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ON NORMAL AW^*-ALGEBRAS

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An AW^*-algebra M is a C^*-algebra with the following two properties: (a) in the set of projections M_p in M, every orthogonal collection has a least upper bound, (b) every maximal abelian $*$-subalgebra is generated by its projections [2].

Kaplansky ([2], [3], [4] (see also [1])) showed that much of the "non-spatial theory" of von Neumann algebras can be extended to AW^*-algebras. Above all, he showed that M_p is a complete lattice.

One of the difficulties in treating AW^*-algebras is that, because of the lack of the strong topology as in von Neumann algebras, there is no guarantees for the fact that whenever $\{f_\beta\}$ is an increasing net of projections with the supremum f in M_p, then f is the supremum of $\{f_\beta\}$ in the partially ordered space M_h of the hermitian part of M.

An AW^*-algebra M is said to be *normal* if, for every increasing net $\{e_\alpha\}$ of projections in M with the supremum e in M_p, e is the supremum of $\{e_\alpha\}$ in M_h (that is, if $a \in M_h$ such that $a \geqq e_\alpha$ for all α, then $a \geqq e$) [8].

It is known that every monotone complete C^*-algebra (a von Neumann algebra, a type 1 AW^*-algebra) is normal. In [8], Wright proved the following interesting result, by using the regular ring, to the effect that every finite AW^*-algebra is normal (a similar result was also proved by Hamana by using the regular monotone completion of AW^*-algebras [9]).

We say that an increasing net $\{e_\alpha\}$ of projections with the supremum e in M_p in a C^*-algebra M is *well-behaved* if, whenever x (in M_h) satisfies $e_\alpha x e_\alpha \geqq 0$ for all α, then $exe \geqq 0$.

In this paper, by using the above concept, we shall show the following theorem which is a nominally more general result on the one hand and is a simple alternative proof of the theorem of Wright and Hamana on the other (see Corollary).

THEOREM. *Let M be an AW^*-algebra. Then M is normal if and only if every increasing net of projections in M is well-behaved.*

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This is, however, an easy consequence of the following proposition:

PROPOSITION. *Let M be an AW^*-algebra and let $\{e_\alpha\}$ be an increasing net in M_p with the supremum e in M_p. Then e is the supremum of $\{e_\alpha\}$ in M_h if and only if $\{e_\alpha\}$ is well-behaved (the "if" part is valid for a general (unital) C^*-algebra).*

We shall break up the proof of Proposition into a sequence of lemmas.

LEMMA 1. *Let M be a C^*-algebra and let $\{e_\alpha\}$ be an increasing net in M_p with the supremum e in M_p. Suppose that $\{e_\alpha\}$ is well-behaved. Then e is the supremum of $\{e_\alpha\}$ in M_h.*

PROOF. We have only to check that $e_\alpha \leqq a$ for all α for some a in M_h implies $e \leqq a$. Let $b_n = (1/n + a)^{-1}a^{1/2}$ (see [5]) for each positive integer n (note that $a \geqq 0$). Then for every pair α and n,

$$\begin{aligned}(e_\alpha b_n)^*(e_\alpha b_n) &= a^{1/2}(1/n + a)^{-1}e_\alpha(1/n + a)^{-1}a^{1/2} \leqq a^{1/2}(1/n + a)^{-1}a(1/n + a)^{-1}a^{1/2} \\ &= (a(1/n + a)^{-1})^2 \leqq 1 .\end{aligned}$$

Thus we get that $\|e_\alpha b_n\| \leqq 1$ and $(e_\alpha b_n)(e_\alpha b_n)^* \leqq 1$ for every α and n, that is, $e_\alpha(1 - b_n b_n^*)e_\alpha \geqq 0$ for every α and n. Since $\{e_\alpha\}$ is well-behaved, this implies that $e(1 - b_n b_n^*)e \geqq 0$ for all n, that is, for each n, $1 \geqq e \geqq (eb_n)(eb_n)^*$.

On the other hand, since, for all n,

$$\begin{aligned}\|e_\alpha(e - eb_n a^{1/2})\|^2 &= \|e_\alpha(1 - (1/n + a)^{-1}a)\|^2 \\ &= \|(1 - (1/n + a)^{-1}a)e_\alpha(1 - (1/n + a)^{-1}a)\| \\ &\leqq \|(1 - (1/n + a)^{-1}a)a(1 - (1/n + a)^{-1}a)\| \\ &\leqq 1/n^2\|(1/n + a)^{-1}\| \leqq 1/n ,\end{aligned}$$

we see that

$$e_\alpha(1/n - (e - eb_n a^{1/2})(e - eb_n a^{1/2})^*)e_\alpha \geqq 0$$

for all α and n. Thus, by the same reasoning, we have that for all n

$$e(1/n - (e - eb_n a^{1/2})(e - eb_n a^{1/2})^*)e \geqq 0 .$$

This implies that for each n

$$\|e - eb_n a^{1/2}\| \leqq (1/n)^{1/2}$$

and that

$$\begin{aligned}\|e - (eb_n a^{1/2})^*(eb_n a^{1/2})\| &\leqq \|(e - eb_n a^{1/2})^*e\| + \|(eb_n a^{1/2})^*(e - eb_n a^{1/2})\| \\ &\leqq 3(1/n)^{1/2}\end{aligned}$$

for all n.

