we have:

(i) The $C^{(r)}$ norm of f is preseved in the above extension.

(ii) $\partial_x^\alpha\partial_t^a\left(\int_0^t f(x, \tau)d\tau\right) = \int_0^t \partial_x^\alpha\partial_t^a f(x, \tau)d\tau$ if $|\alpha| + 2a \leqq [r]$.

(iii) $\int_0^s \partial_x^\alpha \partial_t^a f(x, \tau) d\tau = \int_0^t \partial_x^\alpha \partial_t^a f(x, \tau - t + s) d\tau$ if $|\alpha| + 2a \leqq [r]$ and $0 \leqq s \leqq t \leqq T$.

By (ii) and (iii)

$$\left| \int_0^t \partial_x^\alpha \partial_t^a f(x, \tau) d\tau - \int_0^s \partial_x^\alpha \partial_t^a f(x, \tau) d\tau \right| \Big/ |t - s|^{\mu/2}$$
$$\leqq \int_0^t (|\partial_x^\alpha \partial_t^a f(x, \tau) - \partial_x^\alpha \partial_t^a f(x, \tau - t + s)| / |t - s|^{\mu/2}) d\tau$$

if $|\alpha| + 2a \leqq [r]$, $0 \leqq s \leqq t \leqq T$ and $0 < \mu < 2$. Then the lemma follows from (i).

We introduce a special class of operators.

DEFINITION 8.A.2. Let M and M' be compact C^∞ manifolds. Let $0 \leqq k \leqq r$. Let C and N be positive constants. We say that a linear operator $\mathscr{K}: C_\#^{(r)}(M \times [0, T]) \to C_\#^{(r)}(M' \times [0, T])$ is (k, r, C, N)-balanced (resp. (k, r, C, N)-integral-balanced) if:

(i) $|\mathscr{K} f|_{(k),t} \leqq C |f|_{(k),t}$ $\left(\text{resp. } |\mathscr{K} f|_{(k),t} \leqq C \int_0^t |f|_{(k),\tau} d\tau\right)$ for $t \in [0, T]$.

(ii) $|\mathscr{K} f|_{(r),t} \leqq C |f|_{(r),t} + CN |f|_{(k),t}$ $\left(\text{resp. } |\mathscr{K} f|_{(r),t} \leqq C \int_0^t |f|_{(r),\tau} d\tau + CN \int_0^t |f|_{(k),\tau} d\tau\right)$ for $t \in [0, T]$.

LEMMA 8.A.3. *Let M and M' be compact C^∞ manifolds. Let $0 \leqq k \leqq r$. Let C and N be positive constants. Let*

$$\mathscr{K}: C_\#^{(r)}(M \times [0, T]) \to C_\#^{(r)}(M' \times [0, T])$$

be a (k, r, C, N)-balanced linear operator. Define a linear operator

$$\mathscr{J}: C_\#^{(r)}(M \times [0, T]) \to C_\#^{(r)}(M' \times [0, T])$$

by

$$(\mathscr{J} f)(x, t) = \int_0^t (\mathscr{K} f)(x, \tau) d\tau \quad \textit{for} \quad (x, t) \in M' \times [0, T] .$$

Then $\mathscr{J}$ is (k, r, C, N)-integral-balanced.

PROOF. The lemma follows immediately from Lemma 8.A.1.

LEMMA 8.A.4. *Let M and M' be compact C^∞ manifolds. Let $0 \leqq k \leqq r$. Let C and N be positive constants. Let $\mathscr{K}: C_\#^{(r)}(M \times [0, T]) \to C_\#^{(r)}(M \times [0, T])$ and $\mathscr{K}': C_\#^{(r)}(M' \times [0, T]) \to C_\#^{(r)}(M \times [0, T])$ be (k, r, C, N)-integral-balanced linear operators. Let g belong to $C_\#^{(r)}(M' \times [0, T])$. Then the equation $u - \mathscr{K} u = \mathscr{K}' g$ has a unique solution $u \in C_\#^{(r)}(M \times [0, T])$. Further the linear operator $(I - \mathscr{K})^{-1} \mathscr{K}': C_\#^{(r)}(M' \times [0, T]) \to C_\#^{(r)}(M \times [0, T])$*

is (k, r, C', N)-integral-balanced, where I denotes the identity operator and C' is a constant bounded with C.

PROOF. By induction on i, we have:

(i) $|\mathscr{K}^i\mathscr{K}'g|_{(k),t} \leqq C^{i+1}(t^i/i!)\int_0^t |g|_{(k),\tau}d\tau$ for $i = 0, 1, \cdots$ and $t \in [0, T]$.

(ii) $|\mathscr{K}^i\mathscr{K}'g|_{(r),t} \leqq C^{i+1}(t^i/i!)\int_0^t |g|_{(r),\tau}d\tau + (i+1)C^{i+1}(t^i/i!)N\int_0^t |g|_{(k),\tau}d\tau$ for $i = 0, 1, \cdots$ and $t \in [0, T]$.

Then, from the estimates of the Neumann series $u = \mathscr{K}'g + \mathscr{K}\mathscr{K}'g + \cdots$, the lemma follows.

(B) Throughout (B), we use the following notations.

(a) $\{U_1, \cdots, U_\kappa\}$ is a finite covering of Γ_0 by coordinate neighborhoods and let $(\omega_1, \cdots, \omega_{n-1})$ be local coordinates.

(b) $\{\eta_1, \cdots, \eta_\kappa\}$ is a C^∞ partition of unity subordinate to $\{U_1, \cdots, U_\kappa\}$.

(c) $(N_0^-)_\nu = \{x \in N_0^-; \omega(x) \in U_\nu\}$ for $\nu = 1, \cdots, \kappa$.

(d) Let $\mathscr{H}_1$ be a differential operator on Γ_T of the form $\mathscr{H}_1 = \partial_t + \sum_{i=1}^{n-1} H_{\nu,i}(\omega, t)\partial_{\omega_i}$ for $\nu = 1, \cdots, \kappa$ and $(\omega, t) \in U_\nu \times [0, T]$.

(e) Let $\mathscr{P}_1$ be a differential operator on $\bar{\Omega}_T$ of the form:

(e.i) $$\mathscr{P}_1 = \partial_t + \chi_0(\lambda(x)) \sum_{i=1}^{n-1} H_{\nu,i}(\omega(x), t)\partial_{\omega_i} + \chi_0(\lambda(x))H_\lambda(\omega(x), t)\partial_\lambda$$

for $\nu = 1, \cdots, \kappa$ and $(x, t) \in (N_0^-)_\nu \times [0, T]$. Here χ_0 is the function introduced in Section 1 and ∂_{ω_i} is the partial differentiation with respect to ω_i in the $(\omega_1, \cdots, \omega_{n-1}, \lambda)$ coordinates.

(e.ii) $$\mathscr{P}_1 = \partial_t \quad \text{for} \quad (x, t) \in (\bar{\Omega}_0 - N_0^-) \times [0, T].$$

DEFINITION 8.B.1. Let $r \geqq 0$. Then $|H|_{(r)} = \sum_{\nu=1}^{\kappa} \sum_{i=1}^{n-1} |\eta_\nu H_{\nu,i}|_{(r)}$, where $\eta_\nu H_{\nu,i}$ is regarded as a function on $\boldsymbol{R}^{n-1} \times [0, T]$ in the canonical manner.

DEFINITION 8.B.2. Let $0 \leqq k \leqq r$. Let B be a positive constant. We say that $\mathscr{H}_1$ is a (k, r, B)-fine operator if:

(i) $\eta_\nu H_{\nu,i} \in C_\#^{(r)}(R^{n-1} \times [0, T])$ for $\nu = 1, \cdots, \kappa$ and $i = 1, \cdots, n-1$.

(ii) $|H|_{(k)} \leqq B$.

We say that $\mathscr{P}_1$ is a (k, r, B)-λ-fine operator if:

(i) $\eta_\nu H_{\nu,i} \in C_\#^{(r)}(R^{n-1} \times [0, T])$ for $\nu = 1, \cdots, \kappa$ and $i = 1, \cdots, n-1$.

