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ON POLARIZED VARIETIES OF SMALL Δ-GENERA

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Introduction. The purpose of this paper is to generalize the theory of Δ-genus of polarized varieties (cf. [F1], [F2], [F4] etc.) to the positive characteristic cases. This is accomplished by weakening the assumptions on singularities in our previous results in such a way that we can avoid the use of the strong Bertini theorem and the desingularization theory of Hironaka. Moreover, for the vanishing theorem of Kodaira, we can sometimes substitute the vanishing theorem of Serre. Furthermore, with the help of the theory of liftings to characteristic zero, we obtain most results in [F1], [F2] and [F4] which were proved by transcendental methods.

This article is organized as follows: In §1 we review basic notions. In §2 we prove the fundamental inequality $\dim Bs|L| < \Delta(V, L)$ for any polarized variety (V, L). In §3 we give a couple of sufficient conditions for the existence of a ladder of (V, L). As an application, a criterion for the very ampleness of L is obtained. In §4 we study the case $\Delta(V, L) = 0$ and establish the same classification theorem as that in [F1]. In §5 and §6 we consider the case $\Delta(V, L) = 1$. An appendix is devoted to liftability problems.

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Notation, convention and terminology. Basically we employ the notation in [F1], [F2], [F3] and [F4], which is similar to that of [EGA]. We work usually in the category of $\mathfrak{K}$-schemes of finite type, where $\mathfrak{K}$ is an algebraically closed field of any characteristic. In many statements (especially where we use the word "generic"), $\mathfrak{K}$ is implicitly assumed further to be "sufficiently big", that means, all the objects in concern can be defined over a subfield k such that $\operatorname{tr.deg}(\mathfrak{K}/k) = \infty$. In other

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words, $\mathfrak{K}$ is a so-called universal domain. A *point* means a $\mathfrak{K}$-rational point, even when we say "generic". Thus, our "*generic point*" is similar to that of Weil [W]. A *variety* means an irreducible, reduced, proper $\mathfrak{K}$-scheme. A *manifold* is a non-singular variety. Line bundles are identified with the invertible sheaves of their sections. Tensor products of line bundles are denoted additively. Multiplicative notations are used for intersection products in Chow rings.

We show some examples of symbols.

$[\Lambda]$: The line bundle associated with a linear system Λ of Cartier divisors.

$Bs\Lambda$: The intersection of all the members of Λ.

ρ_Λ: The rational mapping defined by Λ.

$|L|$: The complete linear system associated with a line bundle L.

$\mathscr{F}[L]: = \mathscr{F} \otimes_{\mathscr{O}} \mathscr{L}$, for a coherent sheaf $\mathscr{F}$, where $\mathscr{L}$ is the invertible sheaf corresponding to L.

L_T: The pull-back of L to a space T by a given morphism. However, when there is no danger of confusion, we write simply L instead of L_T.

ω_V: The dualizing sheaf of a locally Macaulay variety V.

K^M: The canonical bundle of a manifold M.

$H_\alpha, H_\beta, \cdots$: The (pull-backs of) $\mathscr{O}(1)$'s of projective spaces $\boldsymbol{P}_\alpha, \boldsymbol{P}_\beta, \cdots$ indicated by the same Greek letters.

1. Basic notions. (1.1) A *prepolarized variety* is a pair (V, L) of a variety V and a line bundle L on V. It is called a *polarized variety* if L is ample (not necessarily very ample).

(1.2) Since the Hilbert polynomial $\chi(V, tL)$ is integer valued for every $t \in \boldsymbol{Z}$, there are integers $\chi_0, \cdots, \chi_n$ such that $\chi(V, tL) = \sum_{j=0}^{n} \chi_j t^{[j]}/j!$, where $n = \dim V$ and $t^{[j]} = t(t+1)(t+2)\cdots(t+(j-1))$. These χ_j's are invariants of (V, L). In particular we set $d(V, L) = \chi_n(V, L)$ and $g(V, L) = 1 - \chi_{n-1}(V, L)$.

By the Riemann-Roch theorem, we have $d(V, L) = L^n$. If V is non-singular, we have $2g(V, L) - 2 = (K^V + (n-1)L)L^{n-1}$.

If V is a curve, then $g(V, L) = h^1(V, \mathscr{O}_V)$. Hence, in view of (1.3) below, we call $g(V, L)$ *the sectional genus* of (V, L).

(1.3) If D is an irreducible reduced member of $|L|$, then $\chi_r(D, L_D) = \chi_{r+1}(V, L)$ for any $r \geqq 0$. In particular, $d(D, L_D) = d(V, L)$ and $g(D, L_D) = g(V, L)$. For a proof, see [F1; Proposition 1.3].

(1.4) The *total deficiency* of (V, L) is defined to be $\Delta(V, L) = n + d(V, L) - h^0(V, L)$. This is also called the *Δ-genus*. More generally, we define $\Delta(V, \Lambda) = n + d(V, [\Lambda]) - (1 + \dim \Lambda)$ for a linear system Λ on V. Then, of course, $\Delta(V, L) = \Delta(V, |L|)$.

(1.5) Let V, L and D be as in (1.3). Then we have $0 \leqq \Delta(V, L) - \Delta(D, L_D) \leqq h^1(V, \mathcal{O}_V)$. Moreover, the following conditions are equivalent to each other:

(a) $\Delta(V, L) = \Delta(D, L_D)$.

(b) $H^0(V, L) \to H^0(D, L_D)$ is surjective.

(c) $|L|_D = |L_D|$.

(1.6) Let (V, L) be a prepolarized variety. An irreducible reduced member D of $|L|$ is called a *rung* of (V, L). It is said to be *regular* if the conditions (a), (b) and (c) in (1.5) are satisfied. A sequence $V = D_n \supset D_{n-1} \supset \cdots \supset D_1$ of subvarieties of V with $j = \dim D_j$ is called a *ladder* of (V, L) if each D_j is a rung of (D_{j+1}, L). This ladder is said to be *regular* if so is each rung D_j. By (1.5), this is equivalent to saying $\Delta(V, L) = \Delta(D_1, L)$.

(1.7) One easily sees $\Delta(V, L) - \Delta(D, L_D) = \dim \operatorname{Coker}(H^0(V, L) \to H^0(D, L))$ for any rung D of (V, L). This was called the deficiency in classical geometry. Now, suppose that (V, L) has a ladder $V = D_n \supset \cdots \supset D_1$ and that $L_{D_1} = [D_0]$ for some divisor D_0 on D_1. Set $\Delta_j = \dim \operatorname{Coker}(H^0(D_j, L) \to H^0(D_{j-1}, L))$. Since $\Delta(D_1, L) = \Delta_1$, we have $\Delta(V, L) = \Delta_1 + \cdots + \Delta_n$. This is why we call it the total deficiency. Obviously $\Delta(V, L) \geqq 0$ if (V, L) has a ladder.

2. **Fundamental inequality.** (2.1) The purpose of this section is to prove the following:

THEOREM. *Let Λ be a linear system on a variety V such that $L = [\Lambda]$ is ample. Then* $\dim Bs\Lambda < \Delta(V, \Lambda)$, *where* $\dim \varnothing$ *is defined to be* -1.

The proof given in this section is an amalgam of the argument in [F1; Theorem 1.9] and that of [MM; Theorem 3]. First we review several preliminary facts.

(2.2) Let S be a proper $\mathfrak{K}$-scheme such that any irreducible component of its support is of dimension n. If there is an embedding of S into a manifold M with $N = \dim M$, we define $\mathscr{D}^q = \mathscr{E}xt_{\mathcal{O}_M}^{N-q}(\mathcal{O}_S, \omega_M)$. Actually, $\mathscr{D}^q$ is independent of the choice of such an embedding and can be defined without assuming the existence of one (see, e.g., [F5; §1]). One easily sees that $\mathscr{D}^q = 0$ unless $0 \leqq q \leqq n$, and that $\dim \operatorname{Supp}(\mathscr{D}^q) \leqq q$. Moreover, $\operatorname{Supp}(\mathscr{D}^n) = \operatorname{Supp}(S)$. If S is locally Macaulay, then $\mathscr{D}^n$ is the dualizing sheaf ω_S of S.

DEFINITION (2.3). Let x be a point on S and let k be an integer with $1 \leqq k \leqq n$. S is said to be *k-Macaulay at x* if $\dim Y \leqq q - k$ for any $0 \leqq q < n$ and any component Y of $\operatorname{Supp}(\mathscr{D}^q)$ such that $x \in Y$, where we define $\dim \varnothing = -\infty$. S is said to be *k-Macaulay* if it is so at every point on S.

S is locally Macaulay if and only if it is n-Macaulay. S has no embedded component if and only if it is 1-Macaulay (cf. [F5; (1.14)]). Using the theory in [G1; §3], we infer that S is k-Macaulay if and only if it is (S_k) in the sense of [EGA; Chap. IV, (5.7.2)]. In particular, a variety is normal if and only if it is 2-Macaulay and non-singular in codimension one (see [EGA; (5.8.6)]).

