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HOMOGENEOUS KÄHLER MANIFOLDS OF COMPLEX DIMENSION TWO

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Introduction. Let M be a connected and simply connected homogeneous Kähler manifold. In this note, by a homogeneous Kähler manifold we mean a Kähler manifold on which the group of all holomorphic isometries acts transitively. The purpose of this note is to prove the following theorem.

THEOREM 1. *If* $\dim_C M = 2$ *and if the canonical hermitian form of* M *is degenerate and non-zero, then the fibering of* M *due to Hano and Kobayashi* [3] *is holomorphic and the fiber with the induced Kähler structure is a homogeneous Kähler manifold with zero Ricci curvature.*

Our proof of Theorem 1 is based on the theory of Kähler algebras developed by Gindikin, Pjateckii-Sapiro and Vinberg [2]. They studied the structure of homogeneous Kähler manifolds and stated the following *Fundamental conjecture*:

Every homogeneous Kähler manifold admits a holomorphic fibering, whose base is analytically isomorphic to a homogeneous bounded domain and whose fiber with the induced Kähler structure is isomorphic to the direct product of a locally flat homogeneous Kähler manifold and a simply connected compact homogeneous Kähler manifold.

Combining Theorem 1 with the results of Alekseevskii and Kimel'fel'd [1] and Shima [7], [8], we see that the above conjecture is true for a complex two dimensional connected and simply connected homogeneous Kähler manifold. As an immediate consequence of this fact, we obtain the following.

THEOREM 2. *Let* M *be a connected homogeneous Kähler manifold of complex dimension two. If* M *contains no complex line, that is, if there are no non-constant holomorphic maps of* $\boldsymbol{C}$ *into* M, *then* M *is homogeneous bounded domain in* $\boldsymbol{C}^2$.

In the theory of hyperbolic complex manifolds in the sense of Kobayashi [4], we have the following basic problem (see [4, Problem 12, p. 133]):

Let M be a homogeneous complex manifold of complex dimension n which is hyperbolic in the sense of Kobayashi. Then is M a homogeneous bounded domain in C^n?

Noting that hyperbolic complex manifolds contain no complex line, we see that Theorem 2 provides an affirmative answer to the above problem when M is a homogeneous Kähler manifold of complex dimension two.

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1. Preliminaries. In this section we recall the definition of Kähler algebras and state several lemmas for later use.

We denote by M a connected homogeneous Kähler manifold on which a connected Lie group G acts transitively as a group of holomorphic isometries, and by K an isotropy subgroup of G at a point o of M. Let I be the G-invariant complex structure tensor on M, let g be the G-invariant Kähler metric on M and let v be the G-invariant volume element corresponding to the Kähler metric g. In terms of a local coordinate system $\{z_1, \cdots, z_n\}$ on M, the form v is expressed by $v = (\sqrt{-1})^n F dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$, where F is a positive function. The G-invariant hermitian form

$$h = \sum \frac{\partial^2 \log F}{\partial z_i \partial \bar{z}_j} dz_i d\bar{z}_j$$

is called the canonical hermitian form of M. It is easy to see that the Ricci tensor of the Kähler manifold M is equal to $-h$.

Let $\mathfrak{g}$ be the Lie algebra of all left invariant vector fields on G and let $\mathfrak{k}$ be the subalgebra of $\mathfrak{g}$ corresponding to K. Let π be the canonical projection of G onto $M = G/K$ and let $T_o(M)$ be the tangent space of M at the point $o = \pi(e)$, where e is the identity element of G. We define a linear mapping π_* of $\mathfrak{g}$ onto $T_o(M)$ as follows:

$$\pi_*(X) = (d\pi)_e(X_e) \quad \text{for} \quad X \in \mathfrak{g} ,$$

where $(d\pi)_e$ is the differential of π at e and X_e is the value of X at e. There exist a linear endomorphism J of $\mathfrak{g}$ and a skew symmetric bilinear form ρ on $\mathfrak{g}$ such that

$$\pi_* JX = I_o(\pi_* X) , \qquad \rho(X, Y) = g_o(\pi_* X, I_o(\pi_* Y)) \qquad \text{for} \quad X, Y \in \mathfrak{g} ,$$

where I_o and g_o are the values of I and g at o, respectively. Then the quadruple $(\mathfrak{g}, \mathfrak{k}, J, \rho)$ satisfies the following properties and is called the Kähler algebra of M (see Gindikin, Pjateckii-Sapiro and Vinberg [2]):

(K.1) $J\mathfrak{k} \subset \mathfrak{k},\ J^2 \equiv -\mathrm{id}\ (\mathrm{mod}\ \mathfrak{k})$,

(K.2) $[W, JX] \equiv J[W, X]\ (\mathrm{mod}\ \mathfrak{k})$,

(K.3) $[JX, JY] \equiv J[JX, Y] + J[X, JY] + [X, Y]\ (\mathrm{mod}\ \mathfrak{k})$,

(K.4) $\rho(W, X) = 0$,

(K.5) $\rho(JX, JY) = \rho(X, Y)$,

(K.6) $\rho(JX, X) > 0,\ X \notin \mathfrak{k}$,

(K.7) $\rho([X, Y], Z) + \rho([Y, Z], X) + \rho([Z, X], Y) = 0$,

where $X, Y, Z \in \mathfrak{g},\ W \in \mathfrak{k}$.

