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GENERIC PROPERTIES OF THE EIGENVALUE OF THE LAPLACIAN FOR COMPACT RIEMANNIAN MANIFOLDS

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Introduction. In this paper, we discuss generic properties of the eigenvalues of the Laplacian for compact Riemannian manifolds without boundary.

Throughout this paper, let M be an arbitrary fixed connected compact C^∞ manifold of dimension n without boundary, and $\mathscr{M}$ the set of all C^∞ Riemannian metrics on M. For $g \in \mathscr{M}$, let Δ_g be the Laplacian (cf. (2.1)) of (M, g) acting on the space $C^\infty(M)$ of all C^∞ real valued functions on M and

$$0 = \lambda_0(g) < \lambda_1(g) \leqq \lambda_2(g) \leqq \cdots \uparrow \infty$$

the eigenvalues of the Laplacian Δ_g counted with their multiplicities. We regard each eigenvalue $\lambda_k(g)$, $k = 0, 1, 2, \cdots$, as a function of g in $\mathscr{M}$. Let us consider the following problem: "*Does each eigenvalue* $\lambda_k(g)$ *depend continuously on* g *in* $\mathscr{M}$ *with respect to the* C^∞ *topology?*"

The continuous dependence of the eigenvalues of the Dirichlet problem upon variations of domains is well known (cf. [CH, p. 290]). Variations of coefficients of elliptic differential operators were dealt with by Kodaira-Spencer [KS] who gave a proof of the continuity of eigenvalues. In this paper, we give a simple proof of the above problem.

To answer the above problem, in § 1, we introduce a complete distance ρ on $\mathscr{M}$ which gives the C^∞ topology. Then, in § 2, we assert that each $\lambda_k(g)$, $k = 1, 2, \cdots$, depends continuously on $g \in \mathscr{M}$ with respect to the topology on $\mathscr{M}$ induced by the distance ρ. More precisely, we have

THEOREM 2.2. *For each positive number* δ *and each* $g, g' \in \mathscr{M}$, *the inequality* $\rho(g, g') < \delta$ *implies that*

$$\exp(-(n+1)\delta) \leqq \lambda_k(g)/\lambda_k(g') \leqq \exp((n+1)\delta) ,$$

for each $k = 1, 2, \cdots$ (*where* $n = \dim M$).

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That is, if two Riemannian metrics g and g' are close to each other with respect to the distance ρ, then the ratio $\lambda_k(g)/\lambda_k(g')$ is close to one *uniformly* in $k = 1, 2, \cdots$. Thus we have immediately the following corollary. A similar result was obtained by [KS].

COROLLARY 2.3. *The multiplicity* $m_k(g)$ *of each eigenvalue* $\lambda_k(g)$, *i.e.*, $m_k(g) = \#\{i; \lambda_i(g) = \lambda_k(g)\}$, *depends upper semi-continuously on* $g \in \mathscr{M}$: *For each* $g \in \mathscr{M}$ *and* $k = 0, 1, 2, \cdots$, *there exists a positive number* δ *such that* $\delta(g, g') < \delta$ *implies* $m_k(g') \leqq m_k(g)$.

These results are useful in investigating generic properties of Riemannian metrics. As one of these applications, we give a simple and constructive proof of the following theorem of Uhlenbeck (cf [U], [T]):

THEOREM 3.1. *Let* M *be a compact connected* C^∞ *manifold of dimension not less than two. Then the set* $\mathscr{S} = \{g \in \mathscr{M}$; *all eigenvalues* $\lambda_k(g)$, $k = 0, 1, 2, \cdots$, *have multiplicity one*$\}$ *is a residual set in the complete metric space* $(\mathscr{M}, \rho)$, *i.e., a countable intersection of open dense subsets.*

Therefore $\mathscr{S}$ is a subset of the second category and dense in $\mathscr{M}$, i.e., for most Riemannian metrics, all the eigenvalues of the Laplacian have multiplicity one. A similar result was obtained by Bleecker-Wilson [BW]. They showed that, for each Riemannian metric g, there exists a residual set of f in $C^\infty(M)$ for which all the eigenvalues of the Riemannian metric $\exp(f)g$ have multiplicity one. Their result implies the density of $\mathscr{S}$ in $\mathscr{M}$, but it does not necessarily imply that $\mathscr{S}$ is residual in $\mathscr{M}$.

Secondly, we show the following proposition.

PROPOSITION 3.4. *Let* M *be a compact connected* C^∞ *manifold of dimension not less than two. If a Riemannian metric* g *belongs to the set* $\mathscr{S}$, i.e., *if all the eigenvalues of the Laplacian* Δ_g *have multiplicity one, then the group of all isometries of* (M, g) *is discrete.*

Combining this with Theorem 3.1, we have:

COROLLARY 3.5. *Let* M *be a compact connected* C^∞ *manifold of dimension not less than two. Then the set of all elements* g *in* $\mathscr{M}$ *with discrete isometry group contains a residual subset of* $\mathscr{M}$.

That is, for most Riemannian metrics of a compact connected C^∞ manifold of dimension not less than two, the isometry groups are trivial. This corollary was obtained by Ebin (cf. [E_1, Proposition 8.3]) in a different manner.

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1. Complete distance on the set of Riemannian metrics. Let M be a compact n-dimensional C^∞ manifold without boundary. Let $S(M)$ be the space of all C^∞ symmetric covariant 2-tensors on M and $\mathscr{M}$ the set of all C^∞ Riemannian metrics on M. In this section, we define a complete distance on $\mathscr{M}$.

1.1. *Fréchet space* $S(M)$. Following [E$_2$] and [GG], we introduce a Fréchet norm $|\cdot|$ on $S(M)$. We fix a finite covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of M such that the closure of U_λ is contained in the open coordinate neighborhood V_λ. For $h \in S(M)$, we denote by h_{ij} the components of h with respect to coordinates $(x_1, \cdots, x_n)$ on V_λ, $\lambda \in \Lambda$. For every non-negative integer k and $\lambda \in \Lambda$, put

$$|h|_{\lambda,k} = \sup_{U_\lambda} \sum_{|\alpha|\leqq k} \sum_{i,j=1}^{n} |\partial^{|\alpha|}(h_{ij})/\partial(x_1)^{\alpha_1} \cdots \partial(x_n)^{\alpha_n}| ,$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ denotes an n-tuple of non-negative integers α_i and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Define a norm $|\cdot|_k$ on $S(M)$ by $|h|_k = \sum_{\lambda\in\Lambda}|h|_{\lambda,k}$, $h \in S(M)$, and a Fréchet norm $|\cdot|$ on $S(M)$ by

$$|h| = \sum_{k=0}^{\infty} 2^{-k}|h|_k(1 + |h|_k)^{-1} , \quad h \in S(M) .$$

We can define a distance ρ' on $S(M)$ by $\rho'(h_1, h_2) = |h_1 - h_2|$, $h_1, h_2 \in S(M)$. Then it is well-known that $S(M)$ is a Fréchet space, that is, the metric space $(S(M), \rho')$ is complete.