Combining these estimates, we see that

$$e \leqq 3(1/n)^{1/2} + (eb_n a^{1/2})^*(eb_n a^{1/2}) = 3(1/n)^{1/2} + a^{1/2}(eb_n)^*(eb_n)a^{1/2}$$
$$\leqq 3(1/n)^{1/2} + a \qquad (\text{because } \|eb_n\| \leqq 1)$$

for all n, that is, $e \leqq a$ and the lemma follows.

The next lemma is due to Hamana ([9]) and is included only for completeness.

LEMMA 2. *Let M be a C^*-algebra and let $\{e_\alpha\}$ be an increasing net in M_p with the supremum e in M_p. If e is the supremum of $\{e_\alpha\}$ in M_h, then for every non-negative a in M, $\{ae_\alpha a\}$ has the supremum aea in M_h.*

LEMMA 3. *Let M be an AW^*-algebra and let $\{e_\alpha\}$ be an increasing net in M_p with the supremum e in M_p. Suppose that e is the supremum of $\{e_\alpha\}$ in M_h. Then $\{e_\alpha\}$ is well-behaved.*

PROOF. We must show that $exe \geqq 0$ for every x (in M_h) with $e_\alpha x e_\alpha \geqq 0$ for all α. To prove this, we may assume that $\|x\| \leqq 1$ and $e = 1$ without loss of generality because $\{e_\alpha\}$ has the supremum e in $(eMe)_h$.

Since $(1+x)(1-e_\alpha)(1+x) - (1-x)(1-e_\alpha)(1-x) = 2x(1-e_\alpha) + 2(1-e_\alpha)x$, we see that

$$\begin{aligned} e_\alpha x e_\alpha - x &= (1-e_\alpha)x(1-e_\alpha) - (1-e_\alpha)x - x(1-e_\alpha) \\ &= (1/2)((1-x)(1-e_\alpha)(1-x) - (1+x)(1-e_\alpha)(1+x)) + (1-e_\alpha)x(1-e_\alpha) \\ &\leqq (1/2)(1-x)(1-e_\alpha)(1-x) + 1 - e_\alpha \quad (\text{because } \|x\| \leqq 1 \text{ and } x \in M_h)\,. \end{aligned}$$

If $x = x^+ - x^-$ ($x^+x^- = 0$, $x^+ \geqq 0$, $x^- \geqq 0$) and $x^- \neq 0$, then by the spectral theory, we can find a non zero projection q in M and a positive number ε such that $x^- \geqq \varepsilon q$ and $(1-q)x^+ = x^+$. By the above estimates, it follows that

$$qe_\alpha x e_\alpha q - qxq \leqq (1/2)q(1-x)(1-e_\alpha)(1-x)q + q(1-e_\alpha)q\,.$$

Thus, noting that $qe_\alpha x e_\alpha q \geqq 0$ for all α, we have

$$\varepsilon q \leqq qx^-q = -qxq \leqq qe_\alpha x e_\alpha q - qxq \leqq (1/2)q(1-x)(1-e_\alpha)(1-x)q + q(1-e_\alpha)q$$

for all α. Since $\{(1/2)q(1-x)(1-e_\alpha)(1-x)q + q(1-e_\alpha)q\}$ has the infimum 0 in M_h (by Lemma 2), we see that $\varepsilon q \leqq 0$ and $q = 0$. This is a contradiction. Thus $x^- = 0$ and $x = x^+ \geqq 0$. The lemma follows.

Combining Lemmas 1 and 3, we get Proposition (and Theorem follows immediately from Proposition).

COROLLARY. ([8], [9]). *Let M be an AW^*-algebra and $\{e_\alpha\}$ be an increasing net in M_p with the supremum e in M_p. Suppose that $e - e_\alpha$ is finite for all α. Then e is the supremum of $\{e_\alpha\}$ in M_h. In particular if M is finite, then M is normal.*

PROOF. By Lemma 1, we have only to show that $\{e_\alpha\}$ is well-behaved. To prove this, we may assume that $e = 1$. Suppose that x in M_h satisfies $e_\alpha x e_\alpha \geqq 0$ for all α.