Putting $\eta(X, Y) = h_o(\pi_* X, \pi_* Y)$ and

$$\psi(X) = \mathrm{Tr}_{\mathfrak{g}/\mathfrak{k}}(\mathrm{ad}\,(JX) - J\,\mathrm{ad}\,(X))\ , \tag{1.1}$$

we have

$$2\eta(X, Y) = \psi([JX, Y]) \quad \text{for} \quad X, Y \in \mathfrak{g}\ (\text{see [6]})\ . \tag{1.2}$$

The forms η and ψ satisfy the following properties:

$$\eta(JX, JY) = \eta(X, Y)\ , \tag{1.3}$$

$$\psi([W, X]) = 0\ , \tag{1.4}$$

$$\psi([JX, JY]) = \psi([X, Y]) \quad \text{for} \quad X, Y \in \mathfrak{g},\ W \in \mathfrak{k}\ . \tag{1.5}$$

We note that if G acts effectively on M, then $\mathfrak{k}$ contains no non-zero ideal of $\mathfrak{g}$.

Now we have the following lemmas which are due to Shima [8].

LEMMA 1 (cf. [8, Lemma 2.4]). *Let $\mathfrak{r}$ be an ideal of $\mathfrak{g}$. Suppose $\psi = 0$ on $\mathfrak{r}$. Then $\mathfrak{r} \subset \{X \in \mathfrak{g}; \eta(X, Y) = 0$ for all $Y \in \mathfrak{g}\}$.*

LEMMA 2 (cf. [8, Lemma 2.3]). *Let $\mathfrak{r}$ be a commutative ideal of $\mathfrak{g}$. If G acts effectively on M and if the center of $\mathfrak{g}$ is zero, then $\mathfrak{k} \cap \mathfrak{r} = \mathfrak{k} \cap J\mathfrak{r} = \{0\}$.*

LEMMA 3 (cf. [8, Lemma 2.6]). *Let $\{E\}$ be a one dimensional ideal of $\mathfrak{g}$. Then we have:*

(a) *If $\psi(E) \neq 0$, then $[E, \mathfrak{k}] = \{0\}$.*

(b) *If $[E, \mathfrak{k}] = \{0\}$ and if G acts effectively on M, then there exists an endomorphism $\tilde{J}$ of $\mathfrak{g}$ such that $\tilde{J} \equiv J\ (\mathrm{mod}\ \mathfrak{k})$ and $[\tilde{J}E, \mathfrak{k}] = \{0\}$.*

LEMMA 4 (cf. [8, Lemma 3.2]). *Let $\{E\}$ be a one dimensional ideal of $\mathfrak{g}$. If $\psi(E) \neq 0$ and $[JE, \mathfrak{k}] = \{0\}$, then $[JE, E] \neq 0$.*

LEMMA 5 (cf. [8, Lemma 3.3]). *Let $\{E\}$ be a one dimensional ideal of $\mathfrak{g}$. Suppose $[E, \mathfrak{k}] = [JE, \mathfrak{k}] = \{0\}$ and $[JE, E] = E$, and put $\mathfrak{p} = \{P \in \mathfrak{g};$*

$[P, E] = [JP, E] = 0\}$. *Then* $\mathrm{ad}\,(JE)\mathfrak{p} \subset \mathfrak{p}$ *and* $\mathfrak{g} = \{JE\} + \{E\} + \mathfrak{p}$ *(direct sum), where* $\{JE\}$, $\{E\}$, $\mathfrak{p}$ *are mutually orthogonal with respect to the form* η, *and* η *is positive definite on* $\{JE\} + \{E\}$.

2. Existence of certain ideals. Throughout this section we use the same notations as in the previous section and assume the following:

(2.1) $$\dim_C M = 2\ .$$

(2.2) The canonical hermitian form h is degenerate and non-zero.

(2.3) G acts effectively on M.

Then, by a result of Hano and Kobayashi [3] there exists a closed subgroup L of G satisfying the following properties:

(2.4) $$L \supset K\ .$$

(2.5) The coset space L/K is a one dimensional connected complex submanifold of $M = G/K$.

(2.6) $T_o(L/K) = \{v \in T_o(M); h_o(v, v') = 0 \text{ for all } v' \in T_o(M)\}$, where $T_o(L/K)$ is the tangent space of L/K at the point $o = \pi(e)$.

It is easy to see that the submanifold L/K of M is a homogeneous Kähler manifold with the Kähler metric induced from M.

