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ABSOLUTE CONVERGENCE OF FOURIER SERIES OF PERIODIC STOCHASTIC PROCESSES AND ITS APPLICATIONS

Dedicated to Professor Tamotsu Tsuchikura on his sixtieth birthday

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1. Introduction. Let $X(t, \omega)$, $t \in R^1$, be a complex-valued stochastic process on a given probability space $(\Omega, \mathscr{F}, P)$. Suppose that $X(t, \omega)$ is measurable $\mathscr{L} \times \mathscr{F}$ on $R^1 \times \Omega$, $\mathscr{L}$ being the class of Lebesgue measurable sets on R^1.

If

$$(1.1) \qquad E|X(t, \omega)|^r = \|X(t, \omega)\|_r^r < \infty, \ t \in (a, b)$$

for some $r \geqq 1$ and for some $-\infty < a < b < \infty$, and

$$(1.2) \qquad \int_a^b \|X(t, \omega)\|_r^r dt < \infty ,$$

then we write $X(t, \omega) \in L^r(a, b)$ and call $X(t, \omega)$ an L^r process on (a, b). In this case, $X(t, \omega)$ is of $L^r(a, b)$ as a function of t almost surely (a.s.). We mention that if $X(t, \omega) \in L^r(a, b)$ for every $-\infty < a < b < \infty$, then the subset with probability one of Ω on which (1.2) holds is taken independently of a and b.

If (1.1) holds for $r = 1$ for every $t \in R^1$ and if

$$(1.3) \qquad E|X(t + 2\pi, \omega) - X(t, \omega)| = 0 , \quad \text{for} \quad t \in R^1 ,$$

then we call $X(t, \omega)$ 2π-periodic. The class of 2π-periodic processes of $L^r(-\pi, \pi)$ is simply denoted by L_P^r.

A stochastic process $X(t, \omega)$ is of L_P^2, if and only if the correlation function $R(s, t) = EX(s, \omega)\overline{X(t, \omega)}$ is 2π-periodic for each variable s and t.

For a stochastic process of L_P^r for some $r \geqq 1$, we consider the Fourier series

$$(1.4) \qquad X(t, \omega) \sim \sum_{n=-\infty}^{\infty} C_n(\omega) e^{int} ,$$

where

$$(1.5) \qquad C_n(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(t, \omega) e^{-int} dt .$$

The main purpose of this paper is to study the almost sure convergence of the series

$$\sum_{n=-\infty}^{\infty} |n|^k |C_n(\omega)| \tag{1.6}$$

for $X(t, \omega) \in L_P^r$, for some nonnegative integer k.

A known argument usually used in the theory of absolute convergence of ordinary Fourier series (Bary [1, I. p. 153], Zygmund [16, I, p. 240] is adapted to our problem and actually the results we are going to give are mostly the analogues of what are well known in that field. However the author believes that they are of particular interest, because some of them are directly applied to get some theorems on sample continuity and differentiability of stochastic processes which seems to provide the simplest way of deriving them at least for periodic case or possibly some other results in the theory of sample properties of stochastic processes. As an example we give a result on quasianalytic class of processes.

2. Continuity modulus of stochastic processes. Let $X(t, \omega) \in L_P^r$, for some $r \geqq 1$. We define, for some integer $p \geqq 1$ and $\delta > 0$,

$$M_r^{(p)}(\delta) = M_r^{(p)}(\delta, X) = \sup_{|h| \leqq \delta, |t| \leqq \pi} \| \Delta_h^{(p)} X(t, \omega) \|_r \tag{2.1}$$

and

$$M_r^{*(p)}(\delta) = M_r^{*(p)}(\delta, X) = \sup_{|h| \leqq \delta} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \| \Delta_h^{(p)} X(t, \omega) \|_r^r dt \right)^{1/r}, \tag{2.2}$$

where $\Delta_h^{(p)} X(t, \omega)$ is the p-th difference with increment h of t, namely

$$\Delta_h^{(p)} X(t, \omega) = \sum_{\nu=0}^{p} (-1)^{p-\nu} \binom{p}{\nu} X(t + \nu h, \omega) . \tag{2.3}$$

$M_r^{(p)}(\delta)$ and $M_r^{*(p)}(\delta)$ are called respectively the continuity modulus and integrated continuity modulus of p-th order of $X(t, \omega) \in L_P^r$.

