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HOLOMORPHIC MAPPINGS ONTO A CERTAIN COMPACT COMPLEX ANALYTIC SPACE

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1. Statement of the result. In this paper we shall prove some finiteness theorems of holomorphic mappings into a certain compact complex analytic space which is hyperbolic in the sense of Kobayashi [8]. Complex analytic spaces (or manifolds) are always assumed to be reduced, connected and countable at infinity, and $\mathrm{Hol}(X, Y)$ stands for the set of all holomorphic mappings of a complex analytic space X into another Y. In Sections 3 and 4 we shall show the following.

THEOREM 1. *Let Y be a Carathéodory-hyperbolic compact complex analytic space (cf. Section 3). Then, for any compact complex analytic space X, there are at most finitely many holomorphic surjections of X to Y.*

COROLLARY 1. *Let X and Y be as in Theorem 1. Then, for any compact connected complex analytic subvarieties A of X and B of Y, the set*

$$\{f \in \mathrm{Hol}(X, Y);\ f(A) = B\}$$

is finite.

THEOREM 2. *Let X be a compact complex analytic space and A a compact connected complex analytic subvariety of X. Let M be a complex analytic manifold with a complete hermitian metric ds^2_M whose holomorphic sectional curvature is bounded above by a negative constant. Let B be a compact connected complex analytic subvariety of M. Then the set*

$$\{f \in \mathrm{Hol}(X, M);\ f(A) = B\}$$

is finite.

Immediately from Theorem 2 we obtain

COROLLARY 2. *Let M be a compact complex analytic manifold with a hermitian metric ds^2_M whose holomorphic sectional curvature is negative. Then, for any compact complex analytic space X, there are* *at most finitely many holomorphic surjections of X to M.*

Our proof of Theorem 1 is based on the complex analytic structure of the space $\mathrm{Hol}(X, Y)$ and the pseudoconvexity of the Carathéodory pseudometric $E_{Y'}$ on the universal covering space Y' of Y. In the proofs of Theorems 1 and 2, Proposition 1 in Section 2 is the key. Further we shall prove the following in Section 5.

THEOREM 3. *Let M be a compact complex analytic manifold whose holomorphic tangent bundle $T(M)$ is negative in the sense of Grauert. Then, for any compact complex analytic space X, there are at most finitely many nonconstant holomorphic mappings of X into M.*

For example, assume that M is a compact quotient of the unit open ball B of the complex Euclidean space C^n by a properly discontinuous group acting freely on B. As is well known, M admits a Kähler metric of negative holomorphic bisectional curvature. Hence the holomorphic tangent bundle $T(M)$ is negative in the sense of Grauert (cf. [5, p. 208]). In this case Theorem 3 slightly generalizes Proposition 4 of Sunada's paper [14].

2. Holomorphic surjections. Let X be a compact complex analytic space and Y a complex analytic space. It is well known that $\mathrm{Hol}(X, Y)$ equipped with the compact-open topology admits a universal structure of a complex analytic space (not necessarily connected) such that the canonical mapping

$$\Phi: X \times \mathrm{Hol}(X, Y) \to Y$$

defined by the formula $\Phi(x, f) = f(x)$ for each $(x, f) \in X \times \mathrm{Hol}(X, Y)$ is holomorphic (cf. [7]).

LEMMA 1. *Let X and Y be compact complex analytic spaces.*

(i) *Let X_0 (resp. Y_0) be an irreducible component of X (resp. Y). Then the set $\{f \in \mathrm{Hol}(X, Y); f(X_0) = Y_0\}$ is open and closed in* $\mathrm{Hol}(X, Y)$.

(ii) *Let S be the set of all holomorphic surjections of X to Y. Then S is open and closed in* $\mathrm{Hol}(X, Y)$ *and, consequently, S is a complex analytic subvariety of* $\mathrm{Hol}(X, Y)$.

PROOF. Since Y is compact and connected, there exists a distance ρ on Y which induces the topology of Y. Define the distance d on $\mathrm{Hol}(X, Y)$ by

$$d(f, g) = \sup\{\rho(f(x), g(x));\ x \in X\}$$

for $f, g \in \mathrm{Hol}(X, Y)$. Then the compact-open topology of $\mathrm{Hol}(X, Y)$

coincides with the metric topology associated with d on $\mathrm{Hol}(X, Y)$. Put $H = \{f \in \mathrm{Hol}(X, Y);\ f(X_0) = Y_0\}$. By virtue of Remmert's proper mapping theorem, the closedness of H and S, i.e., the openness of $\mathrm{Hol}(X, Y) \setminus H$ and $\mathrm{Hol}(X, Y) \setminus S$, in $\mathrm{Hol}(X, Y)$ follows from arguments using the metric d on $\mathrm{Hol}(X, Y)$.