Let $\mathfrak{l}$ be the subalgebra of $\mathfrak{g}$ corresponding to L. Then $\mathfrak{l} \supset \mathfrak{k}$ and $\mathfrak{l}$ is J-invariant. From (2.6), we have

(2.7) $$\mathfrak{l} = \pi_*^{-1}(T_o(L/K)) = \{X \in \mathfrak{g};\ \eta(X, Y) = 0 \text{ for all } Y \in \mathfrak{g}\}\ .$$

We see $\dim \mathfrak{g}/\mathfrak{k} = 4$ by (2.1) and, furthermore, $\dim \mathfrak{g}/\mathfrak{l} = \dim \mathfrak{l}/\mathfrak{k} = 2$ by (2.5).

The purpose of this section is to prove the following.

PROPOSITION. *The Lie algebra* $\mathfrak{g}$ *contains a one dimensional ideal or a two dimensional commutative ideal* $\mathfrak{r}$ *such that* $\mathfrak{l} = \mathfrak{k} + \mathfrak{r}$.

If the center of $\mathfrak{g}$ is not zero, then it is clear that there exists a one dimensional ideal of $\mathfrak{g}$. Therefore it is sufficient to prove the above proposition when the center of $\mathfrak{g}$ is zero. For the purpose, we need the following lemma.

LEMMA 6. *Let* $\mathfrak{r}$ *be a commutative ideal of* $\mathfrak{g}$. *Then* $\mathfrak{l} + \mathfrak{r} \neq \mathfrak{g}$.

PROOF. First, we note that $\psi([A, X]) = 0$ for all $A \in \mathfrak{l}$ and $X \in \mathfrak{g}$. In fact, by (1.2), (1.5) and (2.7) we see $\psi([A, X]) = \psi([JA, JX]) = 2\eta(A, JX) = 0$. Assume $\mathfrak{g} = \mathfrak{l} + \mathfrak{r}$. Then, we have $JX = A + B$ and $X' = A' + B'$ for $X, X' \in \mathfrak{g}$, where $A, A' \in \mathfrak{l}$ and $B, B' \in \mathfrak{r}$. Since $\psi([A, Y]) =$

$\psi([A', Y']) = 0$ for all $Y, Y' \in \mathfrak{g}$ and since $\mathfrak{r}$ is commutative, it follows that $2\eta(X, X') = \psi([JX, X']) = \psi([A + B, A' + B']) = \psi([B, B']) = 0$. This contradicts the assumption (2.2), and hence the lemma is proved.

We now prove Proposition under the assumption that the center of $\mathfrak{g}$ is zero. Suppose that $\mathfrak{g}$ is semi-simple. Then, by a result of Koszul [6], h is non-degenerate, which contradicts the assumption (2.2). Therefore $\mathfrak{g}$ is not semi-simple, i.e., there exists a non-zero commutative ideal $\mathfrak{r}$. Since $\dim \mathfrak{g}/\mathfrak{k} = 4$, we have $\dim \mathfrak{r} = 1, 2, 3, 4$ by Lemma 2. In the case $\dim \mathfrak{r} = 1$, there is nothing to prove. We consider the cases $\dim \mathfrak{r} = 2, 3, 4$. Using Lemma 6, we see $\dim \mathfrak{r} \neq 4$. For, if $\dim \mathfrak{r} = 4$, then $\mathfrak{g} = \mathfrak{k} + \mathfrak{r} = \mathfrak{l} + \mathfrak{r}$ by Lemma 2. This contradicts Lemma 6. We show that $\mathfrak{r}$ contains an ideal satisfying the assertions of Proposition in the cases $\dim \mathfrak{r} = 2, 3$.

First, suppose $\dim \mathfrak{r} = 3$. Since $\dim \mathfrak{g}/\mathfrak{l} = 2$, we see $\dim \mathfrak{l} \cap \mathfrak{r} \neq 0$. Lemma 6 and the fact $\dim \mathfrak{g}/\mathfrak{l} = 2$ yield $\dim \mathfrak{l} \cap \mathfrak{r} \neq 1$. Furthermore, since $\dim \mathfrak{l}/\mathfrak{k} = 2$, we see $\dim \mathfrak{l} \cap \mathfrak{r} \neq 3$ by Lemma 2. Hence $\dim \mathfrak{l} \cap \mathfrak{r} = 2$. We have $J(\mathfrak{l} \cap \mathfrak{r}) \subset \mathfrak{l} = \mathfrak{k} + \mathfrak{l} \cap \mathfrak{r}$, because $\mathfrak{l}$ is J-invariant and $\mathfrak{k} \cap (\mathfrak{l} \cap \mathfrak{r}) = \mathfrak{k} \cap \mathfrak{r} = \{0\}$ by Lemma 2. This implies that there exists an endomorphism $\tilde{J}$ of $\mathfrak{g}$ such that $\tilde{J} \equiv J \pmod{\mathfrak{k}}$ and $\tilde{J}(\mathfrak{l} \cap \mathfrak{r}) \subset \mathfrak{l} \cap \mathfrak{r}$. Therefore we may suppose $J(\mathfrak{l} \cap \mathfrak{r}) \subset \mathfrak{l} \cap \mathfrak{r}$. Then $J^2 = -\mathrm{id}$ on $\mathfrak{l} \cap \mathfrak{r}$ by (K.1). Moreover, we have $\psi \neq 0$ on $\mathfrak{r}$. In fact, suppose $\psi = 0$ on $\mathfrak{r}$. Then $\mathfrak{r} \subset \mathfrak{l}$ by Lemma 1, which contradicts $\dim \mathfrak{l} \cap \mathfrak{r} = 2$. Using these facts, we can select a basis of $\mathfrak{r}$ as follows:

$$\mathfrak{r} = \{JE, E, F\}\,, \qquad \mathfrak{l} \cap \mathfrak{r} = \{JE, E\}$$

and

$$\text{(i)} \qquad \psi(JE) = 0\,, \qquad \psi(E) = 0\,, \qquad \psi(F) \neq 0$$

or

$$\text{(ii)} \qquad \psi(JE) \neq 0\,, \qquad \psi(E) = 0\,, \qquad \psi(F) = 0\,.$$

Put $\mathfrak{g}' = \mathfrak{k} + J\mathfrak{r} + \mathfrak{r}$. Then $\dim_C \mathfrak{g}'/\mathfrak{k} = 1$ or 2, since $\mathfrak{g}'$ is J-invariant, $\mathfrak{k} \cap \mathfrak{r} = \{0\}$ and $\dim_C \mathfrak{g}/\mathfrak{k} = 2$. From $\dim \mathfrak{r} = 3$, we have $\dim_C \mathfrak{g}'/\mathfrak{k} = 2$, which implies $\mathfrak{g} = \mathfrak{g}' = \mathfrak{k} + J\mathfrak{r} + \mathfrak{r}$. Hence $\mathfrak{g} = \mathfrak{k} + \{JF\} + \{JE, E, F\}$ (direct sum). Further $\mathfrak{l} = \mathfrak{k} + \{JE, E\}$.

Case (i). It suffices to show that $\{JE, E\}$ is an ideal of $\mathfrak{g}$. Since $\{JE, E\} = \mathfrak{l} \cap \mathfrak{r}$ is an ideal of $\mathfrak{l}$, we see $[\mathfrak{l}, \{JE, E\}] \subset \{JE, E\}$. The commutativity of $\mathfrak{r}$ implies $[\{F\}, \{JE, E\}] = \{0\} \subset \{JE, E\}$. Hence it is sufficient to show $[\{JF\}, \{JE, E\}] \subset \{JE, E\}$. Since $\{JE, E, F\} = \mathfrak{r}$ is an ideal of $\mathfrak{g}$, we have $[JF, E] = \lambda JE + \mu E + \nu F$ for some $\lambda, \mu, \nu \in \boldsymbol{R}$. The fact $E\in\mathfrak{l}$ yields $\eta(F, E)=0$ by (2.7). From these and from $\psi(JE)=\psi(E)=0$, it follows that $0=2\eta(F, E)=\psi([JF, E])=\lambda\psi(JE)+\mu\psi(E)+\nu\psi(F)=\nu\psi(F)$, which implies $\nu=0$, since $\psi(F)\neq 0$. Therefore $[JF, E]=\lambda JE+\mu E\in\{JE, E\}$. Similarly, we have $[JF, JE]\in\{JE, E\}$. Thus $[\{JF\}, \{JE, E\}]\subset\{JE, E\}$.

Case (ii). It suffices to prove that $\{E\}$ is an ideal of $\mathfrak{g}$. Since $\{JE, E\}$ is an ideal of $\mathfrak{l}$, we can put $[X, E]=\lambda JE+\mu E$ for $X\in\mathfrak{l}$, where $\lambda, \mu\in\boldsymbol{R}$. Then $\psi(JE)\neq 0$, $\psi(E)=0$ and $\eta(X, JE)=0$ yield $[X, E]=\mu E$ as in the case (i), which shows $[\mathfrak{l}, \{E\}]\subset\{E\}$. Since $\mathfrak{r}$ is commutative, we see $[\{F\}, \{E\}]=\{0\}\subset\{E\}$. Therefore it suffices to show $[\{JF\}, \{E\}]\subset\{E\}$. Put $[JF, E]=\lambda JE+\mu E+\nu F$, where $\lambda, \mu, \nu\in\boldsymbol{R}$. Then, using $\psi(JE)\neq 0$, $\psi(E)=0$, $\psi(F)=0$ and $\eta(F, E)=0$, we have $[JF, E]=\mu E+\nu F$, which together with $[E, F]=[JE, F]=0$ and (K.3) implies $[JE, JF]=-\mu JE-\nu JF+W$, where $W\in\mathfrak{k}$. Therefore $\nu JF\in\mathfrak{k}+\{JE, E, F\}$, as $[JF, JE]\in\{JE, E, F\}$. Since the sum $\mathfrak{g}=\mathfrak{k}+\{JF\}+\{JE, E, F\}$ is direct, we see $\nu JF=0$, and hence $\nu=0$. This proves $[\{JF\}, \{E\}]\subset\{E\}$.