1.2. *Complete distance of* $\mathscr{M}$. For each point x in M, let P_x (resp. S_x) be the set of all symmetric positive definite (resp. merely symmetric) bilinear forms on $T_xM \times T_xM$, where T_xM is the tangent space of M at $x \in M$. We define a distance ρ''_x on P_x, $x \in M$, by

$$\rho''_x(\varphi, \psi) = \inf\{\delta > 0; \exp(-\delta)\varphi < \psi < \exp(\delta)\varphi\} ,$$

where, for φ, ψ in S_x, $\varphi < \psi$ means that $\psi - \varphi \in S_x$ is positive definite on $T_xM \times T_xM$. In fact, ρ''_x defines clearly a distance on P_x. Let G_x, $x \in M$, be the group of all non-singular linear mappings of T_xM onto itself. For $A \in G_x$ and $\varphi \in S_x$, put $\varphi^A(u, v) = \varphi(A(u), A(v))$ for $u, v \in T_xM$. We fix a basis $\{e_i\}_{i=1}^n$ of T_xM and identify S_x with the set $S(n)$ of all real symmetric matrices of degree n by $S_x \ni \varphi \mapsto (\varphi(e_i, e_j))_{1\leqq i,j\leqq n} \in S(n)$. Denote by Φ this identification of S_x with $S(n)$. Let $P(n)$ be the set of all positive definite matrices in $S(n)$. Then we have the following lemma immediately.

LEMMA 1.1. (i) $\rho''_x(\varphi^A, \psi^A) = \rho''_x(\varphi, \psi)$ *for every* $A \in G_x$ *and* $\varphi, \psi \in P_x$.

(ii) *Let* $\varphi_0 \in P_x$ *be the element such that* $\Phi(\varphi_0)$ *is the identity matrix. Then we have*

$$\rho''_x(\varphi, \varphi_0) = \|\log \Phi(\varphi)\| , \quad \varphi \in P_x .$$

Here we denote by $\log A$, $A \in P(n)$, *the inverse image of the exponential mapping of* $S(n)$ *onto* $P(n)$ *and by* $\|H\|$, $H \in S(n)$, *the operator norm of* H, *that is,* $\|H\| = \sup\{\|H(x)\|; x \in \boldsymbol{R}^n$ *and* $\|x\| = 1\}$, *where* $\|\cdot\|$ *is the Euclidean norm of* $\boldsymbol{R}^n$.

(iii) *The metric space* (P_x, ρ''_x) *is complete.*

(iv) *Let* $\{\varphi_j\}_{j=1}^{\infty}$ *be a sequence in* P_x *which converges to an element* φ *in* P_x *with respect to the distrance* ρ''_x. *Then* $\lim_{j\to\infty} \varphi_j(u, v) = \varphi(u, v)$ *for every* $u, v \in T_xM$.

DEFINITION. We define a distance ρ'' on $\mathscr{M}$ by

$$\rho''(g_1, g_2) = \sup_{x \in M} \rho''_x((g_1)_x, (g_2)_x) , \quad g_1, g_2 \in \mathscr{M} ,$$

and a distance ρ on $\mathscr{M}$ by

$$\rho(g_1, g_2) = \rho'(g_1, g_2) + \rho''(g_1, g_2) , \quad g_1, g_2 \in \mathscr{M} .$$

Then, by Lemma 1.1, we have:

PROPOSITION 1.2. *The metric space* $(\mathscr{M}, \rho)$ *is complete.*

PROOF. We prove this in the usual manner. Let $\{g_j\}_{j=1}^{\infty}$ be a Cauchy sequence in $(\mathscr{M}, \rho)$. Then it is also a Cauchy sequence in both metric spaces $(S(M), \rho')$ and $(\mathscr{M}, \rho'')$. Since the metric space $(S(M), \rho')$ is complete, there exists an element g in $S(M)$ such that $\lim_{j\to\infty} \rho'(g_j, g) = 0$. In particular, for each $x \in M$ and $u, v \in T_xM$ we have

$$\lim_{j\to\infty} (g_j)_x(u, v) = g_x(u, v) . \tag{1.1}$$

On the other hand, because of $\lim_{i,j\to\infty} \rho''(g_i, g_j) = 0$, for every $\varepsilon > 0$, there exists a positive number N such that

$$\rho''_x((g_i)_x, (g_j)_x) \leqq \rho''(g_i, g_j) < \varepsilon \tag{1.2}$$

for every $i, j \geqq N$ and $x \in M$. Then the sequence $\{(g_j)_x\}_{j=1}^{\infty}$ is a Cauchy sequence in the complete metric space (P_x, ρ''_x), hence it converges to an element $\tilde{g}_x$ in P_x with respect to ρ''_x. By Lemma 1.1 (iv), we have $\lim_{j\to\infty} (g_j)_x(u, v) = \tilde{g}_x(u, v)$, $u, v \in T_xM$, so we obtain $g = \tilde{g} \in \mathscr{M}$. Therefore, combining this with the inequalities (1.2), we have $\rho''_x((g_i)_x, g_x) \leqq \varepsilon$ for all $x \in M$. Thus we obtain $\rho''(g_i, g) \leqq \varepsilon$ for $i \geqq N$, that is, $\lim_{i\to\infty} \rho''(g_i, g) = 0$. Therefore the sequence $\{g_i\}_{i=1}^{\infty}$ converges to $g \in \mathscr{M}$ with respect to the distance ρ. q.e.d.

2. Continuity of eigenvalues. 2.1. *Preliminaries.* For every g in $\mathscr{M}$, let $-\Delta_g$ be the Laplace-Beltrami operator acting on the space $C^\infty(M)$ of all real valued C^∞ functions on M, that is,

$$-\Delta_g = \sum_{i,j=1}^{n} g^{ij}(\partial^2/\partial x_i \partial x_j - \sum_{k=1}^{n} \Gamma_{ij}^k \partial/\partial x_k) . \tag{2.1}$$

Here (g^{ij}) is the inverse matrix of the component matrix (g_{ij}) of the Rimannian metric g with respect to a local coordinate $(x_1, \cdots, x_n)$ on M, and Γ_{ij}^k is the Christoffel symbol:

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{m=1}^{n} g^{km}(\partial g_{mi}/\partial x_j + \partial g_{mj}/\partial x_i - \partial g_{ji}/\partial x_m) . \tag{2.2}$$

Let $(\, , \,)_g$ be the inner product on $C^\infty(M)$ given by

$$(f_1, f_2)_g = \int_M f_1(x) f_2(x) dv_g(x) , \quad f_1, f_2 \in C^\infty(M) , \tag{2.3}$$

and put $\|f\|_g = ((f, f)_g)^{1/2}$ for $f \in C^\infty(M)$. Here $dv_g(x)$ is the canonical measure of (M, g) given locally by

$$dv_g(x) = (\det(g_{ij}))^{1/2} dx_1 \cdots dx_n \quad \text{(cf. [BGM, p. 10])} . \tag{2.4}$$

Define as usual the inner product $(\, , \,)_g$ on the space $A^1(M)$ of all real valued C^∞ 1-forms on M by

$$(\omega_1, \omega_2)_g = \int_M \langle \omega_1, \omega_2 \rangle_g(x) dv_g(x) , \quad \omega_1, \omega_2 \in A^1(M) , \tag{2.5}$$

and put $\|\omega\|_g = ((\omega, \omega)_g)^{1/2}$ for $\omega \in A^1(M)$. The pointwise inner product $\langle \omega_1, \omega_2 \rangle_g(x)$ of $\omega_i \in A^1(M)$, $i = 1, 2$, is given by

$$\langle \omega_1, \omega_2 \rangle_g(x) = \sum_{i,j=1}^{n} g^{ij}(x) a_{1i}(x) a_{2i}(x) , \quad x \in M , \tag{2.6}$$

where $\{a_{ki}(x)\}_{i=1}^n$, $k = 1, 2$, are the components of the cotangent vectors $(\omega_k)_x$, $k = 1, 2$, with respect to the local coordinate $(x_1, \cdots, x_n)$.