(i) Take any $f \in H$ and suppose that $\lim_n f_n = f$ in $\mathrm{Hol}(X, Y)$. We denote by $\mathrm{Sg}(Y_0)$ the set of all singular points of the complex analytic space Y_0. Put $m = \dim_C Y_0$. Since $f|_{X_0}: X_0 \to Y_0$ is a proper holomorphic surjection, f is of maximal rank m at some nonsingular point of $X_0 \setminus f^{-1}(\mathrm{Sg}(Y_0))$ (cf. [11, Chap. VII]). Hence f_n is of maximal rank m at some nonsingular point of $X_0 \setminus f_n^{-1}(\mathrm{Sg}(Y_0))$ for sufficiently large n's. By the proper mapping theorem $f_n(X_0)$ is a compact irreducible complex analytic subvariety of Y such that $\dim_C f_n(X_0) = m$ for sufficiently large n's. Consequently $f_n(X_0) = Y_0$, i.e., $f_n \in H$ for sufficiently large n's. This means that H is open in $\mathrm{Hol}(X, Y)$.

(ii) Take any $f \in S$ and suppose that $\lim_n f_n = f$ in $\mathrm{Hol}(X, Y)$. Take any irreducible component Y_0 of Y. Then we can take an irreducible component X_0 of X such that $f(X_0) = Y_0$. By (i) above we see that $f_n(X_0) = Y_0$ and, consequently, $f_n(X) \supset Y_0$ for sufficiently large n's. Since Y has finitely many irreducible components, we see that $f_n(X) = Y$, i.e., $f_n \in S$, for sufficiently large n's. This shows that S is open in $\mathrm{Hol}(X, Y)$.

Take any complex analytic space X. Throughout this paper we denote by $T(X) \xrightarrow[\tau_X]{} X$ the tangent (complex-linear) space over X in the category of reduced complex analytic spaces (cf. [6], [13]). Each fiber $T_x(X)$ in $T(X)$ is the Zariski tangent space of X at $x \in X$. We note that, given any holomorphic mapping $f: X \to Y$ between complex analytic spaces X and Y, the differential $f_*: T(X) \to T(Y)$ is holomorphic. Let Y be a complex analytic space and H a complex analytic subvariety of $\mathrm{Hol}(X, Y)$ assuming that X is compact. We want to describe the differential $\Phi_*: T(X \times H) \to T(Y)$, where $\Phi: X \times H \to Y$ is the restriction of the canonical mapping $\Phi: X \times \mathrm{Hol}(X, Y) \to Y$. Consider the (complex-linear) fiber space $T(X) \times T(H) \xrightarrow[\tau_X \times \tau_H]{} X \times H$ over $X \times H$. This fiber space is naturally identified with the tangent space $T(X \times H)$ over $X \times H$. In this situation, for any $(\xi, \eta) \in T(X) \times T(H)$ with $(\tau_X \times \tau_H)(\xi, \eta) = (x, h) \in X \times H$, we have

$$\Phi_*(\xi, \eta) = h_*\xi + (\Phi_x)_*\eta\,, \tag{1}$$

where $\Phi_x: H \to Y$ $(x \in X)$ is the holomorphic mapping defined by $\Phi_x(h) = h(x)$ for each $h \in H$.

In this paper we denote by $N(X) \underset{n_X}{\rightarrow} X$ the normalization of a complex analytic space X.

PROPOSITION 1. *Let X and Y be compact irreducible complex analytic spaces. Suppose that $\dim_C S > 0$ for the complex analytic subvariety*

$$S = \{f \in \mathrm{Hol}(X, Y);\ f(X) = Y\}$$

of $\mathrm{Hol}(X, Y)$. *Then there are compact irreducible complex analytic sets Γ of $T(Y)$ and Λ of $T(N(Y))$ which satisfy the following conditions:*

(i) $\tau_Y(\Gamma) = Y$.

(ii) $\Gamma \cap (T(Y) \setminus O)$ *is not empty, where O denotes the zero section of $T(Y)$.*

(iii) $n_{Y*}(\Lambda) = \Gamma$ *for the differential* n_{Y*}: $T(N(Y)) \rightarrow T(Y)$.