Next, suppose $\dim\mathfrak{r}=2$. Since $\dim\mathfrak{g}/\mathfrak{l}=2$, we have $\dim\mathfrak{l}\cap\mathfrak{r}\neq 0$ by Lemma 6. If $\dim\mathfrak{l}\cap\mathfrak{r}=2$, then $\mathfrak{l}=\mathfrak{k}+\mathfrak{r}$ by Lemma 2. This shows that $\mathfrak{r}$ is a two dimensional ideal satisfying the assertions of Proposition. Hence, in the following we may suppose $\dim\mathfrak{l}\cap\mathfrak{r}=1$. Then $\mathfrak{g}=\mathfrak{k}+J\mathfrak{r}+\mathfrak{r}$. For, putting $\mathfrak{g}'=\mathfrak{k}+J\mathfrak{r}+\mathfrak{r}$, we see $\dim_C\mathfrak{g}'/\mathfrak{k}=1$ or 2. If $\dim_C\mathfrak{g}'/\mathfrak{k}=1$, then $J\mathfrak{r}\subset\mathfrak{g}'=\mathfrak{k}+\mathfrak{r}$ by Lemma 2. This contradicts (K.1), since $\mathfrak{l}$ is J-invariant. Therefore $\dim_C\mathfrak{g}'/\mathfrak{k}=2$, and hence $\mathfrak{g}=\mathfrak{g}'=\mathfrak{k}+J\mathfrak{r}+\mathfrak{r}$. Furthermore, we have $\psi\neq 0$ on $\mathfrak{r}$ by Lemma 1. So we can select a basis of $\mathfrak{r}$ as follows:

$$\mathfrak{r}=\{E, F\}\,, \qquad \mathfrak{l}\cap\mathfrak{r}=\{E\}$$

and

$$\text{(iii)}\quad \psi(E)=0,\quad \psi(F)\neq 0 \qquad \text{or} \qquad \text{(iv)}\quad \psi(E)\neq 0,\quad \psi(F)=0\,.$$

Then $\mathfrak{g}=\mathfrak{k}+\{JE, JF\}+\{E, F\}$, since $\mathfrak{g}=\mathfrak{k}+J\mathfrak{r}+\mathfrak{r}$.

In the case (iii), we can show in a method similar to that of (i) in the case of $\dim\mathfrak{r}=3$ that $\{E\}$ is an ideal of $\mathfrak{g}$. Hence, in this case $\mathfrak{g}$ contains a one dimensional ideal.

Finally we show that the case (iv) does not occur. Since $\{E\}$ is an ideal of $\mathfrak{l}$, we can put $[X, E]=\lambda E$ for $X\in\mathfrak{l}$, where $\lambda\in\boldsymbol{R}$. Hence, from $\psi(E)\neq 0$ and $\eta(X, JE)=0$, we have $[X, E]=0$. In particular, we see $[E, \mathfrak{k}]=\{0\}$ and $[JE, E]=0$. From $\psi(E)\neq 0$, $\psi(F)=0$ and $\eta(E, F)=0$, it follows that $[JE, F]=\alpha F$ and $[JF, E]=\beta F$ for some $\alpha, \beta\in\boldsymbol{R}$. Put $[JF, F]=\lambda E+\mu F$, where $\lambda, \mu\in\boldsymbol{R}$. Then, putting $f=\mathrm{ad}\,(JF)-J\,\mathrm{ad}\,(F)$, we have $f(E)=\beta F$ and $f(F)=\lambda E+\mu F$. Noting that, for $X\in\mathfrak{g}$,

$f(JX) \equiv Jf(X) \pmod{\mathfrak{k}}$ by (K.3), we see $f(JE) \equiv \beta JF$ and $f(JF) \equiv \lambda JE + \mu JF \pmod{\mathfrak{k}}$. These facts show $\mathrm{Tr}_{\mathfrak{g}/\mathfrak{k}}(\mathrm{ad}(JF) - J\,\mathrm{ad}(F)) = 2\mu$. Since $0 = \psi(F) = \mathrm{Tr}_{\mathfrak{g}/\mathfrak{k}}(\mathrm{ad}(JF) - J\,\mathrm{ad}(F))$ by (1.1), we obtain $\mu = 0$, which implies $[JF, F] = \lambda E$. By (2.2), η is definite on $\mathfrak{g}/\mathfrak{l}$. Therefore $0 \neq 2\eta(F, F) = \psi([JF, F]) = \lambda\psi(E)$, and hence $\lambda \neq 0$. Consequently, we have the following relations:

$$(2.8) \qquad [JE, E] = 0\ , \quad [JE, F] = \alpha F\ , \quad [JF, E] = \beta F\ , \quad [JF, F] = \lambda E\ , \quad \lambda \neq 0\ .$$