PROPOSITION (2.4). *Let D be a Cartier divisor on S and let x be a point on D. Assume that the natural homomorphism $\mathcal{O}_S \to \mathcal{O}_S[D]$ is injective at x. If D is k-Macaulay at x, then S is k-Macaulay at x. On the other hand, if S is k-Macaulay at x, then D is $(k-1)$-Macaulay at x.*

This follows easily from the definition.

PROPOSITION (2.5). *Let Λ be a linear system on S. Suppose that S is k-Macaulay and that S is $(k+1)$-Macaulay at each point $x \in Bs\Lambda$. Then a general member of Λ is k-Macaulay.*

For a proof, use the following:

LEMMA (2.6). *Let $\mathscr{F}$ be a coherent sheaf on S and let $\varphi: \mathscr{F} \to \mathscr{F}[\Lambda]$ be the homomorphism induced by $\delta \in H^0(S, [\Lambda])$ which corresponds to a general member D of Λ. Then φ is injective outside $Bs\Lambda$.*

This is a consequence of the Noetherian decomposition of $\mathscr{F}$. See [F5; (1.2)].

THEOREM (2.7). *Let Λ be a linear system on a normal variety V such that $Bs\Lambda = \varnothing$. Let W be the image of the rational mapping ρ_Λ defined by Λ and let D be a generic member of Λ. Then:* (a) *D is irreducible if* $\dim W \geqq 2$, *and* (b) *D is reduced if Λ is complete.*

PROOF. (a) is a famous result of Zariski (cf., e.g., [Z; p. 30]). Since V is 2-Macaulay, D is 1-Macaulay by (2.4). So it has no embedded component. Hence, in order to prove (b), it suffices to show that any prime component of D is of multiplicity one. Suppose the contrary. Then, by the theory in [W; Chap. IX], the rational mapping $\rho_\Lambda: V \to \boldsymbol{P}^N$, where $N = \dim \Lambda$, is factored as $\rho_\Lambda = F \circ \rho'$ for a morphism $\rho': V \to \boldsymbol{P}^N$, where F is the Frobenius p-power morphism $\boldsymbol{P}^N \to \boldsymbol{P}^N$ with $p = \mathrm{char}(\mathfrak{K}) > 0$. So Λ is not complete, contradicting the assumption.

THEOREM (2.8). *Let Λ be a linear system on a variety V. Then,* *there exists a normal variety* V' *together with a birational morphism* $\pi: V' \to V$, *an effective Cartier divisor* E *on* V' *and a linear system* Λ' *on* V' *such that* $\pi^*\Lambda = E + \Lambda'$ *and* $Bs\ \Lambda' = \emptyset$. *Moreover, the image* $W = \rho_{\Lambda'}(V')$ *is independent of the choice of such* V'.

This is well-known. A proof is found, e.g., in [F5; (3.1)].

DEFINITION (2.9). Such a variety V' is called a *good graph* of ρ_Λ. W is called the *image* of the rational mapping ρ_Λ.

LEMMA (2.10). *Let* D *be an effective ample divisor on a variety* V *and suppose that* $\mathrm{Supp}(D) = D_1 \cup D_2$ *where both* D_1 *and* D_2 *are proper closed subsets of* $\mathrm{Supp}(D)$. *Then* $\dim(D_1 \cap D_2) \geqq n - 2$, *where* $n = \dim V$.

PROOF. The assertion is trivial for $n = 1$, and follows from the connectedness of $\mathrm{Supp}(D)$ when $n = 2$ (cf. [H; p. 79]). For $n > 2$, we use the induction on n as in [F1; Lemma 6.1].

(2.11) Now we prove the Theorem (2.1) by induction on $n = \dim V$. The inequality is easily proved for $n = 1$. So we consider the case $n \geqq 2$.

The normalization morphism $\nu: V' \to V$ is finite. Therefore ν^*L is ample and $\dim Bs(\nu^*\Lambda) = \dim Bs\ \Lambda$. So it suffices to prove the inequality for $\nu^*\Lambda$, and hence we may assume V to be normal.

$\dim Bs\Lambda - \dim Bs\,|\Lambda| \leqq \dim|\Lambda| - \dim \Lambda$ since L is ample. Hence it suffices to consider the case in which Λ is complete.

Now, take a good graph V' of ρ_Λ, and let E, Λ' and W be as in (2.8). We consider the following five cases separately: (a) $\dim Bs\Lambda = n$, or equivalently, $\Lambda = \emptyset$. (b) $\dim Bs\Lambda = n - 1$ and $\dim W = 1$. (c) $\dim Bs\Lambda \leqq n - 2$ and $\dim W = 1$. (d) $\dim Bs\Lambda = n - 1$ and $\dim W \geqq 2$. (e) $\dim Bs\Lambda \leqq n - 2$ and $\dim W \geqq 2$.

Cases (a), (b) and (c). The proof is the same as that in [F1; p. 114].

Case (d). Let S be a generic member of Λ' and let D be the corresponding member of Λ. Λ' is complete since V is normal and Λ is complete. Hence, by (2.7), S is irreducible and reduced. Thus the argument in [F1; p. 114] works without trouble. Note that we have (2.10) in place of [F1; Lemma 6.1], and that $\dim \Lambda_G = \dim \Lambda - 1$ follows from the normality of V.

Case (e). Similar to that in case (d). q.e.d.

COROLLARY (2.12). $\Delta(V, L) > \dim Bs|L|$ *for any polarized variety* (V, L). *In particular, the* Δ*-genus is non-negative.*

3. Ladders of prepolarized varieties. In this section we establish a sufficient condition for the existence of a ladder (cf. (1.6)), which is similar to that in [F2]. First we make the following:

DEFINITION (3.1). A linear system Λ on a variety V is said to be *degenerate* if $\dim W < \dim V$, where W is the image of the rational mapping ρ_Λ (cf. (2.8)).

PROPOSITION (3.2). *Let Λ be a degenerate linear system on a variety V such that $d(V, L) > 0$ and $d(V, L) \geqq 2\Delta(V, \Lambda) - 1$, where $L = [\Lambda]$. Suppose that $B = Bs\Lambda$ is a finite set and that V is non-singular at each point of B. Then, a generic member D of Λ is irreducible and reduced. Moreover, D is non-singular at each point of B.*

The proof is almost the same as that of [F2; Proposition 3.5]. Indeed, similarly as there, D is shown to be non-singular at each point of B. Moreover, we see also $\dim W = n - 1$, where W is the image of the rational mapping ρ_Λ. Hence, by (2.7), D is irreducible (as a set) if $n \geqq 3$. If $n = 2$, we have $wE_pX = E_pH = E_p(L - E_p - E) = -E_p^2 = 1$ since $\dim \pi(E) \leqq 0$, where the notations are as in [F2]. This implies $w = \deg W = 1$. So D is irreducible because a member of Λ' consists of one prime component. Thus, in either case, D is irreducible. D is 1-Macaulay by (2.5). Hence D has no embedded component. This implies that D is reduced, since D is non-singular at each point of B (Note that $B \neq \emptyset$, since otherwise $d(V, L) = 0$ because Λ is degenerate).

LEMMA (3.3). *Let Λ be a non-degenerate linear system on a variety V such that $d(V, L) \geqq 2\Delta(V, \Lambda) - 1$, where $L = [\Lambda]$. Suppose that $B = Bs\Lambda$ is a finite set and that V is locally Macaulay at each point of B. Then, taking generic members of Λ successively, one obtains a ladder of (V, L).*

PROOF. Let V' be a good graph of ρ_Λ and let $\pi: V' \to V$, E, Λ' and W be as in (2.8). Let D be a generic member of Λ and let S be the corresponding member of Λ'. S is irreducible by (2.7). So $D = \pi(S) \cup \pi(E) = \pi(S)$ is also irreducible. D is 1-Macaulay by (2.5). Hence, to show that D is reduced, it suffices to prove that D is of multiplicity one at its generic point.

Assume the contrary. Then S is not reduced at its generic point. In view of the theory in [W; Chap. IX], we infer that the morphism $\rho_{\Lambda'}: V' \to W \subset \boldsymbol{P}_\alpha^N$ is factored as $F \circ \rho'$, where ρ' is a morphism $V' \to \boldsymbol{P}_\beta^N$ and $F: \boldsymbol{P}_\beta^N \to \boldsymbol{P}_\alpha^N$ is the p-power Frobenius morphism with $p = \mathrm{char}(\mathfrak{K}) > 0$. Let W' be the image of ρ' and set $w' = \deg W'$. Then $d(V', H_\alpha) = p^n d(V', H_\beta) = p^n w' \deg(\rho')$ since $F^* H_\alpha = pH_\beta$. Both L and H_α are c-semipositive in the sense of [F3; Appendix B] as line bundles on V'. Hence $d(V', L) = L^n \geqq L^{n-1}H_\alpha \geqq \cdots \geqq LH_\alpha^{n-1} \geqq H_\alpha^n$ since $L - H_\alpha$ is effective on

V'. Thus we obtain $d(V, L) \geqq p^n w'$. On the other hand, W' is not contained in any hyperplane since otherwise W would be contained in some hyperplane of $\boldsymbol{P}_\alpha^N$. Hence $h^0(W', H_\beta) \geqq 1 + \dim \Lambda$. So, using $0 \leqq \Delta(W', H_\beta) = n + w' - h^0(W', H_\beta)$, we obtain $n + w' \geqq 1 + \dim \Lambda = n + d(V, L) - \Delta(V, \Lambda)$. Thus we get $d(V, L) - \Delta(V, \Lambda) \leqq w' \leqq p^{-n} d(V, L)$. One sees that this contradicts $d(V, L) \geqq 2\Delta(V, \Lambda) - 1$.