For $X(t, \omega) \in L_P^r$, $r \geqq 1$, the Fourier coefficient of $\Delta_h^{(p)} X(t, \omega)$ is

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \Delta_h^{(p)} X(t, \omega) e^{-int} dt & \\ &= \sum_{j=0}^{p} (-1)^j \binom{p}{j} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(t + jh, \omega) e^{-int} dt \\ &= \sum_{j=0}^{p} (-1)^j \binom{p}{j} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(t, \omega) e^{-int} dt \, e^{injh} \\ &= C_n(\omega)(1 - e^{inh})^p . \end{aligned} \tag{2.4}$$

Suppose now $1 < r \leqq 2 \leqq r'$, $r^{-1} + r'^{-1} = 1$. Then by the Hausdorff-Young inequality we have

$$\left[\sum_{n=-\infty}^{\infty} |C_n(\omega)|^{r'} |e^{inh} - 1|^{pr'}\right]^{r/r'} \leqq \frac{1}{2\pi}\int_{-\pi}^{\pi} |\Delta_h^{(p)} X(t, \omega)|^r dt . \tag{2.5}$$

Taking expectations of both sides, we have

$$E\left[\sum_{n=-\infty}^{\infty} |C_n(\omega)|^{r'} |e^{inh} - 1|^{pr'}\right]^{r/r'} \leqq \frac{1}{2\pi}\int_{-\pi}^{\pi} \|\Delta_h^{(p)} X(t, \omega)\|_r^r dt .$$

Using the Minkowski inequality, we see that the left hand side is not less than

$$\left\{\sum_{n=-\infty}^{\infty} [E|C_n(\omega)|^r |e^{inh} - 1|^{rp}]^{r'/r}\right\}^{r/r'}$$
$$\geqq \left[\sum_{n=-\infty}^{\infty} \|C_n(\omega)\|_r^{r'} |2\sin(nh/2)|^{pr'}\right]^{r/r'} .$$

Let $\delta = 1/n$ in (2.3). We then see

$$M_r^{*(p)}(1/n) \geqq n\int_0^{1/n}\left[\sum_{k \geqq 4n} (\|C_k(\omega)\|_r |2\sin(kh/2)|^p)^{r'}\right]^{1/r'} dh$$

which is, again because of the Minkowski inequality, not less than

$$n\left[\sum_{k \geqq 4n}\left(\int_0^{1/n} \|C_k(\omega)\|_r |2\sin(kh/2)|^p dh\right)^{r'}\right]^{1/r'} .$$

Since, for $|k| \geqq 4n$,

$$n\int_0^{1/n}\left|2\sin\frac{kh}{2}\right|^p dh = \frac{n}{|k|}\int_0^{|k|/n} |2\sin(u/2)|^p du \geqq 2^{p-1}/(p+1) ,$$

the last expression is not less than

$$C_p\left[\sum_{|k| \geqq 4n} \|C_k(\omega)\|_r^{r'}\right]^{1/r'} ,$$

where $C_p = 2^{p-1}/(p+1)$ is a constant depending only on p. Thus we have the following lemma.

LEMMA 2.1. *If $1 < r \leqq 2$ and $r^{-1} + r'^{-1} = 1$, then*

$$M_r^{*(p)}(1/n) \geqq C_p\left[\sum_{|k| \geqq 4n} \|C_k(\omega)\|_r^{r'}\right]^{1/r'} . \tag{2.6}$$

3. Absolute convergence of Fourier series of a periodic stochastic process.

Throughout what follows, $\phi(t)$ is supposed to be a nondecreasing continuous nonrandom function on $[0, 1]$ such that either

$$\phi(0) = 0 \quad \text{and} \quad \phi(t)/t \quad \text{is nonincreasing over} \quad (0, 1] \tag{3.1}$$

or

$$\phi(t) = 1\,, \quad \text{for} \quad t \in [0, 1]\,. \tag{3.2}$$

In this section, we consider the process $X(t, \omega) \in L_P^r$, where r is restricted to $1 < r \leqq 2$.