Now, by carrying out the same computation as in Shima [8, Proof of Lemma 4.1], we derive a contradiction. First, we show $\mathfrak{k} = \{0\}$. As indicated above, $[E, \mathfrak{k}] = \{0\}$. Let $W \in \mathfrak{k}$. Put $[W, F] = \mu E + \nu F$, where $\mu, \nu \in \boldsymbol{R}$. Then $\psi(E) \neq 0$, $\psi(F) = 0$ and (1.4) yield $[W, F] = \nu F$. From this, we obtain $\psi([JF,[W,F]]) = \nu\psi([JF,F]) = \lambda\nu\psi(E)$ and $\psi([JF,[W,F]]) = \psi([[JF, W], F]) + \psi([W, [JF, F]]) = \psi([J[F, W], F]) = -\nu\psi([JF, F]) = -\lambda\nu\psi(E)$ by (1.4) and (K.2). Therefore we have $2\lambda\nu\psi(E) = 0$, and hence $\nu = 0$ and $[W, F] = 0$. Thus $[\mathfrak{k}, \mathfrak{r}] = \{0\}$. Since $[\mathfrak{k}, J\mathfrak{r}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ and $\mathfrak{g} = \mathfrak{k} + J\mathfrak{r} + \mathfrak{r}$, we see that $\mathfrak{k}$ is an ideal of $\mathfrak{g}$. By (2.3), we have $\mathfrak{k} = \{0\}$. Next, we show $2\alpha = \beta$. Using the Jacobi identity, (K.3) and $\mathfrak{k} = \{0\}$, we have

$$\begin{aligned} 0 &= [[JE, JF], F] + [[JF, F], JE] + [[F, JE], JF] \\ &= (\alpha - \beta)[JF, F] - \alpha[F, JF] = (2\alpha - \beta)[JF, F] = \lambda(2\alpha - \beta)E\ . \end{aligned}$$

Hence we see $2\alpha = \beta$. From this, (2.8) and (K.7), we have

$$\begin{aligned} 0 &= \rho([JE, F], JF) + \rho([F, JF], JE) + \rho([JF, JE], F) \\ &= \alpha\rho(F, JF) - \lambda\rho(E, JE) + (\beta - \alpha)\rho(JF, F) \\ &= (\beta - 2\alpha)\rho(JF, F) + \lambda\rho(JE, E) = \lambda\rho(JE, E)\ . \end{aligned}$$

This contradicts (K.6). Thus, Proposition has been proved.

3. Proof of Theorem 1. We keep our notations and assumptions in the previous section.

By restricting J and ρ to $\mathfrak{l}$, we see that $(\mathfrak{l}, \mathfrak{k}, J, \rho)$ is the Kähler algebra of the homogeneous Kähler manifold L/K. Let h' be the canonical hermition form of L/K. Then, putting $\eta'(X, Y) = h'_o(\pi_* X, \pi_* X)$ and $\psi'(X) = \mathrm{Tr}_{\mathfrak{l}/\mathfrak{k}}(\mathrm{ad}(JX) - J\,\mathrm{ad}(X))$ for $X, Y \in \mathfrak{l}$, we have (1.2), (1.3), (1.4) and (1.5) for the forms η' and ψ'.

Now, Theorem 1 is stated more precisely as follows:

THEOREM 1′. *The homogeneous Kähler manifold L/K has zero Ricci curvature. Furthermore, if M is simply connected, then, by defining a* *suitable G-invariant complex structure on G/L, the natural projection of G/K onto G/L is holomorphic.*

PROOF. To begin with, we prove $h' = 0$, which shows the first half of the theorem. By the proposition in the previous section, $\mathfrak{g}$ contains a one dimensional ideal or a two dimensional commutative ideal $\mathfrak{r}$ such that $\mathfrak{l} = \mathfrak{k} + \mathfrak{r}$. If $\mathfrak{g}$ contains a two dimensional ideal satisfying the assertions of the proposition, then we see $\eta' = 0$ by (1.2), (1.4) and the commutativity of $\mathfrak{r}$, and hence $h' = 0$. Therefore we consider the case where there exists a one dimensional ideal $\{E\}$.