PROOF. Using Hartogs' theorem of holomorphy (cf. [4]), we see easily that $N(X) \times N(S)$ is normal. Hence $N(X) \times N(S) \xrightarrow[n_X \times n_S]{} X \times S$ is the normalization of $X \times S$ by the uniqueness of the normalization of the complex analytic space $X \times S$. Then we have a unique holomorphic mapping $\widetilde{\Phi}$: $N(X) \times N(S) \rightarrow N(Y)$ such that $n_Y \circ \widetilde{\Phi} = \Phi \circ (n_X \times n_S)$ on $N(X) \times N(S)$ for the canonical holomorphic mapping Φ: $X \times S \rightarrow Y$ (cf. [6, Lemma 1]). Since $\dim_C S > 0$, we can take a point x_0 of X, a nonsingular point h of S and an $\eta \neq 0$ in $T_h(S)$ such that $(\Phi_{x_0})_* \eta \neq 0$ in $T(Y)$; if not so, every irreducible component of S is zero-dimensional and, consequently, $\dim_C S = 0$. We define the mapping α: $X \rightarrow T(Y)$ by

$$\alpha(x) = (\Phi_x)_* \eta$$

for each $x \in X$. From (1) we see that α is holomorphic on X. Hence $\alpha(X)$ is a compact irreducible complex analytic set of $T(Y)$. Clearly $\tau_Y \circ \alpha(X) = h(X) = Y$ and $(\Phi_{x_0})_* \eta \in \alpha(X) \cap (T(Y) \setminus O)$. Thus $\Gamma = \alpha(X)$ satisfies the conditions (i), (ii) of the proposition. Now, take a point p_0 of $N(X)$ such that $n_X(p_0) = x_0$. Since h is a nonsingular point of S, there exists a nonsingular point $\widetilde{h}$ of $N(S)$ and an $\widetilde{\eta} \in T_{\widetilde{h}}(N(S))$ such that

$$n_S(\widetilde{h}) = h\ , \qquad n_{S*}\widetilde{\eta} = \eta\ ,$$

where n_{S*}: $T(N(S)) \rightarrow T(S)$ is the differential of n_S: $N(S) \rightarrow S$. Define the mapping $\widetilde{\alpha}$: $N(X) \rightarrow T(N(Y))$ by $\widetilde{\alpha}(p) = (\widetilde{\Phi}_p)_* \widetilde{\eta}$ for each $p \in N(X)$, where $\widetilde{\Phi}_p$: $N(S) \rightarrow N(Y)$ $(p \in N(X))$ is the holomorphic mapping defined by $\widetilde{\Phi}_p(q) = \widetilde{\Phi}(p, q)$ for each $q \in N(S)$. Obviously $\widetilde{\alpha}$ is holomorphic on $N(X)$. We regard $\widetilde{h} \in N(S)$ as the holomorphic mapping $\widetilde{\Phi}(\cdot, \widetilde{h})$ of $N(X)$ into $N(Y)$. Since $n_Y \circ \widetilde{\Phi} = \Phi \circ (n_X \times n_S)$ on $N(X) \times N(S)$, we have

$$n_Y \circ \tilde{h} = h \circ n_X$$

$$n_{Y_*} \circ \tilde{\alpha} = \alpha \circ n_X \text{ on } N(X) .$$

Hence $\tau_{N(Y)} \circ \tilde{\alpha}(N(X)) = \tilde{h}(N(X)) = N(Y)$. Further, for the compact irreducible complex analytic set $\Lambda = \tilde{\alpha}(N(X))$ of $T(N(Y))$, we have $n_{Y_*}(\Lambda) = \Gamma$ in $T(Y)$. This completes the proof.

LEMMA 2. *Let X be a pure-dimensional complex analytic space and Y an irreducible complex analytic space. Let $f: X \to Y$ be a proper holomorphic surjection with finite fibers over Y. Then there exists a nowhere dense complex analytic subvariety A of X which satisfies the following conditions*:

(a) *$f^{-1}f(A) = A$ in X.*

(b) *$X \setminus A$ and $Y \setminus f(A)$ are nonsingular.*

(c) *$f|_{X \setminus A}: X \setminus A \to Y \setminus f(A)$ is a locally biholomorphic covering of $Y \setminus f(A)$.*

PROOF. By Holmann [6, Proposition 5], $B = \{x \in X; \operatorname{corank}_x(f) > 0\}$ is a complex analytic subvariety of X. Since $f: X \to Y$ is a proper holomorphic surjection with finite fibers over Y, $\dim_C X = \dim_C Y$ and f is of maximal rank at some nonsingular points of each irreducible component of X (cf. [11, Chap. VII]). Hence B is nowhere dense in X. Put

$$A = f^{-1}(f(B \cup \operatorname{Sg}(X)) \cup \operatorname{Sg}(Y))$$

in X, where $\operatorname{Sg}(X)$ (resp. $\operatorname{Sg}(Y)$) is the set of all singular points of X (resp. Y). Then A is the required complex analytic subvariety of X.