2.2. *Max-mini principle.* Since M is compact, the spectrum of the Laplacian Δ_g is a discrete set of non-negative eigenvalues with finite multiplicities. We arrange the eigenvalues as

$$0 = \lambda_0(g) < \lambda_1(g) \leqq \lambda_2(g) \leqq \cdots \leqq \lambda_k(g) \leqq \cdots \uparrow \infty .$$

Here the eigenvalues are counted repeatedly as many times as their multiplicities. For example if the multiplicity of $\lambda_1(g)$ is h and $k \leqq h$, then the k-th eigenvalue $\lambda_k(g)$ of (M, g) is $\lambda_1(g)$, i.e., $\lambda_2(g) = \cdots = \lambda_h(g) = \lambda_1(g)$. Then we have the following useful Max-mini principle.

PROPOSITION 2.1. *For $g \in \mathscr{M}$, the k-th eigenvalue $\lambda_k(g)$ of the Laplacian*

Δ_g is given as follows: For every $(k+1)$-dimensional subspace L_{k+1} in $C^\infty(M)$, put

$$\Lambda_g(L_{k+1}) = \sup\{\|df\|_g^2/\|f\|_g^2;\ 0 \neq f \in L_{k+1}\} .$$

Then we have

$$\lambda_k(g) = \inf_{L_{k+1}} \Lambda_g(L_{k+1}) ,$$

where L_{k+1} varies over all $(k+1)$-dimensional subspaces of $C^\infty(M)$.

REMARK. The usual Mini-max principle is of the following type: For k-dimensional subspace L_k of $C^\infty(M)$, put

$$\tilde{\Lambda}_g(L_k) = \inf\{\|df\|_g^2/\|f\|_g^2 ;\ 0 \neq f \in C^\infty(M) \quad \text{and} \quad f \perp L_k\} ,$$

where $f \perp L_k$ means that f is orthogonal to each element in L_k with respect to the inner product $(\ ,\ )_g$. Then $\lambda_k(g)$ is given by

$$\lambda_k(g) = \sup_{L_k} \tilde{\Lambda}_g(L_k) .$$

Here L_k runs over all k-dimensional subspaces of $C^\infty(M)$. Notice that the orthogonality of f to L_k depends on the Riemannian metric g. So we can not use this Mini-max principle to prove Theorem 2.2.

PROOF OF PROPOSITION 2.1. For completeness, we give here a proof of Proposition 2.1. We take a complete orthonormal basis $\{u_k\}_{k=0}^\infty$ of $C^\infty(M)$ with respect to $(\ ,\ )_g$ so that each u_k is an eigenfunction of Δ_g with the eigenvalue $\lambda_k(g)$, $k = 0, 1, 2, \cdots$. Each $f \in C^\infty(M)$ can be expanded as $f = \sum_{i=0}^\infty x_i(f)u_i$, $x_i(f) \in \mathbf{R}$, in the sense of the uniform convergence or the L^2-convergence with respect to $(\ ,\ )_g$. In the following we omit the subscript g and simply denote $\Lambda(L_{k+1}) = \Lambda_g(L_{k+1})$, $\|\cdot\| = \|\cdot\|_g$, etc.

Let L_{k+1}° be the $(k+1)$-dimensional subspace of $C^\infty(M)$ generated by $\{u_i\}_{i=0}^k$. Then, since $\Lambda(L_{k+1}^\circ) = \lambda_k$, we have $\lambda_k \geqq \inf_{L_{k+1}} \Lambda(L_{k+1})$. Suppose that $\lambda_k > \inf_{L_{k+1}} \Lambda(L_{k+1})$. Then there exists a $(k+1)$-dimensional subspace L_{k+1} of $C^\infty(M)$ such that $\lambda_k > \Lambda(L_{k+1})$. Then by definition each $f \in L_{k+1}$ satisfies $\Lambda(L_{k+1}) \cdot \sum_{i=0}^\infty x_i(f)^2 \geqq \sum_{i=0}^\infty \lambda_i x_i(f)^2$. Thus we have

$$\sum_{\Lambda(L_{k+1}) \geqq \lambda_i} (\Lambda(L_{k+1}) - \lambda_i) x_i(f)^2 \geqq \sum_{\Lambda(L_{k+1}) < \lambda_i} (\lambda_i - \Lambda(L_{k+1})) x_i(f)^2 . \tag{2.7}$$

Now let $m = \max\{i;\ \lambda_i \leqq \Lambda(L_{k+1})\}$. Define a linear mapping Φ of L_{k+1} into $C^\infty(M)$ by

$$\Phi(f) = \sum_{i=0}^m x_i(f) u_i \quad \text{for} \quad f = \sum_{i=0}^\infty x_i(f) u_i \in L_{k+1} .$$

Then the dimension of the image of L_{k+1} under Φ is smaller than $k+1$. Indeed, for each $i = 0, \cdots, m$, the fact that $\lambda_i \leqq \Lambda(L_{k+1}) < \lambda_k$ implies that

$\dim \Phi(L_{k+1}) \leqq m+1 < k+1$. Therefore there exists a non-zero element f_0 in L_{k+1} such that $\Phi(f_0)=0$, that is, $x_i(f_0)=0$ for i with $\lambda_i \leqq \Lambda(L_{k+1})$. We apply (2.7) to this f_0 in L_{k+1}. If the left hand side of (2.7) is equal to zero, then each term on the right hand side is zero. Thus $x_i(f_0)=0$ for i with $\lambda_i > \Lambda(L_{k+1})$. Therefore we obtain $f_0 = \sum_{i=0}^{\infty} x_i(f_0)u_i = 0$, which is a contradiction. q.e.d.

2.3. *Proof of Theorem* 2.2. In this subsection, we show Theorem 2.2. For each positive number δ and $g \in \mathscr{M}$, we denote by $U_\delta(g)$ (resp. $V_\delta(g)$) the set $\{g' \in \mathscr{M};\ \rho(g', g) < \delta\}$ (resp. $\{g' \in \mathscr{M};\ \rho''(g', g) < \delta\}$). We note $U_\delta(g) \subset V_\delta(g)$.

THEOREM 2.2. *Let δ be a positive number and let g be in $\mathscr{M}$. Then*

(2.8) *$g' \in V_\delta(g)$ implies* $\exp(-(n+1)\delta) \leqq \lambda_k(g)/\lambda_k(g') \leqq \exp((n+1)\delta)$, *for each $k=1, 2, \cdots$. Thus*

(2.9) *$g' \in V_\delta(g)$ implies* $|\lambda_k(g') - \lambda_k(g)| \leqq (\exp((n+1)\delta)-1)\lambda_k(g)$, *for each $k=0, 1, 2, \cdots$.*

By Theorem 2.2, we have the following:

COROLLARY 2.3. *The multiplicity $m_k(g)$ of each eigenvalue $\lambda_k(g)$, that is, $m_k(g) = \#\{i;\ \lambda_i(g) = \lambda_k(g)\}$ depends upper semi-continuously on $g \in \mathscr{M}$: For each $g \in \mathscr{M}$ and $k=0, 1, 2, \cdots$, there exists a positive number δ such that*

$$g' \in V_\delta(g) \quad \textit{implies} \quad m_k(g') \leqq m_k(g)\,.$$

PROOF OF THEOREM 2.2. *Let* $(x_1, \cdots, x_n)$ be a local coordinate on an open set U of M. For each $\delta > 0$ and $g' \in V_\delta(g)$, the component matrices (g_{ij}), (g'_{ij}) of g, g' satisfy

$$(\exp(-\delta)g'_{ij}) < (g_{ij}) < (\exp(\delta)g'_{ij})$$

as symmetric matrices on U by the definition of the distance ρ''. Then we have

$$\exp((-n/2)\delta)(\det(g'_{ij}))^{1/2} < (\det(g_{ij}))^{1/2} < \exp((n/2)\delta)(\det(g'_{ij}))^{1/2}$$

and

$$(\exp(-\delta)g'^{ij}) < (g^{ij}) < (\exp(\delta)g'^{ij})\,.$$

Hence, for each $f \in C^\infty(M)$ and $\omega \in A^1(M)$ with support contained in U, we obtain

$$\exp((-n/2)\delta)\|f\|_{g'}^2 \leqq \|f\|_g^2 \leqq \exp((n/2)\delta)\|f\|_{g'}^2 \tag{2.10}$$

and

$$\exp\left(-\left(\frac{n}{2}+1\right)\delta\right)\|\omega\|_{g'}^2 \leqq \|\omega\|_g^2 \leqq \exp\left(\left(\frac{n}{2}+1\right)\delta\right)\|\omega\|_{g'}^2 \tag{2.11}$$

by the definitions of the inner products on $C^\infty(M)$ and $A^1(M)$ and by the above inequalities. Making use of the partition of unity, we have (2.10) and (2.11) for every $f \in C^\infty(M)$ and $\omega \in A^1(M)$. Thus we have

$$\exp(-(n+1)\delta)\|df\|_{g'}^2/\|f\|_{g'}^2 \leqq \|df\|_g^2/\|f\|_g^2 \leqq \exp((n+1)\delta)\|df\|_{g'}^2/\|f\|_{g'}^2$$

for every non-zero element f in $C^\infty(M)$. Therefore, by Proposition 2.1, we obtain

$$\exp(-(n+1)\delta)\lambda_k(g') \leqq \lambda_k(g) \leqq \exp((n+1)\delta)\lambda_k(g') . \qquad \text{q.e.d.}$$

REMARK. From the above proof, for each $g, g' \in \mathscr{M}$, if g' is close to g with respect to the C^0-topology, then the ratio $\lambda_k(g)/\lambda_k(g')$ is close to one for each $k = 1, 2, \cdots$. But notice that the coefficients of the first order terms of the Laplacians Δ_g and $\Delta_{g'}$ are not in general close to each other (cf. 2.1)).

3. Genericity of eigenvalues with multiplicity one. 3.1. *Uhlenbeck's theorem.* A subset S of a topological space X is residual if S is a countable intersection of open dense subsets of X. A topological space X is called a Baire space if any residual subset of X is dense in X. It is well known that a complete metric space (X, ρ) is a Baire space and a residual set in the complete metric space is a subset of the second category. Under these terminologies, we can state Uhlenbeck's theorem:

THEOREM 3.1 (cf. [U] and [T]). *Let M be a compact connected C^∞ manifold of dimension not less than two. Let $\mathscr{M}$ be the set of all C^∞ Riemannian metrics on M and ρ the complete distance on $\mathscr{M}$ as in §1. Let $\mathscr{S}$ be the set of all elements g in $\mathscr{M}$ all of whose eigenvalues of Δ_g have multiplicity one, that is,*

$$\mathscr{S} = \{g \in \mathscr{M};\ \lambda_0(g) < \lambda_1(g) < \lambda_2(g) < \cdots < \lambda_k(g) < \cdots\} .$$

Then $\mathscr{S}$ is a residual set in $(\mathscr{M}, \rho)$.

The proof of Theorem 3.1 can be carried out as follows: Let $\mathscr{S}_k$ be the set of all elements in $\mathscr{M}$ of which the first k eigenvalues have multiplicity one, that is,

$$\mathscr{S}_k = \{g \in \mathscr{M};\ \lambda_0(g) < \lambda_1(g) < \cdots < \lambda_{k-1}(g) < \lambda_k(g)\} ,$$

for each $k = 1, 2, \cdots$. Then we have

$$\mathscr{M} = \mathscr{S}_1 \supset \mathscr{S}_2 \supset \cdots \supset \mathscr{S}_k \supset \cdots \supset \mathscr{S} \quad \text{and} \quad \mathscr{S} = \bigcap_{k=1}^\infty \mathscr{S}_k .$$

Then it remains to prove the following two theorems.

THEOREM 3.2. *Each $\mathscr{S}_k$, $k = 1, 2, \cdots$, is open in $(\mathscr{M}, \rho)$.*

THEOREM 3.3. *Let M be a compact connected C^∞ manifold of dimension not less than two. Then each $\mathscr{S}_{k+1}$, $k = 1, 2, \cdots$, is dense in $\mathscr{S}_k$ with respect to the topology induced by $(\mathscr{M}, \rho)$.*

3.2. *The isometry group.* Before going into the proof of Theorems 3.2 and 3.3, we discuss the genericity of Riemannian metrics with trivial isometry group.

For $g \in \mathscr{M}$, we denote the eigenvalues of Δ_g by

$$0 < \lambda_1(g) = \cdots = \lambda_{j_1}(g) < \lambda_{j_1+1}(g) = \cdots = \lambda_{j_2}(g) < \cdots, \text{ etc .}$$

Put $\lambda_{j_0}(g) = \lambda_0(g) = 0$. Let V_k be the eigenspace of Δ_g with the eigenvalue $\lambda_{j_k}(g)$, $k = 0, 1, 2, \cdots$. Notice that $\dim V_k = j_k - j_{k-1}$. Let $\{u_i\}_{i=0}^\infty$ be a complete basis of $C^\infty(M)$ such that $\Delta_g u_i = \lambda_i(g) u_i$ and $(u_i, u_j)_g = \delta_{ij}$, $i, j = 0, 1, 2, \cdots$. Take a large integer r so that the mapping $\iota\colon M \ni x \mapsto \iota(x) = (u_0(x), u_1(x), \cdots, u_{i_{N-1}}(x)) \in \boldsymbol{R}^N$, $N = 1 + j_1 + \cdots + j_r$, is an embedding of M into $\boldsymbol{R}^N$. The Lie group G of all isometries of (M, g) acts on $C^\infty(M)$ by $\Phi^* u(x) = u(\Phi^{-1}(x))$, $x \in M$, $u \in C^\infty(M)$ and $\Phi \in G$. Then Φ^*, $\Phi \in G$, are linear mappings of $C^\infty(M)$ into itself and satisfy the conditions $(\Phi^* u, \Phi^* v)_g = (u, v)_g$ and $\Phi_1^* \circ \Phi_2^* = (\Phi_1 \circ \Phi_2)^*$ for $u, v \in C^\infty(M)$ and $\Phi, \Phi_1, \Phi_2 \in G$. Moreover, since $\Delta_g(\Phi^* u) = \Phi^*(\Delta_g u)$, we see that Φ^* maps each eigenspace V_k, $k = 0, 1, 2, \cdots, r$, into itself. Then we obtain a Lie group homomorphism ι^* of G into the orthogonal group $O(V)$ of the Euclidean space $(V, (\ ,\ )_g)$, $V = \sum_{k=0}^r V_k$, by $G \mapsto \Phi^* \in O(V)$. Note that the homomorphism ι^* is one to one since so is ι. Now, if $g \in \mathscr{S}$, then each V_k, $k = 0, 1, 2, \cdots$, is one dimensional. Thus the Lie subgroup $\iota^*(G)$ of $O(V)$ is discrete. Since ι^* is injective, G itself is discrete. Therefore we have:

PROPOSITION 3.4. *If $g \in \mathscr{S}$, that is, if all the eigenvalues of Δ_g have multiplicity one, then the group of all isometries of (M, g) is discrete.*

Combining this with Theorem 3.1, we have:

COROLLARY 3.5. *Let M be a compact connected C^∞ manifold of dimension not less than two. Let $\mathscr{M}$ be the set of all C^∞ Riemannian metrics on M and ρ the complete distance on $\mathscr{M}$ as in §1. Then the set of all elements g in $\mathscr{M}$ with discrete isometry group contains a residual subset of $\mathscr{M}$.*

REMARK. The above corollary was obtained in [E_1, Proposition 8.3, p. 35] in a different manner.