Thus we prove that D is a rung of (V, L). By (1.3) we verify that $d(D, L) \geqq 2\Delta(D, \Lambda_D) - 1$. Clearly Λ_D is non-degenerate and D is locally Macaulay at each point of $B = Bs\Lambda_D$. Therefore, we complete the proof by induction on n.

REMARK. The above lemma is slightly weaker than its $\boldsymbol{C}$-version [F2; Proposition 3.4]. In fact, thanks to the strong version of Bertini theorem, we need not assume $d(V, L) \geqq 2\Delta(V, L) - 1$ in case $\mathrm{char}(\mathfrak{K}) = 0$.

THEOREM (3.4). *Let (V, L) be a prepolarized variety such that $d(V, L) > 0$, $d(V, L) \geqq 2\Delta(V, L) - 1$. Suppose that $Bs|L|$ is a finite (possibly empty) set, at each point of which V is non-singular. Then (V, L) has a ladder.*

We prove this by induction on $n = \dim V$. (3.3) applies if $|L|$ is non-degenerate. If L is degenerate, a generic member D of $|L|$ is a rung of (V, L) by (3.2). Moreover, (3.2) enables us to apply the induction hypothesis to (D, L_D). Thus we get a ladder of (V, L).

DEFINITION (3.5). A line bundle L on a variety V is said to be *simply generated* if the graded $\mathfrak{K}$-algebra $G(V, L) = \bigoplus_{t \geqq 0} H^0(V, tL)$ is generated by $H^0(V, L)$. Then, for any linear basis $X_1, \cdots, X_r$ of $H^0(V, L)$, $G(V, L)$ is a homomorphic image of the polynomial ring $\mathfrak{K}[X_1, \cdots, X_r]$. L is said to be *quadratically presented* if it is simply generated and if the relation ideal among $X_1, \cdots, X_r$ is generated by polynomials of degree two.

REMARK. The above definition is apparently different from that in [F2], but they are equivalent to each other. If L is simply generated (resp. quadratically presented) and ample, then it is normally generated (resp. normally presented) in the sense of [Mu]. Of course, L is very ample in this case.

THEOREM (3.6). *Let (V, L) be a prepolarized variety such that $d = d(V, L) > 0$, $\Delta = \Delta(V, L) \leqq g = g(V, L)$ and that (V, L) has a ladder $V = D_n \supset D_{n-1} \supset \cdots D_1$. Then*

(a) *the ladder $D_n \cdots \supset D_1$ is regular if $d \geqq 2\Delta - 1$,*

(b) *$Bs|L| = \varnothing$ if $d \geqq 2\Delta$,*

(c) *$g = \Delta$ and L is simply generated if $d \geqq 2\Delta + 1$,*

(d) *L is quadratically presented if $d \geqq 2\Delta + 2$.*

This theorem is proved similarly as [F2; Theorem 4.1].

COROLLARY (3.7). *Let* (V, L) *be a prepolarized variety such that* $d = d(V, L) > 0$, $\Delta = \Delta(V, L) \leqq g(V, L)$. *Suppose that* $Bs|L|$ *is a finite set, at each point of which* V *is non-singular. Then* (V, L) *has a regular ladder if* $d \geqq 2\Delta - 1$, *and the implications* (b), (c) *and* (d) *in* (3.6) *are valid.*

For a proof, combine (3.4) and (3.6).

REMARK. In order to show the existence of a ladder, we use the assumption that $\mathfrak{K}$ is "sufficiently big". But this is no longer necessary to prove the above assertions (b), (c) and (d), because the statements are compatible with scalar extensions. We need not even assume that $\mathfrak{K}$ is algebraically closed. The same remark applies to many results in this paper, including the fundamental inequality in (2.1).

COROLLARY (3.8). *Let* (V, L) *be as in* (3.6) *and suppose in addition that* L *is ample. Then* $H^q(V, tL) = 0$ *for any* $0 < q < \dim V$ *and* $t \in \boldsymbol{Z}$ *if* $d \geqq 2\Delta + 1$.

PROOF. We use the induction on $n = \dim V$. In order to apply [F3; (2.1)], we must show that the restriction mapping $\rho_t\colon H^0(V, tL) \to H^0(D_{n-1}, tL)$ is surjective for every integer t. This follows from the surjectivity of ρ_1 and the simply generatedness of L_D.

4. **Polarized varieties with $\Delta = 0$.** Throughout this section we let (V, L) be a polarized variety with $\Delta(V, L) = 0$. We put $n = \dim V$ and $d = d(V, L)$.

(4.1) By (2.12) we have $Bs|L| = \varnothing$. So (V, L) has a ladder by (3.4). Then $g(V, L) = g(D_1, L) = h^1(D_1, \mathcal{O}) \geqq 0$, where D_1 is the one-dimensional rung of a ladder of (V, L). Hence we can apply (3.6) to prove the following:

THEOREM (4.2). (V, L) *has a regular ladder,* $g(V, L) = 0$ *and* L *is simply generated. Moreover,* L *is quadratically presented if* $d \geqq 2$.

COROLLARY (4.3). *If* $d = 1$, *then* $(V, L) \cong (\boldsymbol{P}^n, H)$. *If* $d = 2$, *then* V *is a hyperquadric in* $\boldsymbol{P}^{n+1}$ *and* L *is* $\mathcal{O}_V(1)$.

REMARK. As for the characterization of projective spaces as in [F1; §2], the equivalence of the conditions (a), (b), (b′) and (d) in [F1; Theorem 2.1] is now established (See also [F3; §3]). But the implication (c) $\Rightarrow$ (b) seems to be unknown because of the lack of the vanishing theorem of Kodaira. Similarly, the conditions (a) and (b) in [F1; Theorem 2.2] are equivalent to each other, while (c) $\Rightarrow$ (b) is unknown. See also (4.15) below.

(4.4) For the moment we assume V to be non-singular and write M instead of V. Then $H^q(M, K^M + tL) = 0$ for any $q > 0$, $t > 0$ by (3.8) and the Serre duality. Therefore, by the same argument as in [F1; §3], we prove the following two lemmas.

LEMMA (4.5). $h^0(M, K^M + nL) = d - 1$.

LEMMA (4.6). $\dim Bs\,|K^M + nL| < n - 1$ *if* $d \geqq 3$.

LEMMA (4.7). *If* $n = 2$ *and* $d \geqq 3$, *then* $(K^M + 2L)^2 = 0$ *except in the case* $(M, L) \cong (\boldsymbol{P}^2, 2H)$.

PROOF. We infer $(K^M + 2L)^2 \geqq 0$ from (4.6). The left hand side is equal to $(K^M)^2 - 8$ because $K^M L = -d - 2$ by $g(M, L) = 0$. Suppose that the inequality holds. Since L contains a non-singular rational curve, we infer that M is rational. The classification theory of rational surfaces (cf., e.g., [Sa; Chap. V]) says that $(K^M)^2 > 8$ implies $M \cong \boldsymbol{P}^2$. It follows from $d \geqq 3$ and $g(M, L) = 0$ that $(M, L) \cong (\boldsymbol{P}^2, 2H)$.

LEMMA (4.8). *If* $n \geqq 3$ *and* $d \geqq 3$, *then* $(K^M + nL)^2 L^{n-2} = 0$.

PROOF. Similarly as in [F1; Lemma 3.6], we may assume $n = 3$. Since L is very ample, we have a non-singular member D of $|L|$. Assume $(D, L) \cong (\boldsymbol{P}^2, 2H)$. Then $(K^M + 3L)_D = K^D + 2L = H$. So we can apply [F3; (3.9)] to derive a contradiction. Therefore $(K^M + 3L)^2 L = (K^D + 2L_D)^2\{D\} = 0$ by (4.7).

(4.9) Now, using the above results and the techniques in [F3], we obtain the following theorem in the same way as in [F1].

THEOREM. *Let* (M, L) *be a polarized manifold such that* $\Delta(M, L) = 0$ *and* $d(M, L) \geqq 3$. *Then* $(M, L) \cong (\boldsymbol{P}(E), H^E)$ *for some ample vector bundle* E *on* $\boldsymbol{P}^1$ *except the case* $(M, L) \cong (\boldsymbol{P}^2, 2H)$. E *is a direct sum of* n *line bundles of positive degrees.*

(4.10) Since $\deg(\det E) = d(M, L)$ in the above case, it is an easy exercise to classify all the polarized manifolds of the type (4.9) with given $n = \dim M$ and $d(M, L)$. Thus, together with (4.3), we complete the classification of polarized manifolds of Δ-genus zero.