3. Carathéodory-hyperbolic complex analytic spaces. Let X be a complex analytic space. We denote by $\Delta(X)$ the set of all holomorphic functions on X which take values in the unit open disc Δ of the complex line $\boldsymbol{C}$. The Carathéodory pseudometric E_X on $T(X)$ is defined by

$$E_X(\xi) = \sup\{|f_*\xi| \,;\, f \in \Delta(X)\} \tag{2}$$

for each $\xi \in T(X)$, where $|\ \ |$ is the norm associated with the Poincaré-Bergman metric $ds^2 = dzd\bar{z}/(1 - z\bar{z})^2$ on Δ. E_X is a real-valued continuous function on $T(X)$ (cf. Reiffen [12]). Further we have the following (cf. [10]):

(A) Given any holomorphic mapping $f: X \to Y$ between complex analytic spaces X and Y, we have $E_Y(f_*\xi) \leqq E_X(\xi)$ for every $\xi \in T(X)$.

(B) E_X is invariant under the action of the holomorphic automorphism group of X.

(C) $E_\Delta = |\ \ |$ on $T(\Delta)$.

Recall that a real-valued upper semicontinuous function u on a complex analytic space Z is said to be *pseudoconvex* on Z if and only if, for any $\phi \in \mathrm{Hol}(\varDelta, Z)$, $u\circ\phi$ is subharmonic on $\varDelta$ (cf. [1]). We notice that the continuous function $|\ |$ is pseudoconvex on $T(\varDelta) = \varDelta \times \boldsymbol{C}$. Hence, for any $\phi \in \mathrm{Hol}(\varDelta, T(X))$, $E_X\circ\phi = \sup\{|f_*\phi|\,;\, f \in \varDelta(X)\}$ is subharmonic on $\varDelta$. Thus we have:

LEMMA 3. *E_X is pseudoconvex on $T(X)$.*

Consider the Carathéodory pseudodistance c_X and the inner pseudodistance c_X^i induced by c_X on X (cf. [10]). Then c_X^i is given by

$$c_X^i(p, q) = \inf \int E_X(\gamma_*(t, d/dt))dt \tag{3}$$

for given points p, q of X, where the infimum is taken over the family of all piecewise smooth curves γ from p to q in X (cf. [10, p. 354]). By virtue of [12, Satz 2] we see that c_X^i is a distance on X if and only if the Carathéodory pseudodistance c_X on X is non-degenerate in the sense that each point p of X has an open neighborhood U in X such that $c_X(p, q) > 0$ for every q $(\neq p)$ in U. A complex analytic space X is said to be *Carathéodory-hyperbolic* if and only if there exists a covering space X' of X such that $c_{X'}^i$ is a distance on the complex analytic space X' (cf. [10, p. 367]). It is easy to see that X is Carathéodory-hyperbolic if and only if, for the universal covering space X' of X, $c_{X'}^i$ is a distance on the complex analytic space X'. Notice that any connected complex analytic subvariety of a Carathéodory-hyperbolic complex analytic space X is also Carathéodory-hyperbolic. From (3) we have:

LEMMA 4. *Let X be a Carathéodory-hyperbolic, simply connected complex analytic space. If $\phi\colon D \to X$ is a holomorphic mapping defined on a domain D of $\boldsymbol{C}$ such that $E_X(\phi_*(z, \partial/\partial z)) = 0$ for every $z \in D$, then ϕ is constant on D.*

EXAMPLE. Every bounded domain of $\boldsymbol{C}^n$ is Carathéodory-hyperbolic (cf. [10]). And a compact quotient of a bounded domain D of $\boldsymbol{C}^n$ by a properly discontinuous group acting freely on D is Carathéodory-hyperbolic.

Let X be a compact complex analytic space and $X' \underset{\pi}{\to} X$ the universal covering of X. Then we have the covering transformation group on X' for the universal covering $X' \underset{\pi}{\to} X$, i.e., the group of holomorphic automorphisms f on the complex analytic space X' such that $\pi\circ f = \pi$ on X'. By virtue of (B) we have the pseudometric D_X on $T(X)$ such that

$$(4) \qquad E_{X'} = D_X \circ \pi_*$$

on $T(X')$, where $\pi_*: T(X') \to T(X)$ is the differential of π. By Lemma 3 we have the following:

LEMMA 5. *For a compact complex analytic space X, D_X is continuous and pseudoconvex on $T(X)$.*

PROPOSITION 2. *Let X be a Carathéodory-hyperbolic, compact irreducible complex analytic space. Let Γ be a compact irreducible complex analytic set of $T(X)$ such that $\tau_X(\Gamma) = X$. Then Γ is the zero section of $T(X)$ over X.*

PROOF. Since the function D_X is pseudoconvex on $T(X)$ by Lemma 5, D_X takes a constant value c on the compact irreducible complex analytic set Γ of $T(X)$. Since $\tau_X|_\Gamma: \Gamma \to X$ is a proper holomorphic surjection with finite fibers over X, by Lemma 2 we can take a nowhere dense complex analytic subvariety A of Γ which satisfies the following conditions:

(a) $\Gamma \cap \tau_X^{-1}\tau_X(A) = A$, and $\tau_X(A)$ is nowhere dense in X.