THEOREM 3.1. *Let $X(t, \omega) \in L_P^r$, $1 < r \leqq 2$, $r^{-1} + r'^{-1} = 1$. Let k be a given nonnegative integer. If there exists a positive integer p such that*

$$\sum_{n=1}^{\infty} n^{k-1/r'}[\phi(1/n)]^{-1} M_r^{*(p)}(1/n) < \infty\,, \tag{3.3}$$

then

$$\sum_{n=-\infty}^{\infty} |n|^k [\phi(1/n)]^{-1} |C_n(\omega)| < \infty\,, \quad a.s\,. \tag{3.4}$$

PROOF. It is sufficient to prove that

$$\sum_{n=9}^{\infty} n^k [\phi(1/n)]^{-1} E|C_n(\omega)| < \infty\,. \tag{3.5}$$

The same thing is true for the series for negative n, with $|n|$ in place of n except in the subscripts of $C_n(\omega)$. A standard argument gives us that for the left hand side S of (3.5), we have

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \sum_{j=2^{n+2}+1}^{2^{n+3}} j^k [\phi(1/j)]^{-1} E|C_j(\omega)| \\ &\leqq \sum_{n=1}^{\infty} 2^{(n+3)k} [\phi(1/2^{n+3})]^{-1} \sum_{j=2^{n+2}+1}^{2^{n+3}} \|C_j(\omega)\|_r \\ &\leqq \sum_{n=1}^{\infty} 2^{(n+3)k} [\phi(1/2^{n+3})]^{-1} 2^{(n+2)/r} \Big(\sum_{j=2^{n+2}+1}^{\infty} \|C_j(\omega)\|_r^{r'} \Big)^{1/r'} \\ &\leqq 2^k \sum_{n=1}^{\infty} 2^{(n+2)(k+1/r)} [\phi(1/2^{n+3})]^{-1} \Big[\sum_{j=2^{n+2}+1}^{\infty} \|C_j(\omega)\|_r^{r'} \Big]^{1/r'} \\ &= 2^{4k+3/r} \sum_{n=1}^{\infty} \Big(\sum_{m=2^n-1}^{2^n} 1 \Big) 2^{(n-1)(k-1/r')} [\phi(1/2^{n+3})]^{-1} \cdot \Big[\sum_{j=2^{n+2}+1}^{\infty} \|C_j(\omega)\|_r^{r'} \Big]^{1/r'} \cdot \end{aligned}$$

$$\leqq 2^{4k+3/r} \sum_{m=1}^{\infty} m^{k-1/r'} [\phi(1/(16m))]^{-1} \Big[\sum_{j=4m}^{\infty} \|C_j(\omega)\|_r^{r'} \Big]^{1/r'}\,. \tag{3.6}$$

From $\phi(1/m)/16 \leqq \phi(1/(16m))$ and Lemma 2.1, we have

$$S \leqq 2^{4k+7} C_p^{-1} \sum_{m=1}^{\infty} m^{k-1/r'} [\phi(1/m)]^{-1} M_r^{*(p)}(1/m) < \infty\,.$$

This proves (3.5) and the proof of the theorem is complete.

The following corollary is a special case of Theorem 3.1, which gives the analogues of well known theorems of Bernstein and Szasz (See Bary

[1, Chapter IX], Zygmund [16, I, p. 240]) when $\phi(t) = 1$, $k = 0$, $r = r' = 2$.

COROLLARY 3.1.

(i) *If*

$$\sum_{n=1}^{\infty} n^{k-1/r'} M_r^{*(k+1)}(1/n) < \infty , \tag{3.7}$$

or

(ii) *if*

$$M_r^{(k+1)}(1/n) = O(n^{-(k+\alpha)}) , \quad \textit{for} \quad 1 \geqq \alpha > 1/r , \tag{3.8}$$

then

$$\sum_{n=-\infty}^{\infty} |n|^k |C_n(\omega)| < \infty , \quad a.s. \tag{3.9}$$

It is noted that if $p \leqq k$ in the condition (3.3), it turns out to be meaningless and hence the case $p = k + 1$ will be critical in the sense that the condition for a larger $p(\geqq k + 1)$ will be weaker.

4. Bounded variation of a periodic stochastic process.

DEFINITION 4.1. *Let $X(t, \omega)$, $t \in R^1$, be of $L_P^r(r \geqq 1)$. If*

$$\sup_D \sum_{j=1}^{n} \| X(t_j, \omega) - X(t_{j-1}, \omega) \|_r = V_r < \infty , \tag{4.1}$$

where sup *is taken for all divisions*

$$D: -\pi \leqq t_0 < t_1 < \cdots < t_n \leqq \pi ,$$

then we say that $X(t, \omega)$ is of bounded variation in $L^r(\Omega)$ and write $X(t, \omega) \in BV^r$.

The following propositions are easy to show by ordinary arguments.