If $x^- \neq 0$, then there are a non zero projection q in M and a positive number ε such that $x^- \geqq \varepsilon q$ and $(1 - q)x^+ = x^+$. Putting $f_\alpha = e_\alpha \wedge q$, we have

$$0 \leqq f_\alpha e_\alpha x e_\alpha f_\alpha = f_\alpha x f_\alpha = f_\alpha q x q f_\alpha = -f_\alpha x^- q f_\alpha \leqq -\varepsilon f_\alpha q f_\alpha$$

for all α and $qf_\alpha = 0$ for all α. Thus it follows that $e_\alpha \wedge q = 0$ for all α and $q = q - e_\alpha \wedge q \sim e_\alpha \vee q - e_\alpha \leqq 1 - e_\alpha$ for all α. Since $1 - e_\alpha$ is finite for all α, this implies that $q = 0$, because $1 - e_\alpha \downarrow 0$. This is a contradiction. Thus $x \geqq 0$ and $\{e_\alpha\}$ is well-behaved. This completes the proof.

REMARKS. (1) Let M be an AW^*-algebra and suppose that whenever $\{e_\alpha\}$ is an *orthogonal* family of projections in M with the supremum e in M_p, then e is the supremum of $\{e_\alpha\}$ in M_h. Then M is normal. (Note that M is normal if and only if M is an AW^*-subalgebra of its regular monotone completion $\bar{M}$ ([8], [9])). In fact, we have that e is the supremum of $\{e_\alpha\}$ in $(\bar{M})_p$ (see [9]) and M is an AW^*-subalgebra of $\bar{M}$. Thus, by Corollary 3 of [8], M is normal. The fact that if M is normal, then M is an AW^*-subalgebra of $\bar{M}$ was proved in [9].

(2) Let M be a normal, semi-finite AW^*-factor and suppose that M has a faithful state. Then M is a W^*-algebra. In fact, for each increasing sequence of projections $\{e_n\}$ in M with the supremum e in M_p, it follows that for all p in M_p, $\{pe_np\}$ has the supremum pep in M_h. Thus by Remark after the proof of Theorem 1 in [7], we can prove the above statement.

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Added in Proof (Received November 6, 1981). Proposition still holds for general unital C^*-algebras. To prove this, we have only to check that Lemma 3 is valid under the condition that M is a unital C^*-algebra. In fact, in the same way as that in the first half of the proof of Lemma 3, we conclude that for every α, $e_\alpha x e_\alpha - x \leqq (1/2)(1-x)(1-e_\alpha)(1-x)+1-e_\alpha$, because $\|x\| \leqq 1$ and $x \in M_h$. Take $y=(1/2)((x^2)^{1/2}+x)$ and $z=(1/2)((x^2)^{1/2}-x)$. We see that $x=y-z$, y, $z \in M_h$, $zy=0$ and $z \geqq 0$, $y \geqq 0$. Moreover, y and z commute with x. Hence, it follows from the above inequality that

$$ze_\alpha x e_\alpha z - zxz \leqq (1/2)z(1-x)(1-e_\alpha)(1-x)z + z(1-e_\alpha)z .$$

Thus, since $ze_\alpha x e_\alpha z \geqq 0$ for all α and $zxz = -z^3$, we see that $z^3 \leqq (1/2)z(1-x)(1-e_\alpha)(1-x)z + z(1+e_\alpha)z$ for all α and, by the same reasoning as that in Lemma 3, it follows that $z^3 \leqq 0$, that is, $z=0$. The proof is completed.

Using this, we can give a simple proof of the following corollary (the special, but important, case for Corollary 4.10 in [9]).

Corollary A. *Let A be a unital C^*-algebra. Then for an increasing net $\{e_\alpha\}$ of projections in A with the supremum e in A_p, suppose that e is the supremum of $\{e_\alpha\}$ in A_h (or equivalently that $\{e_\alpha\}$ is well behaved). Then xex^* is the supremum of $\{xe_\alpha x^*\}$ in A_h for each $x \in A$.*

We have only to show that $xe_\alpha x^* \leqq a$ for all α for some $a \in A_h$ implies that $xex^* \leqq a$. $xe_\alpha x^* \leqq a$ implies that

$$(a+1/n)^{-1/2} x e_\alpha x^* (a+1/n)^{-1/2} \leqq 1$$

for each α and n because $a \geqq 0$. Thus we get that $\|(a+1/n)^{-1/2} x e_\alpha\| \leqq 1$ for each α and n. This implies that $e_\alpha(e - ex^*(a+1/n)^{-1}xe)e_\alpha \geqq 0$ for all α. Since $\{e_\alpha\}$ is well behaved, it follows that

$$e(e - ex^*(a+1/n)^{-1}xe)e \geqq 0$$

and $\|ex^*(a+1/n)^{-1}xe\| \leqq 1$ for all n. Thus we conclude that

$$(a+1/n)^{-1/2} xex^* (a+1/n)^{-1/2} \leqq 1$$

for all n. This implies that $xex^* \leqq a + 1/n$ for all n and $xex^* \leqq a$. This completes the proof.

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