(b) $\Gamma \setminus A$ and $X \setminus \tau_X(A)$ are nonsingular.

(c) $\tau_X|_{\Gamma \setminus A}: \Gamma \setminus A \to X \setminus \tau_X(A)$ is locally biholomorphic.

Assume that there exists a $\gamma \in \Gamma \setminus A$ such that $\gamma \neq 0$ in $T(X)$; if not so, Γ is obviously the zero section of $T(X)$. Then we can take an open neighborhood U of γ in $\Gamma \setminus A$ such that $V = \tau_X(U)$ is open in $X \setminus \tau_X(A)$ and such that $\tau_X|_U: U \to V$ is biholomorphic. We may regard $\mu = (\tau_X|_U)^{-1}: V \to U \subset T(X)$ as a holomorphic vector field over V. Then there exists a holomorphic mapping $h: \Delta_r \to V$ such that

$$h(0) = \tau_X(\gamma)$$
$$h_*(z, \partial/\partial z) = \mu(h(z)) \text{ for every } z \in \Delta_r \,,$$

where $\Delta_r = \{z \in \boldsymbol{C}; |z| < r\}$ for some positive number r. We can take a holomorphic lifting $g: \Delta_r \to X'$ of h for the universal covering $X' \xrightarrow[\pi]{} X$ of X. Since $D_X(h_*(z, \partial/\partial z)) = c$ for every $z \in \Delta_r$, by (4) we have

(d) $E_{X'}(g_*(z, \partial/\partial z)) = c$

for every $z \in \Delta_r$. Furthermore $\gamma' = g_*(0, \partial/\partial z) \neq 0$ in $T(X')$, because $\pi_*(\gamma') = h_*(0, \partial/\partial z) = \gamma \neq 0$. By Lemma 4 we see that $c > 0$, because g is nonconstant on Δ_r. Since $\Delta(X')$ is a normal family of holomorphic functions on X', we can take a holomorphic function $f \in \Delta(X')$ such that

$$E_{X'}(\gamma') = |f_*\gamma'| = c \,.$$

Furthermore, by (A) and (d), we have

$$|f_*\circ g_*(z, \partial/\partial z)| \leqq E_{X'}(g_*(z, \partial/\partial z)) = c$$

for every $z \in \Delta_r$. Hence, for the holomorphic function $\phi = f\circ g\colon \Delta_r \to \Delta$,

$$|\phi_*(0, \partial/\partial z)| = c$$

$$|\phi_*(z, \partial/\partial z)| = |\phi'(z)|/(1 - |\phi(z)|^2) \leqq c$$

for every $z \in \Delta_r$. Since $|\ \ | = E_\Delta$ is pseudoconvex on $T(\Delta)$, the function $|\phi_*(\cdot, \partial/\partial z)|$ is subharmonic on Δ_r. By the maximum principle we have

$$|\phi'(z)|/(1 - |\phi(z)|^2) = c$$

for every $z \in \Delta_r$. This implies that ϕ is constant on Δ_r. Hence $c = 0$. This is a contradiction. Hence Γ is the zero section of $T(X)$.

PROOF OF THEOREM 1. Let S be the complex analytic subvariety consisting of all holomorphic surjections in $\mathrm{Hol}(X, Y)$. Since Y is Carathéodory-hyperbolic, Y is hyperbolic in the sense of Kobayashi (cf. [10]). Hence $\mathrm{Hol}(X, Y)$ is compact and so is S. If X and Y are irreducible, Theorem 1 immediately follows from Propositions 1 and 2. In the general case, take any irreducible component S_0 of S and any $f \in S_0$. Then, for any irreducible component Y_0 of Y, there exists an irreducible component X_0 of X such that $f(X_0) = Y_0$. Indeed $g(X_0) = Y_0$ for every $g \in S_0$ by Lemma 1. Since the subvariety Y_0 of Y is Carathéodory-hyperbolic, the set $\{h \in \mathrm{Hol}(X_0, Y_0); h(X_0) = Y_0\}$ is finite as previously stated. By the connectedness of S_0 we see that $g = f$ on X_0 for every $g \in S_0$. Since X is compact connected and Y is compact hyperbolic, by [15, Theorem 1] we see that S_0 consists of a single element in $\mathrm{Hol}(X, Y)$. This means that S is finite, because S has finitely many irreducible components.