(4.11) From the preceding results and especially the very ampleness of L (see (4.2)), we obtain the following theorem by the same method as that in [F1; §4].

THEOREM. *Let* (V, L) *be a polarized variety with* $\Delta(V, L) = 0$. *Then* $\rho_{|L|}$ *embeds* V *into* $\boldsymbol{P}^N$, *where* $N = \dim|L|$. *The singular locus* S *of* V *is a linear subspace of* $\boldsymbol{P}^N$ *and* $S = \mathrm{Ridge}(V)$. *For any linear subspace* T *of* $\boldsymbol{P}^N$ *such that* $S \cap T = \varnothing$ *and that* $\dim T = N - \dim S - 1$,

$M = T \cap V$ is a polarized manifold with $\Delta(M, L_M) = 0$. Furthermore, V is the projective join $M*S$.

For the definition of the join operation $*$, see [F1; 4.1]. If in particular S is a point, then $V = M*S$ is the cone over M with vertex S.

(4.12) The result [F1; Corollary 4.10] is generalized in the following way.

PROPOSITION. *Let V be a normal subvariety of $\boldsymbol{P}^N$ such that* Ridge$(V) \neq \varnothing$. *Then* Pic(V) *is a cyclic group generated by $\mathcal{O}_V(1)$.*

PROOF. Let x be a point on Ridge(V) and let P' be the blowing-up of $P = \boldsymbol{P}^N$ with center x. Let V' be the proper transform of V on P' and let H be a hyperplane on P not containing x. Considering the projection from x, we get a morphism $P' \to H$. Let f be the restriction of this morphism to V'. Then $f(V') = V \cap H$, which will be denoted by D. f makes V' a $\boldsymbol{P}^1$-bundle over D, since $V = x*D$. Given any $\mathscr{L} \in$ Pic(V), we have an integer m such that the restriction of $\mathscr{L}(m)$ to each fiber of f is trivial. Then $\mathscr{F} = f_*(\mathscr{L}(m))$ is an invertible sheaf on D, and the natural morphism $f^*\mathscr{F} \to \mathscr{L}(m)$ is an isomorphism. Let E be the exceptional divisor on P' over x and set $X = E \cap V'$. Then $f^*\mathscr{F}_X = \mathscr{L}(m)_X = 0$, since $\mathscr{L}(m)$ comes from Pic(V). This implies $\mathscr{F} = \mathcal{O}$ since $f_X\colon X \to D$ is an isomorphism. Hence $\mathscr{L}(m) = f^*\mathscr{F} = 0$ in Pic(V'). So $\mathscr{L}(m) = 0$ in Pic(V), since V is normal. Thus we have proved the assertion.

COROLLARY (4.13). *Let (V, L) be a polarized variety with $\Delta(V, L) = 0$. Then V is locally Macaulay and normal, L is very ample, and* Pic(V) *is generated by L if V is singular.*

(4.14) In view of the preceding results, we see that the results in [F1; §5] concerning the deformations of polarized varieties of Δ-genus zero are valid in positive characteristic cases too.

(4.15) In the case of curves, $h^1(V, \mathcal{O}_V) = 0$ implies the existence of a line bundle L on V such that $\Delta(V, L) = 0$, and hence $V \cong \boldsymbol{P}^1$. The higher dimensional version of this fact would be the following:

CONJECTURE. *Let (V, L) be a polarized variety with $g(V, L) \leqq 0$. Then $\Delta(V, L) = 0$.*

This seems to be unsolved even in the case char$(\mathfrak{K}) = 0$ and V is non-singular. However, we have the following:

THEOREM (4.16). (Compare [F2; Lemme 5.1]). *Let (V, L) be a polarized variety with $g(V, L) \leqq 0$. Suppose that (V, L) has a ladder $V = D_n \supset$*

$\cdots \supset D_1$ *such that each rung* D_j *is 2-Macaulay for* $j \geqq 2$. *Then* $\Delta(V, L) = 0$.

PROOF. We use the induction on $n = \dim V$. If $n = 1$, the assertion is clear. Next we consider the case $n = 2$. Then, $D = D_1$ is a non-singular rational curve since $h^1(D, \mathcal{O}_D) = g(D, L) = g(V, L)$. So $D \cap S = \emptyset$ for the singular locus S of V. Hence S is a finite set because D is ample. Therefore V is normal by the criterion of Serre (cf. (2.3)). Take a non-singular model $\pi: V' \to V$ of V (cf. [A]). V' is rational since D is lifted to a divisor D' on V' such that $(D')^2 > 0$ and $D' \cong \boldsymbol{P}^1$. On the other hand, we have an injection $H^1(V, \pi_*\mathcal{O}_{V'}) \to H^1(V', \mathcal{O}_{V'})$ thanks to the Leray spectral sequence. Now, putting things together, we obtain $H^1(V, \mathcal{O}_V) = 0$. So $\Delta(V, L) = \Delta(D, L_D) = 0$ by (1.5).

Finally consider the case $n \geqq 3$. Applying the induction hypothesis to $D = D_{n-1}$, we see $\Delta(D, L_D) = 0$. $H^1(V, -tL) = 0$ for any sufficiently large integer t since V is 2-Macaulay. (Recall (2.3) and use the Serre duality.) Therefore, by an argument similar to those in [So; Lemma I-B] and [F3; (2.2)], we infer $H^1(V, \mathcal{O}_V) = 0$ from (3.8). Hence $\Delta(V, L) = \Delta(D, L) = 0$ by (1.5).

5. Polarized varieties with $\Delta = 1$.

Throughout this section let (V, L) be a polarized variety with $\Delta(V, L) = 1$. We put $n = \dim V$, $d = d(V, L)$ and $g = g(V, L)$.

(5.1) By (2.12), $Bs|L|$ is at most a finite set. If V is non-singular at each point of $Bs|L|$, then (V, L) has a ladder by virtue of (3.4).

LEMMA (5.2). *Suppose that* V *is 2-Macaulay and that* (V, L) *has a ladder. Then* $g(V, L) > 0$.

PROOF. Let $V = D_n \supset \cdots \supset D_1$ be a ladder of (V, L) and assume $g(V, L) \leqq 0$. Then $\Delta(D_1, L) = 0$ since $h^1(D_1, \mathcal{O}) = g(D_1, L) = g(V, L)$ (cf. (1.2) and (1.3)). In view of (1.5), we infer that there is an integer j such that $0 < j < n$, $\Delta(D_i, L) = 0$ for any $i \leqq n - j$, and $\Delta(D_i, L) = 1$ for any $i > n - j$. Then, $H^0(V, L) \to H^0(D_i, L)$ is surjective for any $i > n-j$. Hence we can find $f_1, f_2, \cdots, f_j \in H^0(V, L)$ such that, for each $r=1, \cdots, j$, D_r is defined by the equations $f_1 = \cdots = f_r = 0$ in V. $f_1, \cdots, f_j$ form a linear basis of the vector space $\mathrm{Ker}(H^0(V, L) \to H^0(D_{n-j}, L))$. Let Λ be the linear subsystem of $|L|$ on V corresponding to this vector space. Then $Bs\Lambda = D_{n-j}$ and D_{n-j} is a complete intersection in V. On the other hand, D_{n-j} is locally Macaulay by (4.13). Hence V is locally Macaulay at every point on $D_{n-j} = Bs\Lambda$. Taking a general member of Λ successively we find another sequence $D_n \supset D'_{n-1} \supset \cdots \supset D'_{n-j} = D_{n-j}$ which yields another ladder of (V, L) together with D_i's, $i \leqq n - j$. Using (2.5), we infer that D'_i is 2-Macaulay for any $i > n - j$ by the descending induction on i. So $\Delta(V, L) = 0$ by (4.16), contradicting the hypothesis.

COROLLARY (5.3). *Let* (V, L) *be as in* (5.2). *Then*

(a) *any ladder of* (V, L) *is regular,*

(b) $Bs|L| = \emptyset$ *if* $d \geqq 2$,

(c) $g = 1$ *and* L *is simply generated* (*hence very ample*) *if* $d \geqq 3$,

(d) L *is quadratically presented if* $d \geqq 4$.

COROLLARY (5.4). *Let* (V, L) *be as in* (5.2). *Then* V *is a hypercubic if* $d = 3$. V *is a complete intersection of two hyperquadrics if* $d = 4$.

COROLLARY (5.5). *Let* (V, L) *be a polarized manifold with* $\Delta(V, L) = 1$. *Then* (V, L) *has a ladder, and the assertions* (a), (b), (c) *and* (d) *in* (5.3) *are valid. In particular* (5.4) *applies.*

DEFINITION (5.6). A polarized variety (V, L) of dimension n will be called a *Del Pezzo variety* if it satisfies the following conditions.