PROPOSITION 4.1. *If $X(t, \omega) \in BV^r (r \geqq 1)$, then*

$$\int_{-\pi}^{\pi} \| X(t + h, \omega) - X(t, \omega) \|_r dt \leqq 2h V_r . \tag{4.2}$$

PROPOSITION 4.2. *If $X(t, \omega) \in BV^r$, $(r \geqq 1)$, then*

$$\int_{-\pi}^{\pi} \| \Delta_h^{(p)} X(t, \omega) \|_r dt \leqq 2^p h V_r . \tag{4.3}$$

PROPOSITION 4.3. *For $X(t, \omega) \in BV^r$ $(r \geqq 1)$,*

$$\| C_n(\omega) \|_r \leqq (2n)^{-1} V_r . \tag{4.4}$$

THEOREM 4.1. *If $X(t, \omega) \in BV^r$, $(r > 1)$ and*

$$0 < \nu < 1 - 1/r \,, \tag{4.5}$$

then

$$C_n(\omega) = o(|n|^{-\nu}) \,, \quad a.s \,. \tag{4.6}$$

PROOF. For any $A > 0$,

$$P(|C_n(\omega)| > A|n|^{-\nu}) \leqq (A|n|^{-\nu})^{-r} \| C_n(\omega) \|_r^r$$

which is, from (4.4), not greater than $(2A)^{-r}|n|^{(\nu-1)r}$. Since $\sum |n|^{(\nu-1)r} < \infty$,

$$\sum P(|C_n(\omega)| > A|n|^{-\nu}) < \infty$$

from which, by the Borel-Cantelli lemma, (4.6) follows.

Now suppose $1 < r$, $r^{-1} + r'^{-1} = 1$.

$$\begin{aligned} M_r^{*(p)}(\delta) &= \sup_{|h|\leqq\delta} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \| \Delta_h^{(p)} X(t, \omega) \|_r \cdot \| \Delta_h^{(p)} X(t, \omega) \|_r^{r-1} dt \right]^{1/r} \\ &\leqq \sup_{|h|\leqq\delta} \| \Delta_h^{(p)} X(t, \omega) \|_r^{1/r'} \cdot \sup_{|h|\leqq\delta} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \| \Delta_h^{(p)} X(t, \omega) \|_r dt \right]^{1/r} \end{aligned}$$

which is, by (4.3), not greater than

$$[M_r^{(p)}(\delta)]^{1/r'} \cdot C_{p,r} \delta^{1/r} V_r^{1/r} \,, \tag{4.7}$$

where $C_{p,r} = 2^{p/r}(2\pi)^{-1/r}$.

From Theorem 3.1, we now have using (4.7) the following theorem.

THEOREM 4.2. *If $X(t, \omega) \in BV^r$, $1 < r \leqq 2$, $1/r + 1/r' = 1$ and there is a positive integer p such that*

$$\sum_{n=1}^{\infty} n^{k-1} \left[\phi\left(\frac{1}{n}\right) \right]^{-1} \left[M_r^{(p)}\left(\frac{1}{n}\right) \right]^{1/r'} < \infty \,, \tag{4.8}$$

for a given nonnegative integer k, then (3.4) *holds.*

COROLLARY 4.1. *If $X(t, \omega) \in BV^r$, $(1 < r \leqq 2)$ and for some positive integer p,*

$$M_r^{(p)}(\delta) = O(\delta^{\beta}) \quad \textit{for some} \quad \beta > 0 \,, \tag{4.9}$$

then

$$\sum_{n=-\infty}^{\infty} |C_n(\omega)| < \infty \,, \quad a.s \,.$$

This immediately follows from Theorem 4.2 with $k = 0$, $\phi(t) = 1$, and is the analogue of the Zygmund theorem on absolute convergence of Fourier series.

5. Trigonometric approximations. First we suppose $X(t, \omega) \in L_P^2$. Then we can easily show that for every positive integer N,

$$\inf \frac{1}{2\pi} \int_{-\pi}^{\pi} \| X(t, \omega) - \sum_{n=-N}^{N} a_n(\omega) e^{int} \|^2 dt \tag{5.1}$$

is attained by $a_n(\omega) = C_n(\omega)$, a.s., where $C_n(\omega)$ is the Fourier coefficient of $X(t, \omega)$ as before and the inf is taken over all random variables $a_n(\omega) \in L^2(\Omega)$. Writing the quantity (5.1) by $[e_N^{(2)}(X)]^2 = [e_N^{(2)}]^2$, we have

$$e_N^{(2)} = \left[\sum_{|n| \geqq N} \| C_n(\omega) \|^2 \right]^{1/2} . \tag{5.2}$$

Writing the series in (3.5) by S as in the proof of Theorem 3.1, with $r = 2$ we see from (3.3) that

$$S \leqq C_{k,p} \sum_{m=1}^{\infty} m^{k-1/2} [\phi(1/m)]^{-1} e_m^{(2)} ,$$

where $C_{k,p}$ is a constant depending only on k and p. From this we have the following theorem which is seemingly more general than Theorem 3.1 with $r = 2$.