(ii) $H_\lambda \in C_\#^{(r)}(\Gamma_T)$.

(iii) $|H|_{(k)} + |H_\lambda|_{(k)} \leqq B$.

(iv) $H_\lambda(\omega, t) \geqq 0$ for $(\omega, t) \in \Gamma_T$.

Here $\eta_\nu H_{\nu,i}$ is regarded as a function on $\boldsymbol{R}^{n-1} \times [0, T]$ in the canonical manner.

DEFINITION 8.B.3. Let f be a mapping of Γ_T into Γ_0. Then, for $i=1,\cdots,n$, a function f_i on Γ_T is the i-th component of f when it is regarded as a mapping of Γ_T into $\boldsymbol{R}^n$. Further, let $r \geqq 0$ and let $f_1, f_2, \cdots, f_n$ belong to $C^{(r)}(\Gamma_T)$. Then $|f|_{(r)} = \sum_{i=1}^n |f_i|_{(r)}$.

DEFINITION 8.B.4. Let I be an interval in $\boldsymbol{R}$. Let D be a domain with piecewise-C^1 boundary in $\boldsymbol{R}^n \times I$. Let $r \geqq 0$. Let f be a mapping of $\bar{D}$ into $\boldsymbol{R}^n$ whose components $f_1, f_2, \cdots, f_n$ belong to $C^{(r)}(\bar{D})$. Then $|f|_{(r)} = \sum_{i=1}^n |f_i|_{(r)}$.

LEMMA 8.B.5. *Let $2 \leqq k \leqq r$. Let B be a positive constant. Suppose that $\mathscr{H}_1$ is a (k, r, B)-fine operator. Define a mapping ϕ: $\Gamma_T \to \Gamma_0$ so that the mapping $t \mapsto (\phi(\omega, t), t)$ is the characteristic curve of $\mathscr{H}_1$ starting from $\omega \in \Gamma_0$ at $t=0$. Define a mapping ψ: $\Gamma_T \to \Gamma_0$ by $\phi(\psi(\omega, t), t) = \omega$ for $(\omega, t) \in \Gamma_T$. (Since Γ_0 is a compact C^∞ manifold without boundary, ϕ and ψ are well-defined.) Then we have:*

(i) $\phi_1, \cdots, \phi_n \in C^{(r)}(\Gamma_T)$.

(ii) $\partial_t\phi_1, \cdots, \partial_t\phi_n \in C_{\#}^{(r-2)}(\Gamma_T)$.

(iii) $|\phi|_{(r)} \leqq C(1 + |H|_{(r)})$.

Here C is a constant bounded with r and B. Further, suppose that $0 < T \leqq C_B^{-1}$ where C_B is a constant determined by B. Then we have:

(i′) $\psi_1, \cdots, \psi_n \in C^{(r)}(\Gamma_T)$.

(ii′) $\partial_t\psi_1, \cdots, \partial_t\psi_n \in C_{\#}^{(r-2)}(\Gamma_T)$.

(iii′) $|\psi|_{(r)} \leqq C'(1 + |H|_{(r)})$.

Here C' is a constant bounded with r and B.

PROOF. Let D be a bounded domain in $\boldsymbol{R}^n$ with $\Gamma_0 \subset D$. Extend $\mathscr{H}_1$ to $\bar{D} \times [0, T]$ so that:

(i) The extended operator $\tilde{\mathscr{H}}_1$ is of the form $\tilde{\mathscr{H}}_1 = \partial_t + \sum_{i=1}^n \tilde{H}_i(x, t)\partial_{x_i}$ with $\tilde{H}_i \in C_{\#}^{(r)}(\bar{D} \times [0, T])$.

(ii) $\operatorname{supp} \tilde{H}_i \subset D$ for $i = 1, \cdots, n$.

(iii) $|\tilde{H}|_{(q)} \leqq C_1 |H|_{(q)}$ for $q \in [0, r]$.

Here C_1 is a constant bounded with r and $|\tilde{H}|_{(q)} = \sum_{i=1}^n |\tilde{H}_i|_{(q)}$. Define a mapping $\tilde{\phi}$: $\bar{D} \times [0, T] \to \boldsymbol{R}^n$ by:

$$(*.1) \qquad d_t\tilde{\phi}_i(x, t) = \tilde{H}_i(\tilde{\phi}(x, t), t)$$

for $i = 1, \cdots, n$ and $(x, t) \in \bar{D} \times [0, T]$, where $\tilde{\phi}_i$ is the i-th component of $\tilde{\phi}$.

$$(*.2) \qquad \tilde{\phi}(x, 0) = x \quad \text{for} \quad x \in \bar{D}\,.$$

Clearly $\tilde{\phi}$ is an extension of ϕ with $\tilde{\phi}(\bar{D} \times [0, T]) = \bar{D}$. Since $H_i \in C^1$, the mapping $\tilde{\phi}$ is C^1. By (*.1), we easily observe that: If $2 \leqq q \leqq r$

and $\tilde{\phi}_1, \cdots, \tilde{\phi}_n \in C^{(q)}(\bar{D} \times [0, T])$, then $\partial_t\tilde{\phi}_1, \cdots, \partial_t\tilde{\phi}_n \in C_{\#}^{(q-2)}(\bar{D} \times [0, T])$.

Differentiating (*.1) and (*.2) with respect to x_j, we have:

$$(1) \qquad d_t[\partial_{x_j}\tilde{\phi}_i(x, t)] = \sum_{h=1}^{n} [\partial_{x_h}\tilde{H}_i(\tilde{\phi}(x, t), t)][\partial_{x_j}\tilde{\phi}_h(x, t)] .$$

$$(2) \qquad \partial_{x_j}\tilde{\phi}_i(x, 0) = \delta_{ij} .$$

Hence we have

$$(**) \qquad \partial_{x_j}\tilde{\phi}_i(x, t) = \delta_{ij} + \sum_{h=1}^{n} \int_0^t [\partial_{x_h}\tilde{H}_i(\phi(x, \tau), \tau)][\partial_{x_j}\tilde{\phi}_h(x, \tau)]d\tau$$

for $i, j = 1, \cdots, n$ and $(x, t) \in \bar{D} \times [0, T]$.

Let W be the vector space of continuous mappings of $\bar{D} \times [0, T]$ into $\boldsymbol{R}^n$. Define a linear operator $\mathscr{G}: W \rightarrow W$ by

$$(\mathscr{G}f)_i(x, t) = \sum_{h=1}^{n} \int_0^t [\partial_{x_h}\tilde{H}_i(\tilde{\phi}(x, \tau), \tau)][f_h(x, \tau)]d\tau$$

for $i = 1, \cdots, n$ and $(x, t) \in \bar{D} \times [0, T]$. For $i = 1, \cdots, n$, define a mapping δ_i in W by $\delta_i(x, t) = (\delta_{i1}, \cdots, \delta_{in})$ for $(x, t) \in \bar{D} \times [0, T]$. Then, by (**), we have $\partial_{x_i}\tilde{\phi} = \delta_i + \mathscr{G}\delta_i + \mathscr{G}^2\delta_i + \cdots$. This and (*.1) give $|\tilde{\phi}|_{(1)} \leqq C_2$. Here C_2 is a constant bounded with B. Further, by Lemma 3.A.9 and an argument similar to that in the proof of Lemma 8.A.4, we observe that if $1 \leqq q \leqq r - 1$ and $\tilde{\phi}_1, \cdots, \tilde{\phi}_n \in C^{(q)}(\bar{D} \times [0, T])$, then $\partial_{x_i}\tilde{\phi}_1, \cdots, \partial_{x_i}\tilde{\phi}_n \in C^{(q)}(\bar{D} \times [0, T])$ and $|\partial_{x_i}\tilde{\phi}|_{(q)} \leqq C_3(1 + |H|_{(q+1)} + |\tilde{\phi}|_{(q)})$. Here C_3 is a constant bounded with r and B.