(a) $\Delta(V, L) = 1$.

(b) $g(V, L) = 1$.

(c) V is locally Gorenstein and $\omega_V = \mathcal{O}_V[(1-n)L]$.

(d) $H^q(V, tL) = 0$ *for any* $0 < q < n$, $t \in \boldsymbol{Z}$.

REMARK (5.7). It is obvious that (c) implies (b). Moreover, if V is non-singular and if char$(\mathfrak{K}) = 0$, we have the following result (see [F4; (1.9)]):

(a) and (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d).

However, in general, even the following conjectures are still unproven because of the lack of Kodaira's vanishing theorem.

CONJECTURE (5.7.1). (c) *implies* (d).

CONJECTURE (5.7.2). (a) *and* (b) *imply* (d).

At present, we have the following results.

(5.7.3) (a) and (b) imply (d) if $d(V, L) \geqq 3$ and if (V, L) has a ladder. For a proof, apply (3.8).

(5.7.4) (a) and (b) imply (d) if V is 2-Macaulay and if (V, L) has a ladder. For a proof, see (5.11) below.

(5.7.5) (b) and (d) imply (a) and (c). See (5.9) below.

(5.7.6) (c) implies (d) if L is very ample. See (5.12) below.

LEMMA (5.8). *Let* (V, L) *be a polarized variety such that* $H^q(V, tL) = 0$ *for any* $q < n = \dim V$ *and any* $t \ll 0$. *Then* V *is locally Macaulay.*

PROOF. Recall the definition of $\mathscr{D}^q$ in (2.2). Combined with the Serre duality, the local-global Ext-theory gives a spectral sequence with $E_2^{p,q} = H^p(V, \mathscr{D}^{-q}[tL])$ converging to the dual space of $H^{-p-q}(V, -tL)$ for each integer t. $E_2^{p,q} = 0$ for any $p > 0$, $t \gg 0$. Therefore $h^q(V, -tL) = h^0(V, \mathscr{D}^q[tL])$ for $t \gg 0$. By assumption these are zero for $q < n$. Hence $\mathscr{D}^q = 0$ for $q < n$, since L is ample. This implies that V is locally Macaulay.

(5.9) PROOF OF (5.7.5). Given polynomials $\varphi(t)$ and $\psi(t)$, we write $\varphi(t) \sim \psi(t)$ if $\deg(\varphi - \psi) < n - 1$. Then, by the definition of $\chi_j(V, L)$ (cf. (1.2)), we have $\chi(V, tL) \sim dt^n/n! + (d(n-1)/2 + \gamma)t^{n-1}/(n-1)!$, where $d = d(V, L)$ and $\gamma = \chi_{n-1}(V, L)$. Note also that (5.6; b) means $\gamma = 0$. We claim that $H^0(V, \omega_V[sL]) = 0$ if $s < n - 1 + 2\gamma/d$.

Indeed, suppose that there is a non-trivial homomorphism $\mathscr{O}_V \to \omega_V[sL]$. This must be injective because both sheaves are torsion free. Therefore $\chi(V, tL) \leqq \chi(V, \omega_V[(s+t)L])$ for any $t \gg 0$. On the other hand we have $\chi(\omega_V[(s+t)L]) = (-1)^n\chi(V, -(s+t)L) \sim dt^n/n! + (ds - d(n-1)/2 - \gamma)t^{n-1}/(n-1)!$. Comparing the coefficients of t^{n-1}, we obtain $ds \geqq d(n-1) + 2\gamma$. Thus we prove the claim.

By this claim and (b) we infer $H^n(V, -tL) = 0$ for $t < n - 1$. Hence $\chi(V, -tL) = 0$ for $0 < t < n - 1$ by the condition (d). So we may set $\chi(V, tL) = (t+1)\cdots(t+n-2)(dt^2 + bt + c)/n!$ for some constants b and c. By comparison of the coefficients of t^{n-1} we obtain $b = (n-1)d$. From the condition (d) and the above claim we infer $\chi(V, \mathscr{O}_V) = 1$. This implies $c = (n-1)n$. Thus we get:

$$(*)\quad \chi(V, tL) = (t+1)(t+2)\cdots(t+n-2)(dt^2 + (n-1)dt + (n-1)n)/n!\,.$$

Together with the above claim and the condition (d), $(*)$ implies $h^0(V, L) = \chi(V, L) = n + d - 1$. This proves (a). From $(*)$ we infer also $h^0(\omega_V[(n-1)L]) = h^n(V, (1-n)L) = (-1)^n\chi(V, (1-n)L) = 1$. Hence there is a non-trivial homomorphism $f\colon \mathscr{O}_V \to \omega_V[(n-1)L]$, which must be injective. Set $\mathscr{C} = \mathrm{Coker}(f)$. Then we get $\chi(\mathscr{C}[tL]) = \chi(\omega_V[(t+n-1)L]) - \chi(V, tL) = 0$ by a calculation using $(*)$. This implies $\mathscr{C} = 0$ since L is ample. Thus we prove the condition (c).

LEMMA (5.10). *Let (V, L) be a polarized surface satisfying the assumptions of* (5.7.4). *Then* $H^1(V, \mathscr{O}_V) = 0$.

PROOF. We may assume $d \leqq 2$ by (5.7.3). First we consider the case $d = 1$. Any member C of $|L|$ is irreducible because $LC = 1$ and L is ample. Moreover C is reduced because C is locally Macaulay and hence has no embedded component. So, by (3.6; a), C is a regular rung of (V, L). Moreover $H^1(C, L) = 0$. Therefore the mapping $H^1(V, \mathcal{O}_V) \to H^1(V, L)$ is bijective. Now, assuming $H^1(V, \mathcal{O}_V) \neq 0$, we take basis of both $H^1(V, \mathcal{O}_V)$ and $H^1(V, L)$ and consider the determinant $\delta(\varphi)$ of the mapping $H^1(V, \mathcal{O}_V) \to H^1(V, L)$ induced by $\varphi \in H^0(V, L)$. Obviously $\delta(0) = 0$. The above argument proves $\delta(\varphi) \neq 0$ for any $\varphi \neq 0$. Hence the polynomial function δ defined on $H^0(V, L) \cong A^2$ has an isolated zero at the origin. This is absurd. So we conclude that $0 = H^1(V, \mathcal{O}_V)$.

Next we consider the case $d = 2$. Similarly as above, it suffices to show the surjectivity of $\psi: H^0(V, L) \to H^0(C, L_C)$ for every member C of $|L|$. Note that this is equivalent to $h^0(C, L_C) = 2$. C is locally Macaulay since so is V. Hence C has no embedded component. Since $LC = 2$, there are the following three possibilities: (i) C is irreducible and reduced. (ii) C is a union of two prime components C_1 and C_2. (iii) C is not reduced and $X = C_{\mathrm{red}}$ is a curve with $LX = 1$.

In case (i), $h^0(C, L_C) = 2$ because $h^1(C, \mathcal{O}_C) = g(C, L) = 1$.

In case (ii), $LC_1 = LC_2 = 1$. Hence both C_1 and C_2 are non-singular rational curves because $Bs|L| = \varnothing$ by (5.3; b). Let $\mathcal{O}_j$ be the structure sheaf of C_j for $j = 1, 2$. Let $\mathcal{I}_j$ be the defining ideal of C_j in C. Then the product ideal $\mathcal{I}_1\mathcal{I}_2$ is supported at $C_1 \cap C_2$, which is a finite set. Hence $\mathcal{I}_1\mathcal{I}_2 = 0$ because C is locally Macaulay. Therefore $\mathcal{I}_1$ is an $\mathcal{O}_2$-module. Moreover, $\mathcal{I}_1$ has no non-trivial subsheaf supported at finite points, because neither $\mathcal{O}_C$ has. Thus $\mathcal{I}_1$ is torsion free, and hence invertible on C_2. From $g(V, L) = 1$ we infer $\chi(C, \mathcal{O}_C) = 0$. Using the exact sequence $0 \to \mathcal{I}_1 \to \mathcal{O}_C \to \mathcal{O}_1 \to 0$, we obtain $\mathcal{I}_1 = \mathcal{O}_2(-2)$. Then $h^0(C, L_C) = 2$.

In case (iii), $X \cong \boldsymbol{P}^1$ since $Bs|L| = \varnothing$. Let $\mathcal{N}$ be the sheaf of nilpotent functions on C. C is of multiplicity two, so $\mathcal{N}^2 = 0$ at a generic point of X. Hence $\mathcal{N}^2 = 0$, since C is locally Macaulay. Thus $\mathcal{N}$ turns out to be an $\mathcal{O}_X$-module. By an argument similar to that in case (ii), we infer $\mathcal{N} = \mathcal{O}_X(-2)$. So we obtain $h^0(C, L_C) = 2$ using the exact sequence $0 \to \mathcal{N} \to \mathcal{O}_C \to \mathcal{O}_X \to 0$.

Thus, in any case, we have $h^0(C, L_C) = 2$. From this it follows that ψ is surjective, as desired.