THEOREM 5.1. *If $X(t, \omega) \in L_P^2$ and*

$$\sum_{n=1}^{\infty} n^{k-1/2} [\phi(1/n)]^{-1} e_n^{(2)} < \infty , \tag{5.3}$$

then

$$\sum_{n=-\infty}^{\infty} |n|^k [\phi(1/n)]^{-1} |C_n(\omega)| < \infty , \quad a.s .$$

More generally write, for $X(t, \omega) \in L_P^r$, $(r \geqq 1)$,

$$e_N^{(r)} = e_N^{(r)}(X) = \inf \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \| X(t, \omega) - \sum_{n=-N}^{N} a_n(\omega) e^{int} \|_r^r dt \right]^{1/r} . \tag{5.4}$$

Write $\tau_N(t, \omega) = 2N\sigma_{2N-1}(t, \omega) - \sigma_{N-1}(t, \omega)$, where $\sigma_n(t, \omega)$ is the $(C, 1)$ mean of the Fourier series of $X(t, \omega)$. A slight modification of arguments in Zygmund [16, I, p. 115] gives us

$$\left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \| X(t, \omega) - \tau_N(t, \omega) \|_r^r dt \right]^{1/r} \leqq 2^{r+1} e_N^{(r)} . \tag{5.5}$$

Using this, by arguments similar to those used there, we also have

LEMMA 5.1. *Let $r \geqq 1$.*

$$e_{2N-1}^{(r)} \leqq C_r M_r^{*(2)}(2\pi/N) . \tag{5.6}$$

We also note that if $X(t, \omega) \in BV^1$,

$$e_N^{(1)} = O(1/N) . \tag{5.7}$$

This follows from Proposition 4.1 and Lemma 5.1.

We also note that if $X(t, \omega) \in L_P^s$, $1 \leqq r \leqq s$, then

$$\sum_{n=1}^{\infty} n^{-1/s'} e_n^{(s)} \leqq C_{r,s} \sum_{n=1}^{\infty} n^{-1/r'} e_n^{(r)} , \tag{5.8}$$

where $1/r + 1/r' = 1$, $1/s + 1/s' = 1$ and $C_{r,s}$ is a constant depending only on r and s.

This was shown for the nonrandom case in a more general framework by Watari-Okuyama [15]. Passage to our case is immediate. We also mention that the analogue of Theorem 6.35 of Zygmund [16, I, p. 154] to our case holds.

Using (5.8) with $s = 2$, we have, from Theorem 5.1 the following

THEOREM 5.2. *If* $1 \leqq r \leqq 2$, $1/r + 1/r' = 1$ *and*

$$\sum_{n=1}^{\infty} n^{-1/r'} e_n^{(r)} < \infty , \tag{5.9}$$

then

$$\sum_{n=-\infty}^{\infty} |C_n(\omega)| < \infty , \quad a.s .$$

Looking at Theorem 3.1 and Lemma 5.1, the above theorem is apparently a sharpening of Theorem 3.1 with $k = 0$. However it is, in view of (5.8), a consequence of Theorem 5.1 with $k = 0$ and $\phi = 1$ and the proof of it is contained in the proof of Theorem 3.1 so that we may say that Theorems 3.1, 5.1 and 5.2 are equivalent in substance. A similar remark was made by Watari-Okuyama [15] for the case of ordinary Fourier series.

6. Sample continuity and differentiability of stochastic processes of L_P^r. Let $X(t, \omega) \in L_P^r$ $(r > 1)$ and let $\sigma_n(t, \omega)$ be the n-th $(C, 1)$ mean of the Fourier series of $X(t, \omega)$. We begin with following lemma.