3.3. *Proof of Theorem* 3.2. Let g be an arbitrary element in $\mathscr{S}_k$, $k=0, 1, 2, \cdots$. We prove that there exists a positive number δ such that $V_\delta(g)$ is contained in $\mathscr{S}_k$. Let $\varepsilon = \min\{\lambda_{j+1}(g) - \lambda_j(g);\ j=0, 1, \cdots, k-1\} > 0$. We choose $\delta > 0$ so small that $\varepsilon(2\lambda_k(g))^{-1} > \exp((n+1)\delta) - 1$. Then, for $g' \in V_\delta(g)$ and $j = 0, 1, \cdots, k-1$, we have

$$\begin{aligned}\varepsilon &\leqq \lambda_{j+1}(g) - \lambda_j(g)\\ &\leqq |\lambda_{j+1}(g) - \lambda_{j+1}(g')| + |\lambda_{j+1}(g') - \lambda_j(g')| + |\lambda_j(g') - \lambda_j(g)|\\ &\leqq (\exp((n+1)\delta) - 1)(\lambda_{j+1}(g) + \lambda_j(g)) + |\lambda_{j+1}(g') - \lambda_j(g')|\\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Theorem 2.2)}\\ &\leqq 2\lambda_k(g)(\exp((n+1)\delta) - 1) + |\lambda_{j+1}(g') - \lambda_j(g')| .\end{aligned}$$

Thus we obtain

$$0 < \varepsilon - 2\lambda_k(g)(\exp((n+1)\delta) - 1) \leqq |\lambda_{j+1}(g') - \lambda_j(g')| ,$$

$j = 0, 1, \cdots, k-1$, which implies $g' \in \mathscr{S}_k$. We have $V_\delta(g) \subset \mathscr{S}_k$. q.e.d.

4. **Density of $\mathscr{S}_k$ in $\mathscr{M}$.** 4.1. *Preparations.* In this subsection, we prove some lemmas concerning a deformation $g(t)$ of g in $\mathscr{M}$. They will be used in the proof of Theorem 3.3.

LEMMA 4.1 (cf. [B, Lemma 3.15]). *For $g \in \mathscr{M}$ and $h \in S(M)$, let $g(t) = g + th \in \mathscr{M}$, $|t| < \varepsilon$. Let λ be an eigenvalue of Δ_g with multiplicity l. Then there exist $\Lambda_i(t) \in \boldsymbol{R}$* and *$u_i(t) \in C^\infty(M)$, $i = 1, \cdots, l$, such that*

(i) *$\Lambda_i(t)$ and $u_i(t)$ depend real analytically on t, $|t| < \varepsilon$, for each $i = 1, \cdots, l$,*

(ii) *$\Delta_{g(t)}u_i(t) = \Lambda_i(t)u_i(t)$, for each $i = 1, \cdots, l$ and t,*

(iii) *$\Lambda_i(0) = \lambda$, $i = 1, \cdots, l$, and*

(iv) *$\{u_i(t)\}_{i=1}^l$ is orthonormal with respect to $(\ ,\ )_{g(t)}$ for each t.*

For a proof, see [B, p. 137] and also Appendix.

REMARK. Lemma 4.1 does not necessarily imply Theorem 2.2, since the positive number ε may depend on $h \in S(M)$ in general.

LEMMA 4.2. *Let $g \in \mathscr{M}$ and let $a \in C^\infty(M)$ be a positive real valued function on M. Then the Laplacian Δ_{ag} corresponding to the Riemannian metric ag on M is given by*

$$\Delta_{ag} = a^{-1}\Delta_g + (1 - n/2)a^{-2}\nabla_g(a) ,$$

where $n = \dim M$ and $\nabla_g(a)$ is the gradient vector field of the function $a \in C^\infty(M)$ with respect to the Riemannian metric g.

PROOF. Making use of (2.1) and (2.2), we may prove this by a straightforward calculation.

LEMMA 4.3. *For every* $g \in \mathscr{M}$, *we have the following*:

(i) *For* σ, f_1 *and* $f_2 \in C^\infty(M)$, *we have*

$$(\nabla_g(\sigma)f_1, f_2)_g = (\sigma, \delta(f_2 df_1))_g ,$$

where $\delta; A^1(M) \to C^\infty(M)$ *is the codifferential operator with respect to* g.

(ii) $\delta(f_2 df_1) = -\langle df_1, df_2\rangle_g + f_2 \Delta_g f_1$, $f_1, f_2 \in C^\infty(M)$, *where* $\langle \cdot, \cdot\rangle_g$ *is the pointwise inner product in* $A^1(M)$ *relative to* g.

(iii) *Let* V_λ *be the eigenspace of* Δ_g *belonging to the eigenvalue* λ. *For every* u *and* v *in* V_λ, *we have* $\delta(udv) = \delta(vdu)$.

PROOF. (i) Since $\nabla_g(\sigma)f_1 = \langle d\sigma, df_1\rangle_g$, we have $(\nabla_g(\sigma)f_1, f_2)_g = (d\sigma, f_2 df_1)_g = (\sigma, \delta(f_2 df_1))_g$. (ii) For $\omega = \sum_{j=1}^n \omega_j dx_j \in A^1(M)$, $\delta\omega = -\sum_{i,j=1}^n g^{ij}\nabla_i \omega_j$, where $\nabla_i \omega_j$ is the covariant derivative with respect to g of the 1-form ω by the derivative $\partial/\partial x_i$ relative to the coordinate x_i, $i = 1, \cdots, n$. Then we have

$$\begin{aligned}\delta(f_2 df_1) &= -\sum_{i,j=1}^n g^{ij}\nabla_i(f_2 df_1)_j = -\sum_{i,j=1}^n g^{ij}(\partial f_2/\partial x_i)(\partial f_1/\partial x_j) - \sum_{i,j=1}^n g^{ij} f_2 \nabla_i (df_1)_j \\ &= -\langle df_1, df_2\rangle_g + f_2 \Delta_g f_1 .\end{aligned}$$

(iii) $\delta(udv) = -\langle du, dv\rangle_g + u\Delta_g v = -\langle du, dv\rangle_g + v\Delta_g u = \delta(vdu)$, for u, $v \in V_\lambda$. q.e.d.

4.2. *Splitting the eigenvalues.* In the following, we consider a deformation $g(t)$ of $g \in \mathscr{M}$ given by

$$g(t) = g + t\sigma g , \quad \text{for} \quad \sigma \in C^\infty(M) . \tag{4.1}$$

For small enough $\varepsilon(\sigma) > 0$, we have $g(t) \in \mathscr{M}$ for all t with $|t| < \varepsilon(\sigma)$.