(5.11) PROOF OF (5.7.4). We use the induction on $n = \dim V$. First consider the case $n = 2$. For any rung C, we have an exact sequence $H^1(V, (t-1)L) \to H^1(V, tL) \to H^1(C, tL) = 0$ for every positive integer t. Hence we prove $H^1(V, tL) = 0$ for $t \geqq 0$ by induction on t and by (5.10). We also have an exact sequence $H^0(V, tL) \to H^0(C, tL) \to H^1(V, (t-1)L) \to H^1(V, tL)$ for every $t \leqq 0$. From this we obtain $H^1(V, tL) = 0$ for $t \leqq 0$ by induction on t.

Next we consider the case $n \geqq 3$. By (5.3; a), any ladder $V = D_n \supset \cdots \supset D_1$ of (V, L) is regular. Hence $Bs|L| \subset D_1$. D_1 is locally Macaulay since it is an irreducible reduced curve. So V is locally Macaulay at every point of $D_1 \supset Bs|L|$. Hence, using (2.5), we infer that a general member D of L is 2-Macaulay. It is easy to see that the induction hypothesis applies to (D, L_D). So $H^q(D, tL) = 0$ for any integers t, q with $0 < q < n-1$. Hence, applying [F3; (2.1)], we infer $H^q(V, tL) = 0$ for any t, q with $2 \leqq q < n$. On the other hand $H^1(V, -tL) = 0$ for $t \gg 0$ since V is 2-Macaulay. So, similary as in [So; Lemma I-B] and [F3; (2.2)], we infer $H^1(V, tL) = 0$ for any integer t. Thus we proved the condition (5.6; d).

(5.12) PROOF OF (5.7.6). Clearly (V, L) has a ladder since L is very ample. So, by the same method as above, we reduce the problem to the following:

LEMMA. *Let (V, L) be a polarized surface satisfying the assumption of* (5.7.6). *Then $H^1(V, \mathcal{O}_V) = 0$.*

PROOF. When V is normal, the assertion follows from [HW; Theorem 2.2]. So we consider the case in which V is not normal. Let S be the singular locus of V. Then $\dim S > 0$ because V is locally Macaulay. Hence $C \cap S \neq \varnothing$ for any member C of $|L|$. From the condition (c) it follows $\omega_C = \mathcal{O}_C$. C is irreducible and reduced if it is general. So C is a rational curve with one singular point, where C meets S. From this it follows also that S has only one component X of positive dimension, and that $X \cong \boldsymbol{P}^1$, because L is very ample. Take a desingularization $\pi: V^* \to V$ of V and let $\mathscr{C}$ be the cokernel of the natural injective homomorphism $\mathcal{O}_V \to \pi_* \mathcal{O}_{V^*}$. It is easy to see that the support of $\mathscr{C}$ is X. Moreover, $C^* = \pi^{-1}(C)$ is a rational normal curve. So V^* is rational since $(C^*)^2 > 0$. Hence $H^1(V, \pi_* \mathcal{O}_{V^*}) = 0$. This implies $h^0(V, \mathscr{C}) = h^1(V, \mathcal{O}_V)$. We have also $H^1(V^*, -C^*) = 0$, which implies $H^1(\pi_* \mathcal{O}_{V^*}[-L]) = 0$. Therefore $h^0(V, \mathscr{C}[-L]) = h^1(V, -L) = h^1(V, \omega_V) = h^1(V, \mathcal{O}_V)$. On the other hand, we have $h^0(\mathscr{C}[-tL]) \leqq h^1(V, -tL) = h^1(\omega_V[tL]) = 0$ for $t \gg 0$. This implies that $\mathscr{C}$ is torsion free on X. So, if $h^0(V, \mathscr{C}) \neq 0$, then $h^0(\mathscr{C}[-L]) < h^0(\mathscr{C})$ since L is very ample. Hence the preceding equalities imply $h^0(V, \mathscr{C}) = 0$, proving the assertion $H^1(V, \mathcal{O}_V) = 0$.

REMARK (5.13). In (5.7.3) and (5.7.4), it might be unnecessary to assume that (V, L) has a ladder. This is really the case if V is a normal surface. However, in general, (V, L) does not always have a ladder even if $\Delta = 1$ and V is a normal surface. Of course, in this case, V is singular at some point of $Bs|L|$.

6. Structures of Del Pezzo manifolds. (6.1) In this section we study the structure of a Del Pezzo manifold (M, L) such that L is very ample. Since M is non-singular, this is equivalent to saying that $\Delta(M, L) = 1$ and $d(M, L) \geqq 3$ (cf. (5.5)).

As for singular polarized varieties (V, L) with $\Delta(V, L) = 1$ and very ample L, we have the results announced in [F6]. This topic will be treated elsewhere.

(6.2) When $n = \dim M = 2$, M is rational by virtue of Castelnuovo's criterion. Using the structure theory of rational surfaces (cf., e.g., [Sa; Chap. V]), we infer that M is the blowing-up of $\boldsymbol{P}^2$ with center being $(9 - d)$ points on $\boldsymbol{P}^2$ unless $M \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$. In the latter case $d(M, L) = c_1^2 = 8$. So, in particular, we obtain:

$d(M, L) \leqq 9$ if $n = 2$. The equality holds if and only if $(M, L) \cong (\boldsymbol{P}^2, 3H)$.

(6.3) As for the cases $n \geqq 3$, we have the following:

THEOREM. *Let (M, L) be a Del Pezzo manifold with $n = \dim M \geqq 3$ and $d = d(M, L) = L^n \geqq 3$. Then (M, L) is of one of the following types.*

(1) *A hypercubic.* $d = 3$.

(2) *A complete intersection of two hyperquadrics.* $d = 4$.

(3) *A linear section of the Grassmann variety parametrizing lines in $\boldsymbol{P}^4$, embedded in $\boldsymbol{P}^9$ by Plücker coordinates.* $d = 5$ *and* $n \leqq 6$ *in this case.*

(4) *A Segre variety $\boldsymbol{P}^2 \times \boldsymbol{P}^2$ embedded in $\boldsymbol{P}^8$.* $d = 6$.

(5) *A hyperplane section of a fourfold of the above type* (4). *$(M, L) \cong (\boldsymbol{P}(T), \mathcal{O}(1))$ for the tangent bundle T of $\boldsymbol{P}^2$ in this case.*

(6) *A Segre variety $\boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1$ in $\boldsymbol{P}^7$.* $d = 6$.

(7) *A blowing-up of $\boldsymbol{P}^3$ at a point.* $d = 7$.

(8) *A Veronese threefold $(\boldsymbol{P}^3, 2H)$.* $d = 8$.

This theorem was proved in [F4] in case $\mathrm{char}(\mathfrak{K}) = 0$. We can apply (5.4) if $d \leqq 4$. Here we consider the case $d \geqq 6$. The case $d = 5$ will be treated separately because the proof is considerably longer than those in other cases.

(6.4) In order to use the technique of lift, we fix our notation. Given any algebraically closed field $\mathfrak{K}$ with $\mathrm{char}(\mathfrak{K}) = p > 0$, let $W(\mathfrak{K})$ denote the ring of Witt vectors (cf. [Se2], Chap. II, §6). This is a discrete valuation ring with the residue field $\mathfrak{K}$, the maximal ideal is the principal ideal (p) and the quotient field $\mathfrak{K}'$ is a field of characteristic zero.

By S we denote $\mathrm{Spec}(W(\mathfrak{K}))$. We let o (resp. x) be the closed (resp.

generic) point of S. The formal completion of S at o is denoted by $\hat{S}$.

Let M be a manifold defined over $\mathfrak{K}$. We say that M is *liftable* (resp. *formally liftable*) if there is an S-scheme $f: \mathscr{M} \to S$ (resp. formal $\hat{S}$-scheme $f: \hat{\mathscr{M}} \to \hat{S}$) such that the fiber over the closed point o is isomorphic to the given $\mathfrak{K}$-scheme M. In this case $\mathscr{M}$ (resp. $\hat{\mathscr{M}}$) is called a *lift* (resp. *formal lift*) of M. A formal lift is said to be *algebraizable* if it is a restriction to $\hat{S}$ of a lift over S.

Similarly we define the notion of the liftability of various objects defined over $\mathfrak{K}$. For example, given an effective divisor D on M, we say that the pair (M, D) is liftable if there is a lift $f: \mathscr{M} \to S$ of M and if there is an effective Cartier divisor $\mathscr{D}$ on $\mathscr{M}$ whose restriction to M is D.

(6.5) Let D be an effective divisor on a manifold M. Let $\mathscr{I}$ be the $\mathscr{O}_M$-ideal defining D. Then the sheaf Θ_M of vector fields on M can be viewed as the sheaf of derivations of $\mathscr{O}_M$, so we have a natural homomorphism $\Theta_M \to \mathscr{H}om_{\mathscr{O}}(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_M/\mathscr{I})$. The kernel of this homomorphism will be denoted by $\Theta(M, D)$.