PROOF OF COROLLARY 1. Let A (resp. B) be a compact connected complex analytic subvariety of X (resp. Y). Put $F = \{f \in \mathrm{Hol}(X, Y); f(A) = B\}$. Since the subvariety B of Y is Carathéodory-hyperbolic, by Theorem 1 there are at most finitely many holomorphic surjections of the space A to the space B. Hence, for an arbitrary point a of A, there exists a finite subset B_0 of B such that $f(a) \in B_0$ for every $f \in F$. By [15, Theorem 1] we see that F is finite.

4. Proof of Theorem 2.

LEMMA 6. *Let X be a complex analytic space and μ a holomorphic vector field on X, i.e., a holomorphic mapping $\mu\colon X \to T(X)$ satisfying $\tau_X\circ\mu =$ identity on X. If $\mu(p_0) \neq 0$ in $T(X)$ at $p_0 \in X$, then there exists a nonconstant holomorphic mapping $h\colon \Delta_r \to X$ such that*

$$h(0) = p_0$$
$$h_*(z, \partial/\partial z) = \mu(h(z)) \text{ for every } z \in \Delta_r\,,$$

where $\Delta_r = \{z \in \boldsymbol{C}; |z| < r\}$ *for some positive number* r.

PROOF. It suffices to prove the lemma in the case that X is a complex analytic subvariety of a domain of holomorphy D in $\boldsymbol{C}^n$. By [13, Theorem 3.1] there exists a holomorphic vector field ν on D such that $\nu|_X = \mu$ on X. We may assume that $p_0 = 0 \in X \subset D$ in $\boldsymbol{C}^n$. As is well known, we can take local holomorphic coordinates $z_1, \cdots, z_n$ in a relatively compact Stein open neighborhood U of 0 in D such that $\nu = \partial/\partial z_1$ on U. Here $X \cap U$ is given by the intersection of the zeros of some holomorphic functions defined on U. Now, take any holomorphic function $f\colon U \to \boldsymbol{C}$ such that $f = 0$ on $X \cap U$. Then $\mu(f) = \partial f/\partial z_1 = 0$ on $X \cap U$. We inductively see that $\partial^m f/\partial z_1^m = 0$ on $X \cap U$ for every positive integer m. This means that $\{(z, 0, \cdots, 0) \in \boldsymbol{C}^n; z \in \Delta_r\} \subset X$ for some positive number r. Clearly $h(z) = (z, 0, \cdots, 0)$ $(z \in \Delta_r)$ is the required holomorphic mapping.

Now, let M be a complex analytic manifold with a hermitian metric ds^2_M. We denote by $\|\ \ \|$ the norm associated with ds^2_M on $T(M)$.

PROPOSITION 3. *Suppose that the holomorphic sectional curvature of* (M, ds^2_M) *is negative. Let* $f\colon D \to M$ *be a holomorphic mapping defined on a domain* D *of* $\boldsymbol{C}$ *such that*

$$\|f_*(z_0, \partial/\partial z)\| = \sup\{\|f_*(z, \partial/\partial z)\|; z \in D\}$$

for some $z_0 \in D$. *Then* f *is constant on* D.

PROOF. In case $\|f_*(z_0, \partial/\partial z)\| = 0$, clearly f is constant on D. Assume that $\|f_*(z_0, \partial/\partial z)\| > 0$, i.e., $f_*(z_0, \partial/\partial z) \neq 0$ in $T(M)$. Then we can take an open neighborhood U of z_0 in D such that $f_*(z, \partial/\partial z) \neq 0$ for every $z \in U$. Thus we have a nonsingular holomorphic curve $f(U)$ of M. Since $f^*ds^2_M = \|f_*(z, \partial/\partial z)\|^2 dz d\bar{z}$ on U, the Gaussian curvature K of the hermitian submanifold $f(U)$ of M is given by

$$K = \frac{-2}{\|f_*(z, \partial/\partial z)\|^2} \frac{\partial^2}{\partial z \partial \bar{z}} \log \|f_*(z, \partial/\partial z)\|^2$$

at each $f(z) \in f(U)$. Since the function $\|f_*(\cdot, \partial/\partial z)\|^2$ attains its maximum at $z_0 \in U$, we have $K = 0$ at $f(z_0) \in f(U)$. On the other hand, by a theorem of Kobayashi [8, p. 39], the holomorphic sectional curvature of the hermitian submanifold $f(U)$ of M does not exceed that of M. Hence $K < 0$ on $f(U)$. This is a contradiction. This completes the proof.