If $\psi(E) \neq 0$, then, using Lemmas 3, 4 and 5, we have $\mathfrak{g} = \{JE\} + \{E\} + \mathfrak{p}$ (direct sum), where $\{JE\}$, $\{E\}$, $\mathfrak{p}$ are mutually orthogonal with respect to the form η, and η is positive definite on $\{JE\} + \{E\}$. By (2.7), we see $\mathfrak{l} \subset \mathfrak{p}$, and hence $\mathfrak{l} = \mathfrak{p}$, since $\dim \mathfrak{g}/\mathfrak{l} = \dim \mathfrak{g}/\mathfrak{p} = 2$. Therefore we have

$$\mathfrak{g} = \{JE\} + \{E\} + \mathfrak{l} \text{ (direct sum)}, \quad [E, \mathfrak{l}] = \{0\} \quad \text{and} \quad \mathrm{ad}\,(JE)\mathfrak{l} \subset \mathfrak{l}\,. \tag{3.1}$$

Let $X \in \mathfrak{l}$. Then $(\mathrm{ad}\,(JX) - J\,\mathrm{ad}\,(X))(E) = 0$ and $(\mathrm{ad}(JX) - J\,\mathrm{ad}\,(X))(JE) \equiv 0 \pmod{\mathfrak{k}}$ by (3.1) and (K.3). Using this fact and (3.1), we see

$$\psi(X) = \mathrm{Tr}_{\mathfrak{g}/\mathfrak{k}}\,(\mathrm{ad}\,(JX) - J\,\mathrm{ad}\,(X)) = \mathrm{Tr}_{\mathfrak{l}/\mathfrak{k}}\,(\mathrm{ad}\,(JX) - J\,\mathrm{ad}\,(X)) = \psi'(X)\,.$$

Therefore $\psi = \psi'$ on $\mathfrak{l}$. From this and (1.2), we have $2\eta(X, X') = \psi([JX, X']) = \psi'([JX, X']) = 2\eta'(X, X')$ for $X, X' \in \mathfrak{l}$. Since $\eta = 0$ on $\mathfrak{l}$, we see $\eta' = 0$ on $\mathfrak{l}$, and hence $h' = 0$.

If $\psi(E) = 0$, then $\{E\} \subset \mathfrak{l}$ by Lemma 1. Using (2.3) and (K.1), we have

$$\mathfrak{l} = \{JE\} + \{E\} + \mathfrak{k}\,. \tag{3.2}$$

We show $\psi'(E) = 0$. Otherwise $[E, \mathfrak{k}] = \{0\}$ by Lemma 3 (a), and hence, by Lemma 3 (b), $[JE, \mathfrak{k}] = \{0\}$ with a suitable linear endomorphism J of $\mathfrak{g}$ belonging to the Kähler algebra of $M = G/K$. Furthermore, we have $[JE, E] \neq 0$ by (3.2) (cf. Lemma 4). We may assume $[JE, E] = E$ with a suitable $E \neq 0$. From these, it follows by Lemma 5 that $\eta(E, E) > 0$. This contradicts $E \in \mathfrak{l}$. Therefore, we see $\psi'(E) = 0$. Using this fact, we have $2\eta'(E, E) = \psi'([JE, E]) = 0$, which implies $\eta' = 0$ on $\mathfrak{l}$ by (3.2). Thus, $h' = 0$ is proved.

Next, we prove that the natural projection of G/K onto G/L is holomorphic, if M is simply connected and if we define a suitable G-invariant complex structure on G/L. Since M is simply connected, K is connected, and hence so is L by the connectedness of L/K. Therefore, the G-invariant complex structures on G/L are in a natural one-to-one correspondence with the linear endomorphisms $\bar{J}$ of $\mathfrak{g} \pmod{\mathfrak{l}}$ satisfying the following properties (cf. [5, p. 217]):

$$\bar{J}\mathfrak{l} \subset \mathfrak{l}\,, \qquad \bar{J}^2 \equiv -\mathrm{id} \pmod{\mathfrak{l}}\,, \tag{3.3}$$

$$[A, \bar{J}X] \equiv \bar{J}[A, X] \pmod{\mathfrak{l}}\,, \tag{3.4}$$

$$[\bar{J}X, \bar{J}Y] \equiv \bar{J}[\bar{J}X, Y] + \bar{J}[X, \bar{J}Y] + [X, Y] \pmod{\mathfrak{l}}\,, \tag{3.5}$$

where $X, Y \in \mathfrak{g}$, $A \in \mathfrak{l}$. We show that the linear endomorphism J of $\mathfrak{g}$ belonging to the Kähler algebra of $M = G/K$ satisfies the above three properties. If this can be done, then it is easily seen that the natural projection of G/K onto G/L with the G-invariant complex structure corresponding to J is holomorphic.

It is clear that (K.1) and (K.3) imply (3.3) and (3.5), respectively. We show by using the proposition that (K.2) implies (3.4). If $\mathfrak{g}$ contains a two dimensional commutative ideal $\mathfrak{r}$ with $\mathfrak{l} = \mathfrak{k} + \mathfrak{r}$, then we see easily that (K.2) implies (3.4), since $[\mathfrak{r}, \mathfrak{g}] \subset \mathfrak{r} \subset \mathfrak{l}$. Hence we consider the case where there exists a one dimensional ideal $\{E\}$.