LEMMA 6.1. *If $X(t, \omega)$ is stochastically continuous, then $\sigma(t, \omega)$ converges in probability for every t.*

PROOF. We take any positive number $\varepsilon < 1/4$. We see that

$$\sigma_n(t, \omega) - X(t, \omega) = \int_{-\pi}^{\pi} \Delta_u X(t, \omega) K_n(u) du ,$$

where $K_n(u)$ is the Fejer kernel $[2\pi(n+1)]^{-1} \sin^2 [(n+1)u/2]/\sin^2 (u/2)$. Choose δ so that

$$(6.1) \qquad P(|\Delta_u X(t, \omega)| > \varepsilon^r) < \varepsilon^r \eta ,$$

for $|u| \leqq \delta$, where $\eta > 0$ is arbitrary. We then have

$$\begin{aligned} P(|\sigma_n(t, \omega) - X(t, \omega)| > \varepsilon) &\leqq P\Big(\Big|\int_{|u|\leqq\delta} \Delta_u X(t, \omega) K_n(u) du\Big| > \varepsilon/2) \\ &\quad + P\Big(\Big|\int_{|u|>\delta} \Delta_u X(t, \omega) K_n(u) du\Big| > \varepsilon/2\Big) \\ &= I_1 + I_2 , \end{aligned}$$

say. By the Chebyshev-Markov inequality ,

$$\begin{aligned} I_2 &\leqq (2/\varepsilon)^r \Big\|\int_{|u|>\delta} \Delta_u X(t, \omega) K_n(u) du\Big\|_r^r \\ &\leqq (2/\varepsilon)^r \Big[\int_{|u|>\delta} \|\Delta_u X(t, \omega)\|_r K_n(u) du\Big]^r . \end{aligned}$$

Since $\|\Delta_u X(t, \omega)\|_r$ is bounded and $K_n(u) \leqq [2\pi(n+1)\sin^2 \delta/2]^{-1}$, the last member converges to zero as $n \to \infty$.

Define $G_t(u, \omega)$ by

$$\begin{aligned} G_t(u, \omega) &= 1 , \quad \text{for} \quad |\Delta_u X(t, \omega)| > \varepsilon^r , \\ &= 0 , \quad \text{for} \quad |\Delta_u X(t, \omega)| \leqq \varepsilon^r . \end{aligned}$$

Write

$$\begin{aligned} I_1 = P\Big(\Big|\int_{-\delta}^{\delta} G_t(u, \omega) \Delta_u X(t, \omega) K_n(u) du\Big| > \varepsilon/4\Big) \\ + P\Big(\Big|\int_{-\delta}^{\delta} (1 - G_t(u, \omega)) \Delta_u X(t, \omega) K_n(u) du\Big| > \varepsilon/4\Big) . \end{aligned}$$

The second term on the right hand side is zero, since the integral is seen to be less than $\varepsilon^{r+1} \leqq \varepsilon/4$.

We now apply the Chebyshev-Markov inequality to have

$$\begin{aligned} I_1 &\leqq (4/\varepsilon)^r \Big\|\int_{-\delta}^{\delta} G_t(u, \omega) \Delta_u X(t, \omega) K_n(u) du\Big\|_r^r \\ &\leqq (4/\varepsilon)^r \Big(\int_{-\delta}^{\delta} \|G_t(u, \omega) \Delta_u X(t, \omega)\|_r K_n(u) du\Big)^r \\ &\leqq (4/\varepsilon)^r \Big(\int_{-\delta}^{\delta} \|G_t(u, \omega)\|_{r'}^{r'} K_n(u) du\Big)^{r/r'} \cdot \int_{-\delta}^{\delta} \|\Delta_u X(t, \omega)\|_r^r K_n(u) du . \end{aligned}$$

Since there is a constant $C = C(t)$ such that $\|\Delta_u X(t, \omega)\|^r < C$, the second factor is not greater than C. Note that $\|G_t(u, \omega)\|_r^r = E|G_t(u, \omega)| = P(|\Delta_u X(t, \omega)| > \varepsilon^r)$ which is less than $\varepsilon^r \eta$ by (6.1). We thus have

$$I_1 \leqq (4/\varepsilon)^r \varepsilon^r \eta \Big(\int_{-\delta}^{\delta} K_n(u) du \Big)^{r/r'} < 4^r C\eta \, .$$

Altogether we finally have

$$\limsup_{n\to\infty} P(|\sigma_n(t, \omega) - X(t, \omega)| > \varepsilon) < 4^r C\eta \, .$$

This proves the lemma.