Now let λ be a non-zero eigenvalue of Δ_g with multiplicity l and let $\{u_j\}_{j=1}^l$ be an orthonormal system with respect to $(\ ,\ )_g$ such that $\Delta_g u_j = \lambda u_j$, $j = 1, \cdots, l$. Applying Lemma 4.1 to $g(t)$, we obtain $\Lambda_j(t) \in \boldsymbol{R}$ and $u_j(t) \in C^\infty(M)$, $j = 1, \cdots, l$, satisfying the conditions (i)$\sim$(iv) in Lemma 4.1. By (i) in Lemma 4.1 (see also Theorem A.3 in Appendix), we can express $\Lambda_j(t)$ and $u_j(t)$, $j = 1, \cdots, l$, as follows:

$$\Lambda_j(t) = \lambda + t\alpha_j + t^2\beta_j(t) \quad \text{for} \quad |t| < \varepsilon(\sigma) , \tag{4.2}$$

where α_j is a real constant and $\beta_j(t)$ is a real analytic real valued function in t.

$$(u_j(t), v)_g \ \text{ are real analytic functions in } t,\ |t| < \varepsilon(\sigma) , \tag{4.3}$$

for every $v \in C^\infty(M)$. Then we have the following:

LEMMA 4.4. *Let* λ *be a non-zero eigenvalue of* Δ_g *with multiplicity* l *and let* $\{u_j\}_{j=1}^l$ *be an orthonormal system with respect to* $(\ ,\ )_g$ *such that*

$\Delta_g u_j = \lambda u_j$ *for each* $j=1, \cdots, l$. *For* $\sigma \in C^\infty(M)$, *let* $g(t)$ *be a deformation of* $g \in \mathscr{M}$ *given by* (4.1). *Let* $\{\alpha_j\}_{j=1}^l$ *be the real constants given by* (4.2). *Then we have*

$$(((1-n/2)\nabla_g(\sigma)-\lambda\sigma)u_j, u_i)_g = \alpha_j \delta_{ij}\ , \quad 1 \leqq i, j \leqq l\ .$$

PROOF. We apply Lemma 4.2 to $g(t)=a(t)g$ with $a(t)=1+t\sigma>0$ for $|t|<\varepsilon(\sigma)$. Then we have, for every $v \in C^\infty(M)$,

$$(a(t)\Delta_g u_j(t) + (1-n/2)t\nabla_g(\sigma)u_j(t) - \Lambda_j(t)a(t)^2 u_j(t), v)_g = 0\ ,$$

by $\Delta_{g(t)}u_j(t) = \Lambda_j(t)u_j(t)$, $j=1, \cdots, l$, $|t|<\varepsilon(\sigma)$. Differentiating both sides of the above equality at $t=0$, we obtain by (4.2) and (4.3)

$$((\Delta_g - \lambda)v_j + ((1-n/2)\nabla_g(\sigma) - \lambda\sigma - \alpha_j)u_j, v)_g = 0\ , \quad j=1, \cdots, l\ .$$

Thus, for an eigenfunction v of Δ_g belonging to the eigenvalue λ, we have

$$\begin{aligned}(((1-n/2)\nabla_g(\sigma)-\lambda\sigma-\alpha_j)u_j, v)_g &= -((\Delta_g-\lambda)v_j, v)_g \\ &= -(v_j, (\Delta_g-\lambda)v)_g = 0\ . \qquad \text{q.e.d.}\end{aligned}$$

PROPOSITION 4.5. *Assume* $\dim M \geqq 2$. *In the situation of Lemma* 4.4, *there exists a function* σ *in* $C^\infty(M)$ *such that, at least two of* $\{\alpha_i\}_{i=1}^l$ *in* (4.2) *are distinct.*

PROOF. Let P be the orthogonal projection of $C^\infty(M)$ onto the eigenspace V_λ belonging to the eigenvalue λ of Δ_g. For $\sigma \in C^\infty(M)$, define an endomorphism G_σ of V_λ into itself by

$$G_\sigma f = P\circ((1-n/2)\nabla_g(\sigma)-\lambda\sigma)f\ , \quad f \in V_\lambda\ .$$

Let $\{u_i\}_{i=1}^l$ be an arbitrary fixed orthonormal basis of V_λ with respect to $(\ ,\ )_g$. Then we have

$$(G_\sigma u_j, u_i)_g = (((1-n/2)\nabla_g(\sigma)-\lambda\sigma)u_j, u_i)_g = \alpha_j\delta_{ij}\ ,$$

by Lemma 4.4. Thus the endomorphism G_σ can be expressed as a diagonal matrix with respect to $\{u_i\}_{i=1}^l$ whose diagonal entries are α_i, $i=1, \cdots, l$.

Assume that $\alpha_1 = \cdots = \alpha_l$. Then G_σ can be expressed as a constant multiple of the identity matrix with respect to this basis and hence with respect to any basis of V_λ. Therefore, in order to prove Proposition 4.5, we have only to find $\sigma \in C^\infty(M)$ so that $(G_\sigma u_1, u_2)_g \neq 0$.

For $\sigma \in C^\infty(M)$, we have

$$\begin{aligned}(G_\sigma u_1, u_2)_g &= (((1-n/2)\nabla_g(\sigma)-\lambda\sigma)u_1, u_2)_g \\ &= (\sigma, (1-n/2)\delta(u_2 du_1) - \lambda u_1 u_2)_g\ .\end{aligned}$$

Case 1. $(1-n/2)\delta(u_2du_1) - \lambda u_1u_2 \not\equiv 0$. In this case, putting $\sigma =$

$(1-n/2)\delta(u_2du_1)-\lambda u_1u_2$, we have $(G_\sigma u_1, u_2)_g \neq 0$.

Case 2. $(1-n/2)\delta(u_2du_1)-\lambda u_1u_2 \equiv 0$. In this case, we have

$$u_1u_2 \equiv 0 \,. \tag{4.4}$$

In fact, we have

$$\begin{aligned}((1-n/2)\Delta_g-2\lambda)(u_1u_2) &= (1-n/2)\delta d(u_1u_2)-2\lambda u_1u_2\\ &= (1-n/2)\delta(u_1du_2+u_2du_1)-2\lambda u_1u_2\\ &= ((1-n/2)\delta(u_1du_2)-\lambda u_1u_2)+((1-n/2)\delta(u_2du_1)-\lambda u_1u_2)\\ &= 0\,,\end{aligned}$$

by Lemma 4.3 (iii) and the assumption. Since $2-n<0$, if $u_1u_2 \not\equiv 0$, then Δ_g would have a negative eigenvalue, which is a contradiction. (4.4) is thus proved.

We take, as an orthonormal basis of V_λ with respect to $(\ ,\ )_g$,

$$f_1 = 2^{-1}(u_1+u_2)\,,\quad f_2 = 2^{-1}(u_1-u_2)\,,\quad f_3 = u_3, \cdots,\quad f_l = u_l\,.$$

Put $\sigma = (1-n/2)\delta(f_2df_1)-\lambda f_1f_2$. Then we have

$$(G_\sigma f_1, f_2)_g = \int_M \sigma^2 dv_g\,.$$

So we have only to prove $\sigma \not\equiv 0$. Otherwise, we have

$$\begin{aligned}0 \equiv 2\sigma &= (1-n/2)\delta((u_1-u_2)d(u_1+u_2))-\lambda(u_1+u_2)(u_1-u_2)\\ &= (1-n/2)(\delta(u_1du_1)-\delta(u_2du_2))-\lambda(u_1^2-u_2^2) \quad \text{(by Lemma 4.3)}\\ &= (4^{-1}(2-n)\delta d-\lambda)(u_1^2-u_2^2)\,.\end{aligned}$$

Thus, since $2-n \leqq 0$, we have $u_1^2-u_2^2 \equiv 0$. Therefore we obtain

$$0 = \int_M (u_1^2-u_2^2)^2 dv_g = \int_M (u_1^4-2u_1u_2+u_2^4)dv_g = \int_M (u_1^4+u_2^4)dv_g\,,$$

by (4.4), which is a contradiction. We thus obtain $\sigma \not\equiv 0$. q.e.d.