Note that if $f, \partial_{x_1}f, \cdots, \partial_{x_n}f, \partial_t f \in C^{(q)}(\bar{D} \times [0, T])$, then $f \in C^{(q+1)}(\bar{D} \times [0, T])$ and $|f|_{(q+1)} \leqq C_4(|f|_{(q)} + \sum_{i=1}^{n} |\partial_{x_i}f|_{(q)} + |\partial_t f|_{(q)})$. Consequently, with the aid of (*.1), we have: If $1 \leqq q \leqq r - 1$ and $\tilde{\phi}_1, \cdots, \tilde{\phi}_n \in C^{(q)}(\bar{D} \times [0, T])$, then $\tilde{\phi}_1, \cdots, \tilde{\phi}_n \in C^{(q+1)}(\bar{D} \times [0, T])$ and $|\tilde{\phi}|_{(q+1)} \leqq C_5(1 + |H|_{(q+1)} + |\tilde{\phi}|_{(q)})$. Here C_5 is a constant bounded with r and B. From this, we obtain (i)-(iii) in the lemma at once.

Define a mapping $\tilde{\psi}: \bar{D} \times [0, T] \rightarrow \bar{D}$ by

$$(***) \qquad \tilde{\phi}(\tilde{\psi}(x, t), t) = x \quad \text{for} \quad (x, t) \in \boldsymbol{R}^n \times [0, T] .$$

Clearly $\tilde{\psi}$ is well-defined and is an extension of ψ. Since H_i is C^1, the mapping $\tilde{\psi}$ is C^1.

By (*.1) and (*.2), we have $\tilde{\phi}_i(x, t) = x_i + \int_0^t \tilde{H}_i(\tilde{\phi}(x, \tau), \tau)d\tau$ for $i = 1, \cdots, n$ and $(x, t) \in \boldsymbol{R}^n \times [0, T]$. Hence there is a positive constant C_B determined by B such that if $0 < T \leqq C_B^{-1}$, then $|\det[\partial_{x_i}\tilde{\phi}_j(x, t)]| \geqq 1/2$ for $(x, t) \in \bar{D} \times [0, T]$. On the other hand, by (***), we have:

(i) $\sum_{h=1}^{n} [\partial_{x_h}\tilde{\phi}_i(\tilde{\psi}(x, t), t)][\partial_{x_j}\tilde{\psi}_h(x, t)] = \delta_{ij}$ for $i, j = 1, \cdots, n$ and $(x, t) \in \bar{D} \times [0, T]$.

(ii) $\sum_{h=1}^{n}[\partial_{x_h}\tilde{\phi}_i(\tilde{\psi}(x, t), t)][\partial_t\tilde{\psi}_h(x, t)]+\partial_t\tilde{\phi}_i(\tilde{\psi}(x, t), t)=0$ for $i=1, \cdots, n$ and $(x, t)\in\bar{D}\times[0, T]$.

Further, note that $\partial_t\tilde{\phi}_i(\tilde{\psi}(x, t), t) = \tilde{H}_i(\tilde{\phi}(\tilde{\psi}(x, t), t), t)$. Then, with the aid of Lemmas 3.A.9 and 3.A.10, by an argument similar to that in the proof of (i)-(iii) in the lemma, we obtain (i′)-(iii′) in the lemma. This completes the proof of Lemma 8.B.5.

LEMMA 8.B.6. *Let* $2\leqq k\leqq r$. *Let* B *be a positive constant. Suppose that* $0<T\leqq C_B^{-1}$ *where* C_B *is a constant determined by* B. *Suppose that* $\mathscr{P}_1$ *is a* (k, r, B)-λ-*fine operator. Define a mapping* $\phi\colon\tilde{\Omega}_T\to\bar{\Omega}_0$ *so that*:

(i) *The mapping* $t\mapsto(\phi(x, t), t)$ *is the characteristic curve of* $\mathscr{P}_1$ *starting from* $x\in\bar{\Omega}_0$ *at* $t=0$.

(ii) $\tilde{\Omega}_T=\{(x, t);\ x\in\bar{\Omega}_0$ *and* $t\in[0, T_x]\}$.

Here T_x *is the largest number in* $[0, T]$ *such that* $\phi(x, t)$ *can be defined on* $[0, T_x]$. *(We may assume that the boundary of* $\tilde{\Omega}_T$ *in* $\boldsymbol{R}^n\times[0, T]$ *is* C^1.) *Define a mapping* $\psi\colon\bar{\Omega}_T\to\bar{\Omega}_0$ *by* $\phi(\psi(x, t), t)=x$ *for* $(x, t)\in\bar{\Omega}_T$. *(Since* $H_\lambda(\omega, t)\geqq 0$ *for* $(\omega, t)\in\Gamma_T$, *the mapping* ψ *is well-defined.) Then we have*:

(i) $\phi_1, \cdots, \phi_n\in C^{(r)}(\tilde{\Omega}_T)$.

(ii) $\partial_t\phi_1, \cdots, \partial_t\phi_n\in C_\#^{(r-2)}(\tilde{\Omega}_T)$.

(iii) $|\phi|_{(r)}\leqq C(1+|H|_{(r)}+|H_\lambda|_{(r)})$.

(iv) $\psi_1, \cdots, \psi_n\in C^{(r)}(\bar{\Omega}_T)$.

(v) $\partial_t\psi_1, \cdots, \partial_t\psi_n\in C_\#^{(r-2)}(\bar{\Omega}_T)$.

(vi) $|\psi|_{(r)}\leqq C(1+|H|_{(r)}+|H_\lambda|_{(r)})$.

Here C *is a constant bounded with* r *and* B.

PROOF. By an argument similar to the proof of Lemma 8.B.5, the lemma is proved.

LEMMA 8.B.7. *Let* $2\leqq k\leqq r$. *Let* B, C, *and* N *be positive constants. Suppose that*:

(i) *The operator* $\mathscr{H}_1$ *is* (k, r, B)-*fine*.

(ii) *A linear operator* $\mathscr{K}\colon C_\#^{(r)}(\Gamma_T)\to C_\#^{(r)}(\Gamma_T)$ *is* (k, r, C, N)-*balanced*.

(iii) $g\in C_\#^{(r)}(\Gamma_T)$.

(iv) $0<T\leqq C_B^{-1}$.

Here C_B *is the constant introduced in Lemma* 8.B.5. *Consider the Cauchy problem*:

$$(1)\qquad \mathscr{H}_1 u=\mathscr{K}u+g \quad \text{on} \quad \Gamma_T\,.$$

$$(2)\qquad u|_{t=0}=0\,.$$

Then the problem (1)-(2) *has a unique solution* $u\in C_\#^{(r)}(\Gamma_T)$. *Further the linear operator* $(\mathscr{H}_1-\mathscr{K})^{-1}\colon C_\#^{(r)}(\Gamma_T)\to C_\#^{(r)}(\Gamma_T)$ *is* $(k, r, C', 1+N+|H|_{(r)})$-*integral-balanced. Here* C' *is a constant bounded with* r, B, *and* C.

PROOF. We use the notations in Lemma 8.B.5. For $(\omega, t) \in \phi(U_\nu \times [0, T])$, we call the n-tuple $(\psi_1(\omega, t), \cdots, \psi_{n-1}(\omega, t), t)$ the characteristic coordinates of (ω, t). Here ψ_i is the i-th component of the mapping $\psi: \phi(U_\nu \times [0, T]) \to U_\nu$ with respect to the local coordinates in U_ν. As is well known, in the characteristic coordinates, (1) in the lemma is an ordinary differential equation. Then, from Lemmas 3.A.9, 8.A.3, 8.A.4 and 8.B.5, the lemma follows immediately.

LEMMA 8.B.8. *Let $2 \leqq k \leqq r$. Let B, C, and N be positive constants. Suppose that:*

(i) *The operator $\mathscr{P}_1$ is (k, r, B)-λ-fine.*

(ii) *A linear operator $\mathscr{K}: C_\#^{(r)}(\bar{\Omega}_T) \to C_\#^{(r)}(\bar{\Omega}_T)$ is (k, r, C, N)-balanced.*

(iii) $g \in C_\#^{(r)}(\bar{\Omega}_T)$.