We notice that a (complete) hermitian manifold (M, ds_M^2) whose holomorphic sectional curvature is bounded above by a negative constant is (complete) hyperbolic in the sense of Kobayashi (cf. [8, p. 61]).

PROPOSITION 4. *Suppose that the holomorphic sectional curvature of (M, ds_M^2) is negative. Let X be a compact irreducible complex analytic space and Y a compact irreducible complex analytic subvariety of M. Then the set $S = \{f \in \mathrm{Hol}(X, Y); f(X) = Y\}$ is finite.*

PROOF. Since Y is compact, we can take an open neighborhood M' of Y in M such that the holomorphic sectional curvature of M' is bounded above by a negative constant. Since M' is hyperbolic, the subvariety Y of M' is also hyperbolic. Therefore the complex analytic subvariety S of $\mathrm{Hol}(X, Y)$ is compact. Hence it suffices to show $\dim_C S = 0$.

Assume that $\dim_C S > 0$. Then, by Proposition 1, there are compact irreducible complex analytic sets Γ of $T(Y)$ and Λ of $T(N(Y))$ which satisfy the following conditions:

(i) $\tau_Y(\Gamma) = Y$.

(ii) $\Gamma \cap (T(Y) \setminus O)$ is not empty for the zero section O of $T(Y)$.

(iii) $n_{Y*}(\Lambda) = \Gamma$ for the differential n_{Y*}: $T(N(Y)) \to T(Y)$.

Since Γ is compact, the continuous function $\| \ \|$, restricted to Γ, attains its maximum at some $\gamma_0 \in \Gamma$. Here $\gamma_0 \neq 0$ in $T(Y)$ by (ii) above. Since $n_{Y*}(\Lambda) = \Gamma$, there exists a $\lambda_0 \in \Lambda$ such that $n_{Y*}\lambda_0 = \gamma_0$. Put $p_0 = \tau_{N(Y)}(\lambda_0)$ in $N(Y)$. Since $\tau_{N(Y)}|_\Lambda$: $\Lambda \to N(Y)$ is a proper holomorphic surjection with finite fibers over $N(Y)$, we can take an open neighborhood U of λ_0 in Λ which satisfies the following conditions:

(a) $V = \tau_{N(Y)}(U)$ is open in $N(Y)$ and irreducible.

(b) $\tau = \tau_{N(Y)}|_U$: $U \to V$ is proper and surjective.

(c) $\tau^{-1}(p_0) = \{\lambda_0\}$.

Since U is pure-dimensional, by Lemma 2 there exists a nowhere dense complex analytic subvariety K of U such that $\tau|_{U \setminus K}$: $U \setminus K \to V \setminus \tau(K)$ is a locally biholomorphic covering of $V \setminus \tau(K)$. Then we have a holomorphic vector field μ_0 defined on $V \setminus \tau(K)$ by the formula

$$\mu_0(p) = \sum \lambda \qquad (\lambda \in \tau^{-1}(p))$$

for each $p \in V \setminus \tau(K)$. From the fact that τ is a proper holomorphic surjection defined on a complex analytic subvariety U of $\tau_{N(Y)}^{-1}(V)$, we see that μ_0: $V \setminus \tau(K) \to T(N(Y))$ is locally bounded on V. Since V is normal, we have a holomorphic vector field μ defined on V such that

$\mu|_{V\setminus\tau(K)} = \mu_0$ on $V\setminus\tau(K)$. Now, suppose that $\tau|_{U\setminus K}: U\setminus K \to V\setminus\tau(K)$ is an s-sheeted covering of $V\setminus\tau(K)$ for a positive integer s. Since $n_{Y*}(\Lambda) = \Gamma$, we see easily that

$$\mu(p_0) = s\lambda_0$$
$$\|n_{Y*}\mu\| \leqq s\|\gamma_0\| \text{ on } V .$$

Note that $\lambda_0 \neq 0$ in $T(N(Y))$ because of $n_{Y*}\lambda_0 = \gamma_0 \neq 0$. By Lemma 6 there exists a holomorphic mapping $h: \Delta_r \to V$ such that

$$h(0) = p_0 , \qquad h_*(z, \partial/\partial z) = \mu(h(z))$$

for every $z \in \Delta_r$. Consider the holomorphic mapping $g = n_Y \circ h: \Delta_r \to Y \subset M$. Then we have

$$g_*(0, \partial/\partial z) = s\gamma_0 , \qquad \|g_*(z, \partial/\partial z)\| = \|n_{Y*}\mu(h(z))\| \leqq s\|\gamma_0\|$$

for every $z \in \Delta_r$. By Proposition 3, g is constant on Δ_r and hence $\gamma_0 = 0$. This is a contradiction.