4.3. *Proof of Theorem* 3.3. Let $\dim M \geqq 2$. We show $\mathscr{S}_k$ is dense in $\mathscr{S}_{k+1}$. To prove this, we construct, for each $g \in \mathscr{S}_k$, an element g' in $\mathscr{S}_{k+1}$ which is arbitrarily close to g.

Let $g \in \mathscr{S}_k$, that is, $\lambda_0(g) < \lambda_1(g) < \cdots < \lambda_k(g)$. Assume that the k-th eigenvalue $\lambda_k(g)$ has multiplicity l, i.e.,

$$\begin{aligned}&\lambda_k(g) = \cdots = \lambda_{k+l-1}(g) = \lambda \quad \text{and}\\ &\lambda_0(g) < \lambda_1(g) < \cdots < \lambda_{k-1}(g) < \lambda < \lambda_{k+l}(g) \leqq \cdots\,.\end{aligned}$$

Consider a deformation $g(t) = g + t\sigma g \in \mathscr{M}$ of g, $|t| < \varepsilon(\sigma)$, of the type (4.1). Let $\Lambda_j(t)$, $j = 1, \cdots, l$, be such eigenvalues of $\Delta_{g(t)}$ as (4.2).

We apply Proposition 4.5 to the eigenvalue $\lambda = \lambda_k(g)$. Noting that

$$g' \in V_{1/2}(g) \quad \text{implies} \quad \exp(-(n+1)/2)\lambda_m(g) \leqq \lambda_m(g') \,, \quad m = 0, 1, 2, \cdots \,,$$

by (2.8), we may assume

$$\exp((n+1)/2)\cdot(2\lambda) \leqq \lambda_m(g) \quad \text{implies} \quad 2\lambda \leqq \lambda_m(g(t)) \,,$$

for each $m = 0, 1, 2, \cdots$, and $|t| < \varepsilon(\sigma)$. We apply Theorem 2.2 to a finite number of eigenvalues of Δ_g which are smaller than $\exp((n+1)/2)\cdot(2\lambda)$. Then there exists a positive number $\varepsilon'(\sigma) \leqq \varepsilon(\sigma)$ such that

$$\lambda_0(g(t)) < \lambda_1(g(t)) < \cdots < \lambda_{k-1}(g(t)) < \Lambda_j(t) < \lambda_{k+l}(g(t)) \leqq \cdots \,,$$

for each $|t| < \varepsilon'(\sigma)$ and $j = 1, \cdots, l$.

Now, by Proposition 4.5, we can choose $\sigma \in C^\infty(M)$ in such a way that, at least two of $\{\alpha_j\}_{j=1}^l$ in (4.2) are distinct. Let $\alpha_i \neq \alpha_j$, $1 \leqq i, j \leqq l$. For this $\sigma \in C^\infty(M)$, we may choose a positive number $\varepsilon''(\sigma) \leqq \varepsilon'(\sigma)$ in such a way that $\Lambda_i(t) \neq \Lambda_j(t)$ for all $0 < |t| < \varepsilon''(\sigma)$. Therefore all the first k eigenvalues of $\Delta_{g(t)}$, $|t| < \varepsilon''(\sigma)$, have multiplicity one and the k-th eigenvalue $\lambda_k(g(t))$ has multiplicity at most $l-1$. Repeating this process, we can choose $g' \in \mathscr{S}_{k+1}$ as close to g as one wants. q.e.d.

Appendix. In this appendix, we give a proof of Lemma 4.1. The proof given in [B] was based on Kato's perturbation theory [K, p. 375] (See also [RN, p. 373]). In its proof, it was claimed (cf. [B, p. 138]) that the family of the operators $\Delta_{G(\kappa)}$ is of type (A) in the sense of Kato (cf. [K, p. 375]) and $\Delta_{G(\kappa)}$ are self-adjoint. But if we choose the domain of $\Delta_{G(\kappa)}$ as the Sobolev space $H_2(M)$ for a fixed Riemannian metric γ on M, then $\Delta_{G(\kappa)}$ are not self-adjoint with respect to the inner product $(\ ,\ )_\gamma$ in $H_2(M)$. If we require the self-adjointness of $\Delta_{G(\kappa)}$, then we have to choose the inner product $(\ ,\ )_{G(\kappa)}$ on $H_2(M)$. Since the domains of $\Delta_{G(\kappa)}$ vary as Hilbert spaces, the family of $\Delta_{G(\kappa)}$ is not of type (A). Its proof should be modified accordingly.

First we list some notations. Throughout this appendix, let M be an n-dimensional compact connected C^∞ manifold without boundary. Let $C_C^\infty(M)$ be the space of all complex valued C^∞ functions on M. For a fixed Riemannian metric γ on M, let Δ_γ be its Laplacian and $(\ ,\ )_\gamma$ be the inner product on $C_C^\infty(M)$ defined by

$$(\phi, \psi)_\gamma = \int_M \phi(x)\overline{\psi(x)}dv_\gamma \,, \quad \phi, \psi \in C_C^\infty(M) \,,$$

where dv_γ is the canonical measure of (M, γ) (cf. [BGM, p. 10]). For every non-negative integer s, let $H_s(M)$ be the Sobolev space on M (cf. [G, p. 35]) which is the completion of $C_C^\infty(M)$ with respect to the following inner product $[\ ,\ ]_s$:

(A. 1) $$[\phi, \psi]_s = ((I + \Delta_T)^s\phi, \psi)_T\ , \quad \phi, \psi \in C_C^\infty(M)\ .$$

Here I is the identity operator and $(I + \Delta_T)^s$ is the s-ple iteration of the operator $I + \Delta_T$. Put $\|\phi\|_s = [\phi, \phi]_s^{1/2}$, $\phi \in H_s(M)$.

We define the notions of the real analytic families of vectors or bounded operators (cf. [K, p. 365]).

DEFINITION A. 1. Let X, Y be complex Banach spaces. Let D be a domain in $\boldsymbol{R}$. A family of vectors x_t, $t \in D$, in X is said to be *real analytic* if it can be expanded as a convergent power series, i.e., for an arbitrary fixed $t_0 \in D$, there exist elements x_α, $\alpha = 0, 1, 2, \cdots$, in X such that

$$x_t = \sum_{\alpha=0}^{\infty} x_\alpha (t - t_0)^\alpha\ , \quad \text{for every} \quad t \in D,\ |t - t_0| < \varepsilon\ ,$$

where the series converges in the sense of the strong topology of X (cf. [Y, p. 30]). A family of bounded operators A_t, $t \in D$, of X into Y is said to be *real analytic* if it can be expanded as a convergent power series of bounded operators, i.e., for an arbitrary fixed $t_0 \in D$, there exist bounded operators C_α, $\alpha = 0, 1, 2, \cdots$, of X into Y such that

$$A_t = \sum_{\alpha=0}^{\infty} C_\alpha (t - t_0)^\alpha\ , \quad \text{for every} \quad t \in D, |t - t_0| < \varepsilon\ ,$$

where the series converges in the uniform topology (cf. [Y, pp. 111–112]).