(iv) $0 < T \leqq C_B^{-1}$.

Here C_B is the constant introduced in Lemma 8.B.6. Consider the Cauchy problem:

$$\mathscr{P}_1 u = \mathscr{K} u + g \quad in \quad \Omega_T . \tag{1}$$

$$u|_{t=0} = 0 . \tag{2}$$

Then the problem (1)-(2) has a unique solution $u \in C_\#^{(r)}(\bar{\Omega}_T)$. Further, the linear operator $(\mathscr{P}_1 - \mathscr{K})^{-1}: C_\#^{(r)}(\bar{\Omega}_T) \to C_\#^{(r)}(\bar{\Omega}_T)$ is $(k, r, C', 1 + N + |H|_{(r)} + |H_\lambda|_{(r)})$-integral-balanced. Here C' is a constant bounded with r, B, and C.

PROOF. We can easily obtain modifications of Lemmas 8.A.3 and 8.A.4 for the $C_\#^{(r)}(\bar{\Omega}_T)$ case. Then, by using Lemma 8.B.6 in place of Lemma 8.B.5, the lemma is proved as in Lemma 8.B.7.

LEMMA 8.B.9. *Let $\mathscr{L}$ be a differential operator on $\bar{\Omega}_T$ which is expressed as in (B) in Section 3. Let k and i be integers with $2 \leqq k \leqq i$. Let σ, B, C, and N be positive constants. Suppose that:*

(i) *The operator $\mathscr{L}$ is $(\varepsilon_0 + k, \varepsilon_0 + i, \sigma, B)$-#-parabolic.*

(ii) *The operator $\mathscr{P}_1$ is $(\varepsilon_0 + k, \varepsilon_0 + i, B)$-$\lambda$-fine.*

(iii) *A linear operator $\mathscr{K}: C_\#^{(\varepsilon_0+i)}(\bar{\Omega}_T) \to C_\#^{(\varepsilon_0+i)}(\bar{\Omega}_T)$ is $(\varepsilon_0 + k, \varepsilon_0 + i, C, N)$-balanced.*

(iv) $g \in C_\#^{(\varepsilon_0+i)}(\Gamma_T)$.

(v) $0 < T \leqq C_B^{-1}$.

Here C_B is a constant introduced in Lemma 8.B.6. Consider the problem:

$$\mathscr{L} u = 0 \quad in \quad \Omega_T . \tag{1}$$

$$u|_{t=0} = 0 . \tag{2}$$

$$u = 0 \quad on \quad J_T . \tag{3}$$

$$\mathscr{P}_1 u = \mathscr{K} u + g \quad on \quad \Gamma_T . \tag{4}$$

Then the problem (1)-(4) *has a solution* $u \in C_{\#}^{(\varepsilon_0+i)}(\bar{\Omega}_T)$. *Further, the mapping* $g \mapsto u$, *where* u *is the obtained solution of* (1)-(4), *is an* $(\varepsilon_0 + k, \varepsilon_0 + i, C', 1 + N + |A|_{(\varepsilon_0+i)} + |H|_{(\varepsilon_0+i)} + |H_\lambda|_{(\varepsilon_0+i)})$*-integral-balanced linear operator on* $C_{\#}^{(\varepsilon_0+i)}(\Gamma_T)$ *into* $C_{\#}^{(\varepsilon_0+i)}(\bar{\Omega}_T)$. *Here* C' *is a constant bounded with* i, σ, B, *and* C *and* $|A|_{(\varepsilon_0+i)} = \sum_{g,h=1}^{n} |A_{gh}|_{(\varepsilon_0+i)} + \sum_{h=1}^{n} |A_g|_{(\varepsilon_0+i)} + |A_0|_{(\varepsilon_0+i)}$.

PROOF. If u belongs to $C_{\#}^{(\varepsilon_0+i)}(\bar{\Omega}_T)$, then u satisfies (1)-(4) if and only if it satisfies:

$$\mathscr{L}\mathscr{P}_1 u = [\mathscr{L}, \mathscr{P}_1]u \quad \text{in} \quad \Omega_T . \tag{1'}$$

$$(\mathscr{P}_1 u)|_{t=0} = 0 . \tag{2'}$$

$$\mathscr{P}_1 u = 0 \quad \text{on} \quad J_T . \tag{3'}$$

$$\mathscr{P}_1 u = \mathscr{K} u + g \quad \text{on} \quad \Gamma_T . \tag{4'}$$

Here $[\mathscr{L}, \mathscr{P}_1]$ denotes the commutator, i.e., $[\mathscr{L}, \mathscr{P}_1] = \mathscr{L}\mathscr{P}_1 - \mathscr{P}_1\mathscr{L}$. By Lemma 8.B.3, we observe that $\mathscr{P}_1 u$ is expressed as $\mathscr{P}_1 u = \mathscr{M}u + \mathscr{N}g$. Here $\mathscr{M}: C_{\#}^{(\varepsilon_0+i)}(\bar{\Omega}_T) \to C_{\#}^{(\varepsilon_0+i)}(\bar{\Omega}_T)$ (resp. $\mathscr{N}: C_{\#}^{(\varepsilon_0+i)}(\Gamma_T) \to C_{\#}^{(\varepsilon_0+i)}(\bar{\Omega}_T)$) is an $(\varepsilon_0 + k, \varepsilon_0 + i, C_1, N + |A|_{(\varepsilon_0+i)} + |H|_{(\varepsilon_0+i)} + |H_\lambda|_{(\varepsilon_0+i)})$-balanced (resp. $(\varepsilon_0 + k, \varepsilon_0 + i, C_2, |A|_{(\varepsilon_0+i)})$-balanced) linear operator where C_1 (resp. C_2) is a constant bounded with i, σ, B and C (resp. i, σ and B). Then, the lemma is proved by Lemma 8.B.8.

9. Inversion of $D\mathscr{F}(\rho)$. Finally we verify the condition (III) in Nash's implicit function theorem.

We solve the equation $D\mathscr{F}(\rho)\delta\rho = \delta G$ where ρ and δG are given and $\delta\rho$ is unknown. By the definition of $D\mathscr{F}(\rho)$, the above equation means:

$$\mathscr{L}_\rho(\delta U_\rho) = (\delta \mathscr{E}_\rho) U_\rho \quad \text{in} \quad \Omega_T . \tag{$1_{\delta U}$}$$

$$\delta U_\rho|_{t=0} = 0 . \tag{$2_{\delta U}$}$$

$$\delta U_\rho = 0 \quad \text{on} \quad J_T . \tag{$3_{\delta U}$}$$

$$\delta U_\rho = 0 \quad \text{on} \quad \Gamma_T . \tag{$4_{\delta U}$}$$

$$\partial_t \delta\rho + c_0\{(\partial_\lambda U_\rho)\delta S_\rho + [\partial_\lambda(\delta U_\rho)]S_\rho\} = \delta G \quad \text{on} \quad \Gamma_T . \tag{$5_{\delta U}$}$$

Throughout this section, suppose that:

(i) $i = 1, \cdots, 11$.

(ii) $\rho \in V_T \cap C_{\#}^{(r_0-4+4i)}(\Gamma_T)$.

(iii) $\delta G \in C_{\#}^{(r_0-7+4i)}(\Gamma_T)$.

Under these hypotheses, we seek $\delta\rho \in C_{\#}^{(r_0-8+4i)}(\Gamma_T)$ which satisfies $(1_{\delta U})$-$(5_{\delta U})$ when $i \geqq 2$ and set $\mathscr{I}(\rho)\delta G = \delta\rho$.

(A) We eliminate $\delta\rho$ from the problem.