If $\psi(E) \neq 0$, then $[E, \mathfrak{l}] = \{0\}$ and $\mathrm{ad}\,(JE)\mathfrak{l} \subset \mathfrak{l}$ by (3.1). From this, we have $[A, JE] \equiv J[A, E]$ and $[A, J(JE)] \equiv J[A, JE] \pmod{\mathfrak{l}}$ for $A \in \mathfrak{l}$. Since $\mathfrak{g} = \{JE\} + \{E\} + \mathfrak{l}$ by (3.1), this implies (3.4).

If $\psi(E) = 0$, then $\mathfrak{l} = \{JE\} + \{E\} + \mathfrak{k}$ by (3.2). Since $\{E\}$ is an ideal, we see $[E, JX] \equiv J[E, X] \pmod{\mathfrak{l}}$ for $X \in \mathfrak{g}$. This implies $[JE, JX] \equiv J[JE, X] \pmod{\mathfrak{l}}$, since $[JE, JX] - J[JE, X] \equiv J([E, JX] - J[E, X]) \pmod{\mathfrak{l}}$ by (K.3). From these and from (K.2), we have (3.4). Thus, the theorem is established.

REMARK. By the above theorem, we see that a complex two dimensional connected and simply connected homogeneous Kähler manifold with degenerate and non-zero canonical hermitian form is a holomorphic fiber bundle whose base space is the unit disk or the Riemann sphere and whose fiber is the complex plane.

4. **Known results and their consequence.** Let $M = G/K$ be a complex n-dimensional connected homogeneous Kähler manifold with the canonical hermitian form h, where G acts effectively on M. In the investigation of M, the form h plays an important role. Now, we state the known results about the structure of M.

When h is either definite or zero, the following hold:

(a) If the Ricci curvature of M is negative, then M is a homogeneous bounded domain in $\boldsymbol{C}^n$.

(b) If M has zero Ricci curvature, then M is a locally flat homogeneous Kähler manifold, and hence M is obtained by factoring $\boldsymbol{C}^n$ by some lattice (cf. [1, Theorem 1]).

(c) If the Ricci curvature of M is positive, then G is compact and semi-simple, and hence M is a simply connected compact homogeneous Kähler manifold (see [7, Corollary]).

When h is non-degenerate and not definite, the following are valid:

(d) Suppose that M is simply connected and that the signature of h is $(2, 2(n-1))$. Then, if either G is semi-simple or G contains a one parameter normal subgroup, $M = G/K$ is a holomorphic fiber bundle whose base space is the unit disk and whose fiber is a homogeneous Kähler manifold of a compact semi-simple Lie group (see [8, Theorem 1]).

(e) If $\dim_{\boldsymbol{C}} M = 2$ and if the signature of h is $(2, 2)$, then G is semi-simple or G contains a one parameter normal subgroup (see [8, Theorem 2]).

Using these results and Theorem 1′ with its remark, we see that the types of complex two dimensional connected and simply connected homogeneous Kähler manifolds M are the following six ones (cf. [8, Section 5]):

(i) Homogeneous bounded domains in $\boldsymbol{C}^2$. Hence M is $\{z \in \boldsymbol{C}; |z| < 1\} \times \{z \in \boldsymbol{C}; |z| < 1\}$ or $\{(z_1, z_2) \in \boldsymbol{C}^2; |z_1|^2 + |z_2|^2 < 1\}$.

(ii) Complex two dimensional compact hermitian symmetric spaces. Hence M is $\boldsymbol{P}_1(\boldsymbol{C}) \times \boldsymbol{P}_1(\boldsymbol{C})$ or $\boldsymbol{P}_2(\boldsymbol{C})$, where $\boldsymbol{P}_n(\boldsymbol{C})$ is the complex n-dimensional projective space.

(iii) A holomorphic fiber bundle whose base space is the unit disk and whose fiber is $\boldsymbol{P}_1(\boldsymbol{C})$.

(iv) A holomorphic fiber bundle whose base space is the unit disk and whose fiber is $\boldsymbol{C}$.

(v) A holomorphic fiber bundle whose base space is $\boldsymbol{P}_1(\boldsymbol{C})$ and whose fiber is $\boldsymbol{C}$.

(vi) $\boldsymbol{C}^2$.

From these, we obtain the following.

THEOREM 2. *Let M be a connected homogeneous Kähler manifold of complex dimension two. If M contains no complex line, then M is a homogeneous bounded domain in $\boldsymbol{C}^2$.*

REMARK. It should be remarked that Shima [9] proved the following theorem:

Let M be a connected homogeneous Kähler manifold admitting a simply transitive solvable Lie group. Assume that M contains no com-

plex line. Then M is a homogeneous bounded domain.

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