LEMMA 6.2. *Let $\phi(t)$ be a function in* 3. *Then*

$$|\sin xh| \leqq \phi(h)/\phi(1/x) \, , \quad x \geqq 1 \, . \tag{6.2}$$

The proof is simple. (Kawata-Kubo [11])

Denote by Λ_ϕ, the Lipschitz class of functions f such that $\sup_{|h|\leqq\delta} |f(t+h) - f(t)| = O(\phi(\delta))$ for small δ, when $\phi(t)$ satisfies (3.1). When $\phi(t) = 1$, let us denote by Λ_ϕ the class of continuous functions.

Now we shall prove

THEOREM 6.1. (i) *Let $X(t, \omega) \in L_P^r$, $(1 < r \leqq 2)$. Let k be a given nonnegative integer. Suppose there exists a positive integer p such that* (3.3) *holds. If $X(t, \omega)$ is stochastically continuous, then there is a modification $X_0(t, \omega)$ of $X(t, \omega)$ with the property that $X_0(t, \omega)$ has almost surely the k-th derivative belonging to Λ_ϕ.* (ii) If $r = 2$, *the condition* (3.3) *can be replaced by* (5.3).

PROOF. From Theorem 3.1, (3.4) holds. The subset of Ω on which the series in (3.4) converges is denoted by Ω_1, $P(\Omega_1) = 1$. Now define, for $\omega \in \Omega_1$,

$$X_0(t, \omega) = \sum_{n=-\infty}^{\infty} C_n(\omega) e^{int} \, . \tag{6.3}$$

Then, for $\omega \in \Omega_1$,

$$X_0^{(k)}(t, \omega) = \sum_{n=-\infty}^{\infty} (in)^k C_n(\omega) e^{int} \tag{6.4}$$

and

$$|\Delta_h^{(1)} X_0^{(k)}(t, \omega)| = \Big| \sum_{n=-\infty}^{\infty} (in)^k C_n(\omega) \Delta_h^{(1)} e^{int} \Big| \leqq 2 \sum_{n=-\infty}^{\infty} |n|^k |C_n(\omega)| \, |\sin nh/2|$$

which is by Lemma 6.2 not greater than

$$\Big\{ 2 \sum_{n=-\infty}^{\infty} |n|^k |C_n(\omega)| [\phi(1/|n|)]^{-1} \Big\} \phi(h/2) = C(\omega)\phi(h/2) \leqq C(\omega)\phi(h) \, ,$$

where $C(\omega)$ is independent of h. Hence for $\omega \in \Omega_1$, $X_0(t, \omega)$ has the k-th derivative which belongs to Λ_ϕ.

Now since $X(t, \omega)$ is stochastically continuous for all t, the $(C, 1)$ mean $\sigma_n(t, \omega)$ of the Fourier series of $X(t, \omega)$, the right hand side of (6.3), converges in probability to $X(t, \omega)$. Therefore there is, for each t, a subsequence $\{n_k\}$ of subscripts ($n_k = n_k(t)$) and a set $\Omega_2(t)$ depending on each t with $P(\Omega_2(t)) = 1$, such that $\sigma_{n_k}(t, \omega) \to X(t, \omega)$, $k \to \infty$, for $\omega \in \Omega_2(t)$.

On the other hand, for $\omega \in \Omega_1$, (6.3) holds. Accordingly $\sigma_{n_k}(t, \omega) \to X_0(t, \omega)$ for $\omega \in \Omega_1$. Hence we should have

$$X(t, \omega) = X_0(t, \omega) \quad \text{for} \quad \omega \in \Omega_1 \cap \Omega_2(t) .$$

This means that $X_0(t, \omega)$ is a modification of $X(t, \omega)$ and completes the proof of (i). The proof of (ii) is carried out just in the same way.

COROLLARY. 6.1. *If $X(t, \omega)$ is of $L_P^r(1 < r \leqq 2)$ and stochastically continuous, and (3.7) or (3.8) holds, then there is a modification $X_0(t, \omega)$ of $X(t, \omega)$ which has almost surely the continuous k-th derivative.*

THEOREM 6.2. *If $X(t, \omega) \in BV^r(1 < r \leqq 2)$ and is stochastically continuous and (4.8) holds, then there is a modification $X_0(t, \omega)$ of $X(t, \omega)$ which has almost surely the k-th derivative belonging to Λ_ϕ.*

THEOREM 6.3. *If $X(t, \omega) \in L_P^r(1 \leqq r \leqq 2)$, is stochastically continuous and (5.9) holds, then there is a modification $X_0(t, \omega)$ of $X(t, \omega)$ which is continuous almost surely.*