For $\delta\rho \in C_{\#}^{(r_0)}(\Gamma_T)$, define a mapping $\delta e_\rho \colon \bar{\Omega}_T \to \boldsymbol{R}^n \times [0, T]$ by:

(9.1.i) $$\delta e_\rho(x(\omega, \lambda), t) = (\partial_\lambda x(\omega, \lambda + \chi_0(\lambda)\rho(\omega, t))\delta\rho(\omega, t), 0) \quad \text{for} \quad (x, t) = (x(\omega, \lambda), t) \in N_0^- \times [0, T] .$$

(9.1.ii) $$\delta e_\rho(x, t) = (0, 0) \quad \text{for} \quad (x, t) \in (\bar{\Omega}_0 - N_0^-) \times [0, T] .$$

(See (1.2).) Let

(9.2) $$u_\rho = U_\rho \circ e_\rho^{-1} .$$

Let

(9.3) $$\delta V = \delta U_\rho - \langle (\operatorname{grad} u_\rho) \circ e_\rho, \delta e_\rho \rangle .$$

First we show that

(9.4) $$\mathscr{L}_\rho \delta V = 0 \quad \text{in} \quad \Omega_T .$$

Fix ρ and $\delta\rho$. Let D be a relatively compact subdomain with C^∞ boundary in Ω_T. Let ε be a sufficiently small positive number. By (7.3), (9.2), and (9.3), $u_{\rho+\varepsilon\delta\rho} \circ e_{\rho+\varepsilon\delta\rho} - u_\rho \circ e_\rho - \varepsilon\delta V - \varepsilon\langle (\operatorname{grad} u_\rho) \circ e_\rho, \delta e_\rho \rangle = \varepsilon^2 R_\varepsilon$, where R_ε is bounded in $C^{(r_0)}(\bar{\Omega}_T)$. Hence, by (9.1),

(9.5) $$\begin{aligned} &u_{\rho+\varepsilon\delta\rho} \circ e_{\rho+\varepsilon\delta\rho} - u_\rho \circ e_{\rho+\varepsilon\delta\rho} - \varepsilon\delta V \\ &\quad = -[u_\rho \circ e_{\rho+\varepsilon\delta\rho} - u_\rho \circ e_\rho - \varepsilon\langle (\operatorname{grad} u_\rho) \circ e_\rho, \delta e_\rho \rangle] + \varepsilon^2 R_\varepsilon \\ &\quad = \varepsilon^2 R'_\varepsilon \quad \text{on} \quad \bar{D} , \end{aligned}$$

where R'_ε is bounded in $C^{(r_0-2)}(\bar{D})$. Note that $\mathscr{L}_{\rho+\varepsilon\delta\rho}(u_{\rho+\varepsilon\delta\rho} \circ e_{\rho+\varepsilon\delta\rho}) = \mathscr{L}_{\rho+\varepsilon\delta\rho}(u_\rho \circ e_{\rho+\varepsilon\delta\rho}) = 0$ on $\bar{D}$. Hence, by (9.5), $\mathscr{L}_{\rho+\varepsilon\delta\rho}\delta V = \varepsilon R''_\varepsilon$ on $\bar{D}$, where R''_ε is bounded in $C^{(r_0-4)}(\bar{D})$. Taking $\varepsilon \to 0$, we obtain (9.4).

Secondly we eliminate $\delta\rho$ from the boundary condition on Γ_T. In a neighborhood of Γ_T, we have $U_\rho(x(\omega, \lambda), t) = u_\rho(e_\rho(x(\omega, \lambda), t), t) = u_\rho(x(\omega, \lambda + \rho(\omega, t)), t)$, so that $\langle (\operatorname{grad} u_\rho) \circ e_\rho, \delta e_\rho \rangle = [(\partial_\lambda u_\rho) \circ e_\rho]\delta\rho = (\partial_\lambda U_\rho)\delta\rho$. Hence $(4_{\delta U})$ and $(5_{\delta U})$ can be rewritten in the form:

$(4_{\delta U})$ $$\delta V + (\partial_\lambda U_\rho)\delta\rho = 0 \quad \text{on} \quad \Gamma_T .$$

$(5_{\delta U})$ $$\mathscr{H}_\rho \delta\rho + c_0 S_\rho \partial_\lambda \delta V = \delta G \quad \text{on} \quad \Gamma_T .$$

Here

(9.6) $$\begin{aligned} \mathscr{H}_\rho = \partial_t + c_0[(\partial_\lambda U_\rho)|_{\lambda=0}] \sum_{j=1}^{n-1} \{[\partial/\partial(\partial_{\omega_j}\rho)]S\}\partial_{\omega_j} \\ + c_0\{[(\partial_\lambda U_\rho)|_{\omega=0}][(\partial/\partial\rho)S] + [(\partial_\lambda^2 U_\rho)|_{\lambda=0}]S\} . \end{aligned}$$

(See (7.5).) For $g \in C^0(\Gamma_T)$, define $\mathscr{H}_\rho^{-1}g$ by:

(1) $$\mathscr{H}_\rho(\mathscr{H}_\rho^{-1}g) = g \quad \text{on} \quad \Gamma_T .$$

(2) $$(\mathscr{H}_\rho^{-1}g)|_{t=0} = 0 .$$

Then, by $(5_{\delta U})$, we have

$$\delta\rho = -c_0\mathscr{H}_\rho^{-1}(S_\rho\partial_\lambda\delta V) + \mathscr{H}_\rho^{-1}\delta G . \tag{9.7}$$

Inserting this in $(4_{\delta U})$, we eliminate $\delta\rho$ from the boundary condition on Γ_T.

Consequently we have:

$$\mathscr{L}_\rho\delta V = 0 \quad \text{in} \quad \Omega_T . \tag{$1_{\delta V}$}$$

$$\delta V|_{t=0} = 0 . \tag{$2_{\delta V}$}$$

$$\delta V = 0 \quad \text{on} \quad J_T . \tag{$3_{\delta V}$}$$

$$\delta V - c_0(\partial_\lambda U_\rho)\mathscr{H}_\rho^{-1}(S_\rho\partial_\lambda\delta V) = -(\partial_\lambda U_\rho)\mathscr{H}_\rho^{-1}\delta G \quad \text{on} \quad \Gamma_T . \tag{$4_{\delta V}$}$$

This completes the elimination of $\delta\rho$.

(B) Extend S_ρ to $\bar{\Omega}_T$ by:

$$S_\rho(x(\omega, \lambda), t) = \chi_0(\lambda)S_\rho(\omega, t) + 1 - \chi_0(\lambda) \quad \text{for} \quad (x, t) = (x(\omega, \lambda), t) \in N_0^- \times [0, T] . \tag{9.8.i}$$

$$S_\rho(x, t) = 1 \quad \text{for} \quad (x, t) \in (\bar{\Omega}_0 - N_0^-) \times [0, T] . \tag{9.8.ii}$$

Note that the extended S_ρ also has a positive constant lower bound. Extend $\mathscr{H}_\rho$ to $\bar{\Omega}_T$ by:

$$\begin{aligned}\mathscr{H}_\rho = \partial_t &+ c_0\chi_0(\lambda)[(\partial_\lambda U_\rho)|_{\lambda=0}]\sum_{j=1}^{n-1}\{[\partial/\partial(\partial_{\omega_j}\rho)]S\}\partial_{\omega_j} \\ &+ c_0\chi_0(\lambda)\{[(\partial_\lambda U_\rho)|_{\lambda=0}][(\partial/\partial\rho)S] + [(\partial_\lambda^2 U_\rho)|_{\lambda=0}]S\} \\ &\qquad \text{on} \quad N_0^- \times [0, T] .\end{aligned} \tag{9.9.i}$$

$$\mathscr{H}_\rho = \partial_t \quad \text{on} \quad (\bar{\Omega}_0 - N_0^-) \times [0, T] . \tag{9.9.ii}$$

(See (9.6).) For $g \in C^0(\bar{\Omega}_T)$, define $\mathscr{H}_\rho^{-1}g$ by:

$$\mathscr{H}_\rho(\mathscr{H}_\rho^{-1}g) = g \quad \text{in} \quad \Omega_T . \tag{1}$$

$$(\mathscr{H}_\rho^{-1}g)|_{t=0} = 0 . \tag{2}$$

Set

$$\delta X = \mathscr{H}_\rho^{-1}(S_\rho\delta V) . \tag{9.10}$$

We transform $(1_{\delta V})$-$(4_{\delta V})$ into a problem for δX.