PROOF OF THEOREM 2. Note that M is hyperbolic in the sense of Kobayashi and so is B. Hence $S = \{f \in \mathrm{Hol}(A, B);\ f(A) = B\}$ is a compact complex analytic subvariety of $\mathrm{Hol}(A, B)$. Take any irreducible component S_0 of S and any $f \in S_0$. Then, for any irreducible component B_0 of B, there exists an irreducible component A_0 of A such that $f(A_0) = B_0$. In addition, $g(A_0) = B_0$ for every $g \in S_0$ by Lemma 1. On the other hand, since B_0 is a compact irreducible complex analytic subvariety of M, $\{h \in \mathrm{Hol}(A_0, B_0);\ h(A_0) = B_0\}$ is finite by Proposition 4. Therefore, by the connectedness of S_0, we have $g = f$ on A_0 for every $g \in S_0$. Then, by [15, Theorem 1] and the connectedness of S_0, we see that S_0 consists of a single element in $\mathrm{Hol}(A, B)$. This means that S is finite, because S has finitely many irreducible components. Now, put $F = \{f \in \mathrm{Hol}(X, M);\ f(A) = B\}$. By the finiteness of S above, for an arbitrary point a of A there exists a finite subset B_0 of B such that $f(a) \in B_0$ for every $f \in F$. We notice that M is complete hyperbolic and so taut (cf. [10, p. 378]). Hence, by [15, Theorem 1], F is finite.

5. Proof of Theorem 3. Denote by O the zero section of $T(M)$ over the compact complex analytic manifold M. The negativity (in the sense of Grauert) of $T(M)$ means that there exists a smooth exhaustion function ϕ on $T(M)$ which is strongly plurisubharmonic outside the zero section of $T(M)$; in this case O is the only maximal compact connected complex analytic subvariety of positive dimension in $T(M)$. It has been proved by Kobayashi [9] that a compact complex analytic manifold M with the negative $T(M)$ is hyperbolic. Here we give another proof of this assertion.

PROPOSITION 5. *Let M be a compact complex analytic manifold such that $T(M)$ is negative in the sense of Grauert. Then M is hyperbolic in the sense of Kobayashi.*

PROOF. Let h be a hermitian metric on M. We denote by $\| \ \|$ the norm associated with h on $T(M)$. Let ϕ be a smooth exhaustion function on $T(M)$ which is strongly plurisubharmonic on $T(M)\setminus O$. Assume that M is not hyperbolic. Then, by a theorem of Brody [2], there exists a nonconstant holomorphic mapping $f: \boldsymbol{C} \to M$ such that

$$\|f_*(z, \partial/\partial z)\| \leqq 1$$

for every $z \in \boldsymbol{C}$. Then $\phi(f_*(\cdot, \partial/\partial z))$ is a bounded subharmonic function on the complex line $\boldsymbol{C}$, because $\{\xi \in T(M);\ \|\xi\| \leqq 1\}$ is compact. Hence this function is constant on $\boldsymbol{C}$. Since ϕ is strongly plurisubharmonic on $T(M)\setminus O$, in consideration of small perturbations of ϕ on $T(M)$ we obtain that $\|f_*(\cdot, \partial/\partial z)\| = 0$ identically on $\boldsymbol{C}$. Consequently f is constant on $\boldsymbol{C}$. This is a contradiction.

Since M is compact and hyperbolic by Proposition 5, $\mathrm{Hol}(X, M)$ is compact and so every irreducible component of $\mathrm{Hol}(X, M)$ is compact. We notice that the set H_0 of all constant mappings of X into M is a connected component of $\mathrm{Hol}(X, M)$. Let H be an arbitrary irreducible component of $\mathrm{Hol}(X, M)\setminus H_0$ such that $\dim_C H > 0$. Then there exists an $x_0 \in X$ such that $\Phi_{x_0}: H \to M$ (see Section 2) is nonconstant. Further we can take an $h \in H$ and an $\eta \in T_h(H)$ such that $(\Phi_{x_0})_*\eta \neq 0$ in $T(M)$. Then, for the holomorphic mapping $\alpha: X \to T(M)$ defined by $\alpha(x) = (\Phi_x)_*\eta$ for each $x \in X$, $\alpha(X)$ is a compact connected complex analytic subvariety of $T(M)$ which is not contained in the zero section of $T(M)$. Since $\tau_M \circ \alpha(X) = h(X)$ in M and h is nonconstant on X, we have $\dim_C \alpha(X) > 0$. This contradicts the negativity of $T(M)$. Hence $\dim_C(\mathrm{Hol}(X, M)\setminus H_0) = 0$ and, consequently, $\mathrm{Hol}(X, M)\setminus H_0$ is finite.

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