If D has no singularities other than normal crossings, then $\Theta(M, D)$ is locally free and is the dual of the sheaf $\Omega^1_M(\log D)$ of rational 1-forms with only logarithmic poles along D. If D is non-singular, we have a natural exact sequence $0 \to \Theta(M, D) \to \Theta_M \to \mathscr{N} \to 0$ where $\mathscr{N}$ is the normal sheaf $\mathscr{O}_D[D]$ of D in M.

PROPOSITION (6.6). *Let (M, L) be a Del Pezzo threefold defined over $\mathfrak{K}$ with $d = d(M, L) \geqq 5$ and let D be a general member of $|L|$. Then the pair (M, D) is liftable.*

PROOF. D is non-singular since L is very ample. So D is a Del Pezzo surface as in (6.2). We will prove $H^2(M, \Theta(M, D)) = 0$ and then apply the criterion (A1) in the appendix. By virtue of the exact sequence $0 \to \Theta(M, D) \to \Theta_M \to \mathscr{O}_D[L] \to 0$, it suffices to show $H^2(M, \Theta_M) = 0$. For this purpose we use a couple of lemmas.

LEMMA (6.7). $H^1(S, \Theta_S) = 0$ *for any Del Pezzo surface with* $d \geqq 5$.

This is obvious if $S \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$. So we may assume that S is the blowing-up of $\boldsymbol{P}^2$ at $9 - d$ points. No three of them are collinear because $-K^S$ is ample. Hence these $9 - d$ ($\leqq 4$) points are in a general position. The assertion follows easily from this.

LEMMA (6.8). $H^1(D, \Theta_D[tL]) = 0$ *for any* $t \geqq 0$.

PROOF. Any general member C of L is a non-singular elliptic curve.

Using the exact sequence $0 \to \Theta_C \to (\Theta_D)_C \to \mathcal{O}_C[L] \to 0$, we obtain $H^1(C, \Theta_D[tL]_C) = 0$ for any $t \geqq 1$. Hence, in view of the exact sequence $0 \to \Theta_D[(t-1)L] \to \Theta_D[tL] \to \Theta_D[tL]_C \to 0$, we infer that $h^1(D, \Theta_D[tL])$ is a decreasing function on t for $t > 0$. So the assertion is proved by induction on t since (6.7) applies in case $t = 0$.

(6.9) PROOF OF (6.6), CONTINUED. From (6.8) and the exact sequence $0 \to \Theta_D \to (\Theta_M)_D \to \mathcal{O}_D[L] \to 0$, we infer that $H^1(D, \Theta_M[tL]_D) = 0$ for any $t \geqq 0$. Therefore, [F3; (2.1)] applies and we get $H^2(M, \Theta_M) = 0$.

(6.10) Let M, L, D be as in (6.6) and let $(\mathscr{M}, \mathscr{D})$ be a lift of (M, D) over $S = \mathrm{Spec}(W(\mathfrak{K}))$. Let (M', D') be its generic fiber over x and set $\mathscr{L} = [\mathscr{D}] \in \mathrm{Pic}(\mathscr{M})$ and $L' = [D'] \in \mathrm{Pic}(M')$. Then:

LEMMA. *(M', L') is a Del Pezzo manifold with $d(M', L') = d(M, L)$.*

PROOF. Clearly $f: \mathscr{M} \to S$ is smooth and flat. So the Hilbert polynomials of (M, L) and (M', L') are the same. Hence $d(M', L') = d(M, L)$ and $g(M', L') = g(M, L) = 1$. Moreover, $H^1(M, L) = 0$ implies that $h^0(M, L) = \mathrm{rank}(f_*\mathscr{L}) = h^0(M', L')$. So $\Delta(M', L') = \Delta(M, L) = 1$.

COROLLARY (6.11). *Let (M, L) be a Del Pezzo manifold with $n = \dim M \geqq 3$. Then $d = d(M, L) \leqq 8$.*

PROOF. We derive a contradiction assuming $d \geqq 9$. Taking general hyperplane sections successively, we reduce the problem to the case $n = 3$. Take a general member D of $|L|$ further. Then D is a Del Pezzo surface with $d \geqq 9$, so $(D, L_D) \cong (\boldsymbol{P}^2, 3H)$. The pair (M, D) is liftable by (6.6). But we cannot have such a pair in the characteristic zero case, contradicting the above Lemma (6.10).

REMARK. The impossibility of the existence of such a pair (M, D) was first proved by Tango [T] by a completely different method. The above argument is due to Bădescu [B], who considered the case $d = 8$ too (see (6.17) below).

(6.12) In order to proceed further, we need to study the restriction mapping $\mathrm{Pic}(M) \to \mathrm{Pic}(D)$ more precisely. Letting things be as in (6.10), we will reduce the problem to zero-characteristic cases. We begin with the following:

LEMMA. *Let $(\hat{\mathscr{M}}, \hat{\mathscr{D}})$ be the restriction of $(\mathscr{M}, \mathscr{D})$ over the formal completion $\hat{S}$. Then* $\mathrm{Pic}(\hat{\mathscr{M}}) \to \mathrm{Pic}(M)$ *and* $\mathrm{Pic}(\hat{\mathscr{D}}) \to \mathrm{Pic}(D)$ *are bijective.*

PROOF. The injectivity (resp. surjectivity) follows from $H^1(M, \mathcal{O}_M) = H^1(D, \mathcal{O}_D) = 0$ (resp. $H^2(M, \mathcal{O}_M) = H^2(D, \mathcal{O}_D) = 0$) by standard arguments based on a natural exact sequence $0 \to \mathcal{O}_M \to \mathcal{O}_j^\times \to \mathcal{O}_{j-1}^\times \to 0$, where $\mathcal{O}_j^\times$ is the sheaf of (multiplicative) groups consisting of invertible sections of $\mathcal{O}_{\mathcal{M}}/p^{j+1}\mathcal{O}_{\mathcal{M}}$.

LEMMA (6.13). *The natural mappings* $\mathrm{Pic}(\mathcal{M}) \to \mathrm{Pic}(\hat{\mathcal{M}})$ *and* $\mathrm{Pic}(\mathcal{D}) \to \mathrm{Pic}(\hat{\mathcal{D}})$ *are bijective.*

PROOF. The injectivity is clear. To prove the surjectivity, let F be any line bundle on $\hat{\mathcal{M}}$. Since $\hat{\mathcal{L}}$ is ample, we can find an exact sequence $A \to B \to F \to 0$ such that A and B are direct sums of line bundles of the forms $u_i\hat{\mathcal{L}}$, u_i's being integers. Extending each $u_i\hat{\mathcal{L}}$ to $u_i\mathcal{L}$ on $\mathcal{M}$, we extend A and B to vector bundles $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{M}$. We easily see that the homomorphism $A \to B$ is extended to a homomorphism $\psi\colon \mathcal{A} \to \mathcal{B}$. Setting $\mathcal{F} = \mathrm{Coker}(\psi)$, we have $\mathcal{F}_{\hat{\mathcal{M}}} = F$. This implies that $\mathcal{F}$ is invertible on an open subscheme of $\mathcal{M}$ containing the closed fiber, hence $\mathcal{F}$ is invertible on the whole space $\mathcal{M}$. Similarly we prove the surjectivity of $\mathrm{Pic}(\mathcal{D}) \to \mathrm{Pic}(\hat{\mathcal{D}})$.

LEMMA (6.14). *The restriction mappings* $\mathrm{Pic}(\mathcal{M}) \to \mathrm{Pic}(M')$ *and* $\mathrm{Pic}(\mathcal{D}) \to \mathrm{Pic}(D')$ *are bijective.*

PROOF. The surjectivity follows from the smoothness of $\mathcal{M}$ and $\mathcal{D}$. The injectivity follows from the fact that $\mathcal{M} - M' = M$ (resp. $\mathcal{D} - D' = D$) is a Cartier divisor defined by the principal ideal $p\mathcal{O}_{\mathcal{M}}$ (resp. $p\mathcal{O}_{\mathcal{D}}$) $\cong \mathcal{O}_{\mathcal{M}}$ (resp. $\mathcal{O}_{\mathcal{D}}$).

(6.15) Let $\mathfrak{K}^*$ be the algebraic closure of $\mathfrak{K}'$ and let M^* and D^* be the scalar extensions of M' and D' respectively over $\mathrm{Spec}(\mathfrak{K}^*)$. Then we have:

LEMMA. *The natural mappings* $\mathrm{Pic}(M') \to \mathrm{Pic}(M^*)$ *and* $\mathrm{Pic}(D') \to \mathrm{Pic}(D^*)$ *are bijective.*

PROOF. From the preceding lemmas we obtain a bijection $\mathrm{Pic}(D') \to \mathrm{Pic}(D)$. Moreover, this is compatible with the intersection pairings. Therefore $\mathrm{rank}(\mathrm{Pic}(D')) = 10 - d = \mathrm{rank}(\mathrm{Pic}(D^*))$ and the determinant of the intersection number matrix with respect to an integral base of $\mathrm{Pic}(D')$ is $(-1)^{9-d}$. From these observations we infer that $\mathrm{Pic}(D') \to \mathrm{Pic}(D^*)$ is bijective.