First we transform $(1_{\delta V})$ into an equation for δX. By $(2_{\delta V})$, we have $(\mathscr{L}_\rho\delta X)|_{t=0} = 0$. Hence, by $(1_{\delta V})$, we have

$$\begin{aligned}\mathscr{L}_\rho\delta X &= \mathscr{H}_\rho^{-1}\mathscr{H}_\rho\mathscr{L}_\rho\mathscr{H}_\rho^{-1}(S_\rho\delta V) \\ &= \mathscr{H}_\rho^{-1}[\mathscr{H}_\rho, \mathscr{L}_\rho]\delta X + \mathscr{H}_\rho^{-1}\mathscr{L}_\rho(S_\rho\delta V) \\ &= \mathscr{H}_\rho^{-1}[\mathscr{H}_\rho, \mathscr{L}_\rho]\delta X + \mathscr{H}_\rho^{-1}[(\mathscr{L}_\rho S_\rho)S_\rho^{-1}(\mathscr{H}_\rho\delta X)] \\ &\quad - 2\mathscr{H}_\rho^{-1}\Big\{\sum_{g,h=1}^{n} A_{\rho,gh}[\partial_{x_g}(S_\rho)][\partial_{x_h}(S_\rho^{-1}(\mathscr{H}_\rho\delta X))]\Big\} .\end{aligned} \tag{9.11}$$

Clearly

$$(9.12)\qquad \partial_{x_h}[S_\rho^{-1}(\mathscr{H}_\rho\delta X)] = -S_\rho^{-2}[\partial_{x_h}(S_\rho)](\mathscr{H}_\rho\delta X) + S_\rho^{-1}([\partial_{x_h}, \mathscr{H}_\rho]\delta X) + S_\rho^{-1}(\mathscr{H}_\rho\partial_{x_h}\delta X)\ .$$

Let $\mathscr{H}_{\rho,1}$ be the homogeneous part of the first order of $\mathscr{H}_\rho$. By the obvious formula $\mathscr{H}_\rho(fg) = (\mathscr{H}_{\rho,1}f)g + (\mathscr{H}_\rho g)$, we obtain

$$(9.13)\qquad \mathscr{H}_\rho^{-1}(FG) = F(\mathscr{H}_\rho^{-1}G) - \mathscr{H}_\rho^{-1}[(\mathscr{H}_{\rho,1}F)(\mathscr{H}_\rho^{-1}G)]\ .$$

By (9.11), (9.12) and (9.13), we have

$$\begin{aligned}(9.14)\qquad & \mathscr{L}_\rho\delta X - (\mathscr{L}_\rho S_\rho)S_\rho^{-1}\delta X - 2\sum_{g,h=1}^{n} A_{\rho,gh}[\partial_{x_g}(S_\rho)][\partial_{x_h}(S_\rho)]S_\rho^{-2}\delta X \\ & + 2\sum_{g,h=1}^{n} A_{\rho,gh}[\partial_{x_g}(S_\rho)]S_\rho^{-1}(\partial_{x_h}\delta X) \\ & = \mathscr{H}_\rho^{-1}\Big\{[\mathscr{H}_\rho, \mathscr{L}_\rho]\delta X - [\mathscr{H}_{\rho,1}((\mathscr{L}_\rho S_\rho)S_\rho^{-1})]\delta X \\ & - 2\sum_{g,h=1}^{n}[\mathscr{H}_{\rho,1}(A_{\rho,gh}(\partial_{x_g}(S_\rho))(\partial_{x_h}(S_\rho))S_\rho^{-2})]\delta X \\ & - 2\sum_{g,h=1}^{n} A_{\rho,gh}[\partial_{x_g}(S_\rho)]S_\rho^{-1}([\partial_{x_h}, \mathscr{H}_\rho]\delta X) \\ & + 2\sum_{g,h=1}^{n}[\mathscr{H}_{\rho,1}(A_{\rho,gh}(\partial_{x_g}(S_\rho))S_\rho^{-1})](\partial_{x_h}\delta X)\Big\}\ .\end{aligned}$$

This is the equation for δX.

Denote the left-hand side (resp. right-hand side) of (9.14) by $\tilde{\mathscr{L}}_\rho\delta X$ (resp. $\mathscr{R}_\rho\delta X$). Define a differential operator $\mathscr{P}_\rho$ on $\bar{\Omega}_T$ by

$$(9.15.\mathrm{i})\qquad \mathscr{P}_\rho = \mathscr{H}_\rho - c_0\chi_0(\lambda)[(\partial_\lambda U_\rho)|_{\lambda=0}]S_\rho\partial_\lambda \quad \text{on} \quad N_0^- \times [0, T]\ .$$

$$(9.15.\mathrm{ii})\qquad \mathscr{P}_\rho = \partial_t \quad \text{on} \quad (\bar{\Omega}_0 - N_0^-) \times [0, T]\ .$$

From (9.8) and (9.9), we observe that in a neighborhood of Γ_T:

(i) $\partial_\lambda(S_\rho f) = S_\rho(\partial_\lambda f)$.

(ii) $\partial_\lambda\mathscr{H}_\rho^{-1} = \mathscr{H}_\rho^{-1}\partial_\lambda$.

Then the problem $(1_{\delta V})$-$(4_{\delta V})$ is transformed into the problem:

$$(1_{\delta X})\qquad \tilde{\mathscr{L}}_\rho\delta X = \mathscr{R}_\rho\delta X \quad \text{in} \quad \Omega_T\ .$$

$$(2_{\delta X})\qquad \delta X|_{t=0} = 0\ .$$

$$(3_{\delta X})\qquad \delta X = 0 \quad \text{on} \quad J_T\ .$$

$$(4_{\delta X})\qquad \mathscr{P}_\rho\delta X = -(\partial_\lambda U_\rho)S_\rho\mathscr{H}_\rho^{-1}\delta G \quad \text{on} \quad \Gamma_T\ .$$

This is the problem for δX.

(C) By the assumption (A.3) in Theorem and the maximum principle of the heat equation,

$$(9.16)\qquad -c_0(\partial_\lambda U_\rho)S_\rho = -c_0[(\partial_\lambda u_\rho)\circ e_\rho]S_\rho \geqq 0 \quad \text{on} \quad \Gamma_T .$$

Now we can easily observe that:

(i) On Γ_T, if $i \neq 11$ (resp. $i = 11$), then $\mathscr{H}_{\rho,1}$ is an (r_0-1, r_0-5+4i, B)-fine (resp. (r_0-1, r_0+38, B)-fine) operator. See (9.6) and the assumption (T).

(ii) On $\bar{\Omega}_T$, if $i \neq 11$ (resp. $i = 11$), then $\mathscr{H}_{\rho,1}$ is an (r_0-1, r_0-5+4i, B)-λ-fine (resp. (r_0-1, r_0+38, B)-λ-fine) operator. See (9.9).

(iii) The operator $\tilde{\mathscr{L}}_\rho$ is $(r_0-3, r_0-7+4i, \sigma_0, B')$-#-parabolic. See (9.14).

(iv) The composition $\mathscr{H}_\rho\mathscr{R}_\rho$ is a spatial differential operator of the second order whose coefficients belong to $C_\#^{(r_0-9+4i)}(\bar{\Omega}_T)$. See (9.14).

(v) If $i \neq 11$ (resp. $i = 11$), then the homogeneous part of the first order of $\mathscr{P}_\rho$ is an (r_0-1, r_0-5+4i, B)-λ-fine (resp. (r_0-1, r_0+38, B)-λ-fine) operator (see (9.15) and (9.16)).

Here B' is a constant and B is a constant bounded with $|a_0|_{r_0}$ and $|b_0|_{(r_0)}$.