Then we have:

THEOREM A. 2. *Let D be a small bounded domain in $\boldsymbol{R}$ containing the origin* 0. *Let $s_1 > s_0$ be non-negative integers. Let A_t, $t \in D$, be a real analytic family of bounded operators of $H_{s_1}(M)$ into $H_{s_0}(M)$. Assume that*

(1) *each operator A_t, $t \in D$, is self-adjoint with the domain $H_{s_1}(M)$ contained in $H_{s_0}(M)$ with respect to the inner product $[\ ,\ ]_{s_0}$* (cf. [Y, p. 197]), *and*

(2) *A_0 is bounded below, i.e., there exists a positive constant C such that $[A_0(x), x]_{s_0} \geqq C[x, x]_{s_0}$ for all $x \in H_{s_1}(M)$.*

Let λ be an eigenvalue of the operator A_0. Then

(I) *the kernel of $A_0 - \lambda I$ is finite dimensional.*

(II) *Put $l = \dim \ker(A_0 - \lambda I)$. Then there exists a subdomain D' in D containing the origin and l real analytic families of vectors ϕ_t^i, $i = 1, \cdots, l$, in $H_{s_1}(M)$ and l real analytic real valued functions λ_t^i, $i = 1, \cdots, l$, in $t \in D'$ such that*

(3) $A_t\phi_t^i = \lambda_t^i\phi_t^i$, $i = 1, \cdots, l$, $t \in D'$,
(4) $[\phi_t^i, \phi_t^j]_{s_0} = \delta_{ij}$, $i, j = 1, \cdots, l$, $t \in D'$ *and*
(5) $\lambda_0^i = \lambda$, $i = 1, \cdots, l$.

The assertion (I) is well known since the bounded self-adjoint operator A_0 is bounded below. The similar assertion as (II) was stated in [RN, p. 376, Theorem], [K, p. 392, Theorem 3.9] and [R, p. 57, Theorem 1, p. 74, Theorem 3]. It can be proved by the similar way, so we omit its proof.

We apply Theorem A. 2 to prove Lemma 4.1. Let g_t, $|t| < \varepsilon$, be a one-parameter family of Riemannian metrics on M depending real analytically on the parameter t. In the following, we denote merely by Δ_t (resp. $(,)_t$) the Laplacian Δ_{g_t} (resp. the inner product $(,)_{g_t}$ on $C_C^\infty(M)$) of (M, g_t). Then we have:

THEOREM A. 3. *Let g_t, $|t| < \varepsilon$, be the one-parameter family of Riemannian metrics on M depending real analycally on the parameter t. For any eigenvalue λ of Δ_0 with multiplicity l, there exist l families of $\phi_t^i \in C_C^\infty(M)$, $i = 1, \cdots, l$, which are real analytic in $H_0(M)$, and l real analytic real valued functions λ_t^i, $i = 1, \cdots, l$, in t such that*
(6) $\Delta_t\phi_t^i = \lambda_t^i\phi_t^i$, $i = 1, \cdots, l$, *and* t,
(7) $(\phi_t^i, \phi_t^j)_t = \delta_{ij}$, $i, j = 1, \cdots, l$, *and* t, *and*
(8) $\lambda_0^i = \lambda$, $i = 1, \cdots, l$.

For the proof of Theorem A. 3, we need the following:

LEMMA A. 4. *Let L_t, $|t| < \varepsilon$, be differential operators of order m which can be expressed locally as*

$$L_t = \sum_{|\alpha| \leqq m} a_\alpha(t, x) D_x^\alpha .$$

Here $D_x^\alpha = \partial^{|\alpha|}/\partial(x_1)^{\alpha_1} \cdots \partial(x_n)^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for an n-tuple $\alpha = (\alpha_1, \cdots, \alpha_n)$ of non-negative integers, and $a_\alpha(t, x)$ is real analytic in t, $|t| < \varepsilon$, where x belongs to the local coordinate open subset. Then the family of bounded operators L_t of $H_m(M)$ into $H_0(M)$ is real analytic.

PROOF. By assumption, $a_\alpha(t, x)$ can be expressed as $a_\alpha(t, x) = \sum_{k=0}^\infty a_{\alpha,k}(x)t^k$, where $a_{\alpha,k}(x)$ satisfy the following inequalities:

$$|a_{\alpha,k}(x)| \leqq Cr^k \quad \text{for all} \quad \alpha\,, \quad |\alpha| \leqq m\,, \quad k = 0, 1, 2, \cdots\,, \quad \text{and} \quad x\,.$$

Here the positive constants C and r do not depend on α, k and x. Using the partition of unity, define differential operators L_k, $k = 0, 1, 2, \cdots$, of order m which can be expressed locally as $L_k = \sum_{|\alpha| \leqq m} a_{\alpha,k}(x)D_x^\alpha$. Since L_k satisfy the inequalities

$$\| L_k f \|_0 \leqq m^n C' r^k \| f \|_m , \quad f \in H_m(M) ,$$

for a certain constant C', they are bounded operators of $H_m(M)$ into $H_0(M)$ and the series $\sum_{k=0}^{\infty} L_k t^k$ converges to L_t in the uniform topology. q.e.d.

PROOF OF THEOREM A. 3. For a function f on M and for t with $|t| < \varepsilon$, put

$$(U_t f)(x) = (\det(g_{tij}(x))/\det(g_{0ij}(x)))^{1/4} f(x) , \quad x \in M ,$$

where g_{tij}, $|t|<\varepsilon$, are the components of g_t with respect to the local coordinate $(x_1, \cdots, x_n)$ around x. Then the operators U_t, $|t|<\varepsilon$, define a real analytic family of bounded operators of $H_s(M)$ into itself for every non-negative integer s. By definition they are isometries of the Hilbert space $(H_0(M), (\ , \ )_t)$ into the Hilbert space $(H_0(M), (\ , \ )_0)$. Since the Laplacian Δ_t, $|t| < \varepsilon$, are self-adjoint operators of $H_0(M)$ with respect to the inner product $(\ , \ )_t$, the operators $\tilde{\Delta}_t$ defined by the composition $U_t \circ \Delta_t \circ U_t^{-1}$ are self-adjoint with respect to the inner product $(\ , \ )_0$. Moreover by Lemma A. 4 the family of $A_t = \tilde{\Delta}_t + I$, $|t| < \varepsilon$, is a real analytic family of bounded operators of $H_2(M)$ into $H_0(M)$ and satisfies (1), (2) of Theorem A. 2. Therefore by Theorem A. 2 there exist l real analytic families of vectors $\tilde{\phi}_t^i$, $i = 1, \cdots, l$, in $H_2(M)$ and l real analytic real valued functions $1 + \lambda_t^i$, $i = 1, \cdots, l$, in t satisfying (3), (4) and (5). Then the vectors ϕ_t^i, $i = 1, \cdots, l$, in $H_0(M)$ defined by $\phi_t^i = U_t^{-1} \tilde{\phi}_t^i$ satisfy $\Delta_t \phi_t^i = \lambda_t^i \phi_t^i$ in the sense of distribution and the condition (7) in Theorem A. 3. By hypoellipticitiy of Δ_t (cf. [G, p. 30]), ϕ_t^i belong to $C_C^\infty(M)$ and satisfy (6). Theorem A. 3 is proved.

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