It is clear that $\mathrm{Pic}(M') \to \mathrm{Pic}(M^*)$ is injective. Moreover, its image is the largest space on which $G = \mathrm{Gal}(\mathfrak{K}^*/\mathfrak{K}')$ acts trivially (cf., e.g., [MR; Lemma 4]). On the other hand, $\mathrm{Pic}(M^*) \to \mathrm{Pic}(D^*)$ is injective by Lefschetz' theorem and G acts trivially on $\mathrm{Pic}(D^*)$ by the above argument. Therefore G acts trivially on $\mathrm{Pic}(M^*)$, which implies our assertion.

(6.16) By virtue of the preceding lemmas altogether we obtain the following natural commutative diagram:

$$\begin{array}{ccc} \mathrm{Pic}(M) & \longrightarrow & \mathrm{Pic}(D) \\ \wr\Vert & & \wr\Vert \\ \mathrm{Pic}(M^*) & \longrightarrow & \mathrm{Pic}(D^*) \end{array}.$$

Now we will prove the theorem (6.3) in case $d \geqq 6$.

(6.17) In view of (6.11), we first consider the case $d = 8$.

Suppose that $n = 3$. A general member D of $|L|$ is a Del Pezzo surface with $d = 8$. The pair (M, D) is liftable by (6.6) and we let the notation be as in the above lemmas. Using (6.10) and [F4; (5.6)], we obtain $(M^*, L^*) \cong (\boldsymbol{P}^3, 2H)$. So $L = 2F$ for some $F \in \mathrm{Pic}(M)$ by (6.16). It is now easy to show $(M, F) \cong (\boldsymbol{P}^3, \mathcal{O}(1))$.

Next we derive a contradiction assuming $n \geqq 4$. Similarly as in (6.11), we may assume $n = 4$. We sketch here two different proofs.

FIRST PROOF. Take a general member D of $|L|$. Then (D, L_D) is a Del Pezzo threefold with $d = 8$, hence we have $(D, L_D) \cong (\boldsymbol{P}^3, 2H)$. By [H; p. 178] we infer that $\mathrm{Pic}(M) \to \mathrm{Pic}(D)$ is bijective. In particular $L = 2F$ for some $F \in \mathrm{Pic}(M)$. Then $8 = d(M, L) = L^4 = (2F)^4 = 16F^4$, which is absurd.

SECOND PROOF. We see $H^2(M, \Theta_M) = 0$ by a method similar to that in (6.9). From this we infer that the pair (M, D) is liftable similarly as in (6.6). But such a pair does not exist in the characteristic zero case.

REMARK. This second argument works in several other cases too. In the following we consider chiefly variants of the above first proof.

(6.18) Here we consider the case $d = 7$.

Suppose first that $n = 3$. In view of (6.2) we infer that a general member D of $|L|$ is isomorphic to the blowing-up of $\boldsymbol{P}^2$ with center being two points. Let C be the strict transform of the line passing these points. Using (6.16) and [F4; (5.8)], we infer that $F_D = [C]$ in $\mathrm{Pic}(D)$ for some $F \in \mathrm{Pic}(M)$. We have $H^1(D, F_D - tL) = 0$ for any $t \geqq 0$ by a technique similar to that in (6.8). Hence, by [F3; (2.3)], we have a member E of $|F|$ such that $E_D = C$, which is an exceptional curve on D. In view of [F3; §5] we infer that E can be blown down to a smooth point. Similarly as in [F4; (5.8)], we see that the manifold thus obtained is $\boldsymbol{P}^3$. So M is the blowing-up of $\boldsymbol{P}^3$ at a point.

Next we derive a contradiction assuming $n \geqq 4$. We may assume $n = 4$. Then a general member D of $|L|$ is a Del Pezzo threefold of the above type. By the Lefschetz theorem $\mathrm{Pic}(M) \to \mathrm{Pic}(D)$ is bijective (see [H; p. 178]). We can prove also $H^1(D, E - tL) = 0$ for any $t \geqq 0$, where E is the exceptional divisor on D as above. From this we infer that E is the restriction to D of an effective divisor on M, and that this divisor can be contracted to a non-singular point. Similarly as in [F4; (5.5)], we get a Del Pezzo fourfold with $d = 8$ by this contraction. This is impossible by (6.17).

(6.19) Here we consider the case $d = 6$.

Suppose first that that $n = 3$. In view of (6.2) we infer that a general member D of $|L|$ is isomorphic to the blowing-up of $\boldsymbol{P}^2$ with center being three different non-collinear points q_1, q_2, q_3. Let H be the pull-back of $\mathcal{O}_{P^2}(1)$ to D and let E_j be the exceptional curve over q_j for each $j = 1, 2, 3$. Let things be as in (6.16). Then M^* is either of the type (6.3; 5) or of the type (6) by [F4].

If M^* is of the type (5), then we infer that H comes from $\mathrm{Pic}(M)$ by (6.16). Similarly as above and as in [F4], we can extend the birational morphisms $\rho_{|H|}: D \to \boldsymbol{P}^2$ and $\rho_{|L-H|}: D \to \boldsymbol{P}^2$ to morphisms defined on M, which give rise to an inclusion $M \subset \boldsymbol{P}^2 \times \boldsymbol{P}^2$. So M is of the type (5).

If M^* is of the type (6), then we infer that $H - E_1$, $H - E_2$ and $H - E_3$ come from $\mathrm{Pic}(M)$. Similarly as in [F4], the rational mappings defined by the linear systems associated with these line bundles give an isomorphism $M \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1$. Thus M is of the type (6).

In case $n = 4$, we see $M \cong \boldsymbol{P}^2 \times \boldsymbol{P}^2$ similarly as in [F4]. If $n \geqq 5$, we derive a contradiction as above or as in [F4].

REMARK (6.20). In case $d = 5$, we can show that $\mathrm{Pic}(M)$ is generated by L. However, this is not sufficient to prove (6.3; 3). This case will be treated in a forthcoming paper.

Appendix. Here we prove a couple of liftability criteria. We use the notation introduced in (6.4) and (6.5).

THEOREM (A1). *Let D be a reduced effective divisor on a manifold M defined over $\mathfrak{K}$. Then the pair (M, D) is formally liftable if $H^2(M, \Theta(M, D)) = 0$. If in addition there is an ample divisor on M whose support is contained in D, then any formal lift of (M, D) is algebraizable.*

PROOF. The formal liftability is proved quite similarly as in [G2; Exposé III, Cor. 6. 10], where the case $D = \varnothing$ is considered. Indeed, as a substitute for [G2; Prop. 6.1], one easily shows that the sheaf of infinitesimal automorphisms of the pair (M, D) is $\Theta(M, D)$. The assertion about the algebraizability is well-known.

(A2) Given a prepolarized manifold (M, L), we let $P = \boldsymbol{P}(\mathcal{O}_M \oplus L)$ and let D_- and D_+ be the sections of the $\boldsymbol{P}^1$-bundle $\pi: P \to M$ corresponding to the quotient bundles $\mathcal{O}_M$ and L respectively. Let H be the tautological line bundle $\mathcal{O}_P(1)$. Then $[D_-] = H - \pi^* L$, $[D_+] = H$ in $\mathrm{Pic}(P)$ and $H_{D_-} = \mathcal{O}$, $H_{D_+} = L$. Hence the normal bundles of D_- and D_+ are $-L$ and L respectively. Set $D = D_- + D_+$.

We have a natural surjective homomorphism $f: \Theta(P, D) \to \pi^* \Theta_M$, and it is easy to see $\mathrm{Ker}(f) \cong \mathcal{O}_P$. Taking π_* we obtain the following exact sequence: $0 \to \mathcal{O}_M \to \pi_* \Theta(P, D) \to \Theta_M \to 0$. We denote $\pi_* \Theta(P, D)$ by $\Theta(M, L)$. The extension class $c \in \mathrm{Ext}^1(\Theta_M, \mathcal{O}_M) \cong H^1(M, \Omega_M^1)$ defined by the above sequence is essentially the Chern class of L.

THEOREM (A3). *(M, L) is formally liftable if $H^2(M, \Theta(M, L)) = 0$. If L is ample, any formal lift of (M, L) is algebraizable.*

PROOF. Let the notation be as above. Then $H^2(P, \Theta(P, D)) \cong H^2(M, \Theta(M, L))$ follows from a calculation of the Leray spectral sequence. So the assumption implies that (P, D) is formally liftable by (A1). Let $\hat{P}$ and $\hat{D}_+$ be a formal lift of P and D_+ (on $\hat{P}$) respectively. Then the pair ($\hat{D}_+$, the normal bundle of $\hat{D}_+$ in $\hat{P}$) is a formal lift of (M, L).

REMARK. The above sheaves $\Theta(M, D)$ in (A1) and $\Theta(M, L)$ in (A2) play important roles in the deformation theory too. This is no wonder because there is a close relation between the lift theory and the deformation theory.

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