(D) Note that we can apply Lemma 8.B.9 to $(1_{\delta X})$-$(4_{\delta X})$ if the right-hand side of $(1_{\delta X})$ is replaced by 0. To realize this idea, we introduce three linear operators as follows. The compatibility of these definitions is verified in (E).

Define a linear operator $\mathscr{Y}_\rho$: $C_\#^{(r_0-7+4i)}(\bar{\Omega}_T) \to C_\#^{(r_0-7+4i)}(\bar{\Omega}_T)$ by:

$$(1_{\mathscr{Y}})\qquad \tilde{\mathscr{L}}_\rho(\mathscr{Y}_\rho\delta X) = \mathscr{R}_\rho\delta X \quad \text{in} \quad \Omega_T .$$

$$(2_{\mathscr{Y}})\qquad (\mathscr{Y}_\rho\delta X)|_{t=0} = 0 .$$

$$(3_{\mathscr{Y}})\qquad \mathscr{Y}_\rho\delta X = 0 \quad \text{on} \quad J_T .$$

$$(4_{\mathscr{Y}})\qquad \partial_\lambda(\mathscr{Y}_\rho\delta X) = 0 \quad \text{on} \quad \Gamma_T .$$

Here δX belongs to $C_\#^{(r_0-7+4i)}(\bar{\Omega}_T)$. Define a linear operator $\mathscr{X}_\rho$: $C_\#^{(r_0-7+4i)}(\bar{\Omega}_T) \to C_\#^{(r_0-7+4i)}(\bar{\Omega}_T)$ by:

$$(1_{\mathscr{X}})\qquad \tilde{\mathscr{L}}_\rho(\mathscr{X}_\rho\delta X) = 0 \quad \text{in} \quad \Omega_T .$$

$$(2_{\mathscr{X}})\qquad (\mathscr{X}_\rho\delta X)|_{t=0} = 0 .$$

$$(3_{\mathscr{X}})\qquad \mathscr{X}_\rho\delta X = 0 \quad \text{on} \quad J_T .$$

$$(4_{\mathscr{X}})\qquad \mathscr{P}_\rho(\mathscr{X}_\rho\delta X) = -\mathscr{H}_\rho\mathscr{Y}_\rho\delta X \quad \text{on} \quad \Gamma_T .$$

Here δX belongs to $C_\#^{(r_0-7+4i)}(\bar{\Omega}_T)$. Define a linear operator $\tilde{\mathscr{X}}_\rho$: $C_\#^{(r_0-7+4i)}(\Gamma_T) \to C_\#^{(r_0-7+4i)}(\bar{\Omega}_T)$ by:

$$(1_{\tilde{\mathscr{X}}})\qquad \tilde{\mathscr{L}}_\rho(\tilde{\mathscr{X}}_\rho\delta G) = 0 \quad \text{in} \quad \Omega_T .$$

$$(2_{\tilde{\mathscr{X}}})\qquad (\tilde{\mathscr{X}}_\rho\delta G)|_{t=0} = 0 .$$

$$(3_{\tilde{\mathscr{X}}})\qquad \tilde{\mathscr{X}}_\rho\delta G = 0 \quad \text{on} \quad J_T .$$

$$(4_{\tilde{\mathscr{X}}}) \qquad \mathscr{P}_\rho(\tilde{\mathscr{X}}_\rho \delta G) = -(\partial_\lambda U_\rho) S_\rho \mathscr{H}_\rho^{-1} \delta G \quad \text{on} \quad \Gamma_T .$$

Here δG belongs to $C_{\#}^{(r_0-7+4i)}(\Gamma_T)$.

Clearly, for δX in $C_{\#}^{(r_0-7+4i)}(\bar{\Omega}_T)$, the problem $(1_{\delta X})$-$(4_{\delta X})$ is equivalent to the equation

$$(9.17) \qquad \delta X = \mathscr{Y}_\rho \delta X + \mathscr{X}_\rho \delta X + \tilde{\mathscr{X}}_\rho \delta G .$$

(E) Let T be so small that:

(i) For $\mathscr{H}_{\rho,1}$, we can use Lemmas 8.B.7 and 8.B.8.

(ii) For $\tilde{\mathscr{L}}_\rho$ and the homogeneous part of the first order of $\mathscr{P}_\rho$, we can use Lemma 8.B.9.

First we consider $(1_{\mathscr{Y}})$-$(4_{\mathscr{Y}})$. By Lemma 8.B.8,

$$(9.18) \qquad |\mathscr{R}_\rho \delta X|_{(r_0-9+4i),t} \leqq C[|\delta X|_{(r_0-7+4i),t} + (1 + |\rho|_{(r_0-4+4i),T})|\delta X|_{(r_0-3),t}]$$

for $t \in [0, T]$. Here C is a constant bounded with $|a_0|_{r_0+39}$ and $|b_0|_{(r_0+39)}$. By Lemma 3.B.3 and (9.18), the operator $\mathscr{Y}_\rho$ is well-defined and satisfies

$$(9.19) \qquad |\mathscr{Y}_\rho \delta X|_{(r_0-7+4i),t} \leqq C[|\delta X|_{(r_0-7+4i),t} + (1 + |\rho|_{(r_0-4+4i),T})|\delta X|_{(r_0-3),t}]$$

for $t \in [0, T]$. The operator $\mathscr{Y}_\rho$ is $(r_0 - 3, r_0 - 7 + 4i, C, 1 + |\rho|_{(r_0-4+4i),T})$-balanced. Here C is a constant bounded with $|a_0|_{r_0+39}$ and $|b_0|_{(r_0+39)}$. From $(1_{\mathscr{Y}})$-$(4_{\mathscr{Y}})$ we obtain:

$$(1) \qquad \tilde{\mathscr{L}}_\rho(\mathscr{H}_\rho \mathscr{Y}_\rho \delta X) = [\tilde{\mathscr{L}}_\rho, \mathscr{H}_\rho]\mathscr{Y}_\rho \delta X \quad \text{in} \quad \Omega_T .$$

$$(2) \qquad (\mathscr{H}_\rho \mathscr{Y}_\rho \delta X)|_{t=0} = 0 .$$

$$(3) \qquad \mathscr{H}_\rho \mathscr{Y}_\rho \delta X = 0 \quad \text{on} \quad J_T .$$

$$(4) \qquad \partial_\lambda(\mathscr{H}_\rho \mathscr{Y}_\rho \delta X) = 0 \quad \text{on} \quad \Gamma_T .$$

Hence, by Lemma 3.B.3 and (9.19),

$$(9.20) \qquad |\mathscr{H}_\rho \mathscr{Y}_\rho \delta X|_{(r_0-7+4i),t} \leqq C[|\delta X|_{(r_0-7+4i),t} + (1 + |\rho|_{(r_0-4+4i),T})|\delta X|_{(r_0-3),t}]$$

for $t \in [0, T]$. Here C is a constant bounded with $|a_0|_{r_0+39}$ and $|b_0|_{(r_0+39)}$. By Lemma 8.B.8 and (9.20),

$$(9.21) \qquad |\mathscr{Y}_\rho \delta X|_{(r_0-7+4i),t} \leqq C\Big[\int_0^t |\delta X|_{(r_0-7+4i),\tau} d\tau + (1 + |\rho|_{(r_0-4+4i),T}) \int_0^t |\delta X|_{(r_0-3),\tau} d\tau\Big]$$

for $t \in [0, T]$. The operator $\mathscr{Y}_\rho$ is $(r_0 - 3, r_0 - 7 + 4i, C, 1 + |\rho|_{(r_0-4+4i),T})$-integral-balanced. Here C is a constant bounded with $|a_0|_{r_0+39}$ and $|b_0|_{(r_0+39)}$.

Secondly we consider $(1_{\mathscr{X}})$-$(4_{\mathscr{X}})$ and $(1_{\tilde{\mathscr{X}}})$-$(4_{\tilde{\mathscr{X}}})$. By Lemma 8.B.9 and (9.20), the operator $\mathscr{X}_\rho$ is well-defined and satisfies

$$(9.22)\qquad |\mathscr{X}_\rho\delta X|_{(r_0-7+4i),t} \leqq C\Big[\int_0^t |\delta X|_{(r_0-7+4i),\tau}d\tau + (1+|\rho|_{(r_0-4+4i),T})\int_0^t |\delta X|_{(r_0-3),\tau}d\tau\Big]$$

for $t\in[0, T]$. Here C is a constant bounded with $|a_0|_{r_0+39}$ and $|b_0|_{(r_0+39)}$. By Lemmas 8.B.7 and 8.B.9, the operator $\tilde{\mathscr{X}}_\rho$ is well-defined and satisfies

$$(9.23)\qquad |\tilde{\mathscr{X}}_\rho\delta G|_{(r_0-7+4i),t} \leqq C\Big[\int_0^t |\delta G|_{(r_0-7+4i),\tau}d\tau + (1+|\rho|_{(r_0-4+4i),T})\int_0^t |\delta G|_{(r_0-3),\tau}d\tau\Big]$$

for $t\in[0, T]$. Here C is a constant bounded with $|a_0|_{r_0+49}$ and $|b_0|_{(r_0+39)}$.

Hence, by Lemma 8.A.4, (9.21), (9.22) and (9.23), we obtain the solution δX of (9.17) which satisfies

$$(9.24)\qquad |\delta X|_{(r_0-7+4i),T} \leqq C[|\delta G|_{(r_0-7+4i),T} + (1+|\rho|_{(r_0-4+4i),T})|\delta G|_{(r_0-3),T}]\ .$$

Here C is a constant bounded with $|a_0|_{r_0+39}$ and $|b_0|_{(r_0+39)}$.

(F) Define a linear operator $\mathscr{J}(\rho)$: $C_\#^{(r_0-7+4i)}(\Gamma_T)\to C_\#^{(r_0-8+4i)}(\Gamma_T)$ by

$$(9.25)\qquad \mathscr{J}(\rho)\delta G = -c_0\partial_\lambda\delta X + \mathscr{H}_\rho^{-1}\delta G\ ,$$

where δX is the solution of (9.17) in view of (9.7) and (9.10). By (9.24), (9.25) and (4.5.ii), we easily observe that:

(i) $D\mathscr{F}(\rho)\mathscr{J}(\rho)\delta G=\delta G$ when $i\geqq 2$.

(ii) $|\mathscr{J}(\rho)\delta G|_{(r_0-8+4i)} \leqq C[|\delta G|_{(r_0-7+4i)} + (1+|\rho|_{(r_0-4+4i)})|\delta G|_{(r_0-3)}]$.

(iii) $|\mathscr{J}(\rho)\mathscr{F}(\rho)|_{(r_0-8+4i)} \leqq C(1+|\rho|_{(r_0-4+4i)})$.

Here C is a constant bounded with $|a_0|_{r_0+39}$ and $|b_0|_{(r_0+39)}$. This proves (III). On the condition $i\geqq 2$ in (i), remember that we assume $\delta\rho$ belongs to $C_\#^{(r_0)}(\Gamma_T)$ in (A).

REMARK. The "right inverse" $\mathscr{J}(\rho)$ thus constructed is also the "left inverse" of $D\mathscr{F}(\rho)$.

(G) We have proved the conditions (I)–(III) in Nash's implicit function theorem for our setting. On the other hand, it is easy to see that

$$|\mathscr{F}(\rho)|_{(r_0-2-(\varepsilon_0/2)),T} \leqq C|\mathscr{F}(\rho)|_{(r_0-2),T}T^{\varepsilon_0/4}\ ,$$

where C is a constant. Hence, by Nash's implicit function theorem, Theorem′ is proved.

10. Elimination of the technical assumption. Finally, in this section, we show how our proof is modified when the assumption (T) in Section 2 is replaced by the assumption (A.4) in Theorem in Section 1. Let a_0 be a Hestenes-Whitney extension of a_0 to $\boldsymbol{R}^n$. Define a function

u_0 on $\boldsymbol{R}^n \times [0, T_0)$ by solving

$$(\partial_t - \Delta)u_0 = 0$$

and $u_0|_{t=0} = a_0$. Observe that the definitions of e_ρ, S_ρ and $\mathscr{L}_\rho$ in Sections 2 and 4 can be extended for $\rho \in C^{(r_0)}(\Gamma_T)$ with small $|\rho|_0$. Recall that $a_0 \in C^{r_0+43}(\bar{\Omega}_0)$. By the theory of nonlinear first-order equations (see e.g., Courant and Hilbert [8, Chapter 2]) and the weighted Hölder estimate for characteristic coordinate transforms (refer to Lemma 8.B.5), taking T sufficiently small, we get $\rho_0 \in C^{(r_0+40)}(\Gamma_T)$ with small $|\rho_0|_0$ such that $\rho_0|_{t=0} = 0$ and

$$\partial_t \rho_0 + c_0[\partial_\lambda(u_0 \circ e_{\rho_0})]S_{\rho_0} = 0 \quad \text{on} \quad \Gamma_T ,$$

because the right-hand side of the characteristic system (see [8, p. 97]) consists of $C^{(r_0+40)}$ functions. Then our problem is to seek $\rho \in V_T$ and $U \in C^{(r_0)}(\bar{\Omega}_T)$ which satisfy:

$$(1) \qquad \mathscr{L}_{\rho_0+\rho} U = 0 \quad \text{in} \quad \Omega_T .$$

$$(2) \qquad U|_{t=0} = a_0 .$$

$$(3) \qquad U = b_0 \quad \text{on} \quad J_T .$$

$$(4) \qquad U = 0 \quad \text{on} \quad \Gamma_T .$$

$$(5) \qquad \partial_t(\rho_0 + \rho) + c_0(\partial_\lambda U)S_{\rho_0+\rho} = 0 \quad \text{on} \quad \Gamma_T .$$

Here $\rho_0 + \rho$ corresponds to ρ in Theorem in Section 1. Taking T and δ_0 sufficiently small, we can suppose that $\mathscr{L}_{\rho_0+\rho}$ is parabolic and $S_{\rho_0+\rho}$ has a positive lower bound for $\rho \in V_T$. The assumption (A.4) implies that there exists a solution $U \in C^{(r_0+39)}(\bar{\Omega}_T)$ of the above parabolic system (1)-(4) for $\rho \in V_T \cap C_\#^{(r_0+39)}(\Gamma_T)$. Now we can define a mapping $\mathscr{F}: V_T \to C_\#^{(r_0-2)}(\Gamma_T)$ by

$$\mathscr{F}(\rho) = \partial_t(\rho_0 + \rho) + c_0(\partial_\lambda U)S_{\rho_0+\rho} ,$$

where U is the solution of the system (1)-(4). In this setting, replacing the heat operator by $\mathscr{L}_{\rho_0}$ and modifying the technical definitions and lemmas slightly, the rest of the proof can be developed in the same manner as in the case of the technical assumption (T).

NOTE ADDED IN PROOF. The author recently learned from the referee and a few other persons that there exist two announcements "On the classical solutions of the Stefan multidimensional problem, by B. M. Budak and M. Z. Moskal, Dokl. Akad. Nauk SSSR Tom, 184 (1969), 1263-1266= Soviet Math. Dokl., Vol. 10 (1969), 219-223" and "On classical solvability of the multidimensional Stefan problem, by A. M. Meirmanov, Dokl. Akad. Nauk SSSR Tom, 249 (1979)=Soviet Math. Dokl., Vol. 20 (1979), 1426-1429."

The former claims the local (-in-time) existence of the classical solutions for the initial value problem of the two-phase multidimensional Stefan problem, provided that the free boundary can be parametrized by flat space variables. However, no proofs are given. Furter it seems that the detailed proof has not been published yet. The latter claims that, for the initial value problem of the two-phase multidimensional Stefan problem, the local (-in-time) existence of the classical solutions can be proved by parabolic regularization method, provided that the initial normal gradient of the thermal distribution on the interface between ice and water has a positive lower bound. Also no detailed proofs are given.

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