We remark that the conditions (3.3) and (4.8) are respectively equivalent to

$$\int_0^1 y^{-k-1-1/r}[\phi(y)]^{-1}M_r^{*(p)}(y)dy < \infty , \tag{6.5}$$

and

$$\int_0^1 y^{-k-1}[\phi(y)]^{-1}M_r^{*(p)}(y)dy < \infty . \tag{6.6}$$

These conditions are of the forms mostly used in rather recent works on sample properties. The classical results on sample continuity are included in Cramér-Leadbetter [3]. Further results are found in Delporte [4], Garsia [5], Garsia-Romig-Rumsey, Jr. [6], Hahn [7], Hahn-Klass [8], Kôno [12], [13], [14]. Furthermore, Ciesielski [2] gave a result which is substantially similar to Theorem 6.1 and Kôno [14] has shown Theorem 6.1 as a generalization of Ciesielski's result and the author's previous result [10], even for more general nonperiodic case. Their argument basically depends on the approximation of the process by splines. On the other hand the proof of this paper is thought of as a simple application of the absolute convergence of Fourier series. We also note that the genralization of Theorem 6.1 to nonperiodic case by periodic continuation is possible although it is not quite obvious. See Kawata [10].

7. Quasianalytic class. Let $f(x)$ be a complex valued function on $[-\pi, \pi]$ which is indefinitely differentiable. The class of those functions satisfying

$$\sup_{-\pi \leqq x \leqq \pi} |f^{(n)}(x)| \leqq AK^n m_n , \tag{7.1}$$

for some sequence $\{m_n, n=0, 1, 2, \cdots\}$ of positive numbers, ∞ being allowed, where A and K are constants depending only on f, is denoted by $C(m_n)$. $C(m_n)$ is called a quasianalytic class, if $f(x) \in C(m_n)$ and $f^{(n)}(x_0) = 0$, $n = 0, 1, \cdots$ for some $x_0 \in (-\pi, \pi)$ implies that $f(x) = 0$ throughout $[-\pi, \pi]$. We consider also the class $C_2(l_n)$ of indefinitely differentiable functions $f(x)$ such that

$$\sup_{-\pi \leqq x \leqq \pi} |f^{(2n)}(x)| \leqq AK^n l_n , \quad n = 0, 1, 2, \cdots \tag{7.2}$$

for some sequence $\{l_n, n = 0, 1, \cdots\}$ of positive numbers. Writing $m_{2n} = l_n$, $m_{2n+1} = \infty$, $n = 0, 1, 2, \cdots$, $C_2(l_n) = C(m_n)$. When this class $C(m_n)$ is quasianalytic, $C_2(l_n)$ is called quasianalytic.

Let $X(t, \omega)$ be a periodic weakly stationary process, that is, $X(t, \omega) \in L_P^2$, $EX(t, \omega) = m$, a constant independent of t and the covariance function

$$E[X(s, \omega) - m][\overline{X(t, \omega)} - \bar{m}] = \rho(s - t) \tag{7.3}$$

is a function of $s - t$ alone. Ivanova [9] gave a result to the effect that if, for a weakly stationary process not necessarily periodic, $\rho(u)$ belongs to a quasianalytic class $C_2(l_n)$ on every finite interval, then $X(t, \omega)$ belongs to a quasianalytic class $C(l_n^{1/2})$ almost surely. The author, however, thinks that the proof of it was incomplete. We, in this section, formulate the result in a more exact form and give a complete proof for a periodic weakly stationary process.

THEOREM 7.1. *If $X(t, \omega)$ is a 2π-periodic weakly stationary process and its covariance function $\rho(u)$ belongs to a quasianalytic class $C_2(l_n)$, then there is a modification $X_0(t, \omega)$ of $X(t, \omega)$ with the property that $X_0(t, \omega)$ belongs to a quasianalytic class $C(l_{n+1}^{1/2})$.*

Before proving the theorem, we give some remarks. Suppose $EX(t, \omega) = 0$, throughout from now on. A 2π-periodic weakly stationary process can be represented by

$$X(t, \omega) = \sum_{n=-\infty}^{\infty} \xi_n(\omega) e^{int} , \tag{7.4}$$

where $\{\xi_n(\omega), n = 0, \pm 1, \pm 2, \cdots\}$ is an orthogonal sequence in $L^2(\Omega)$ with $\sum \|\xi_n(\omega)\|_2^2 < \infty$, and the series on the right hand side of (7.4) is $L^2(\Omega)$ convergent for each t.

Writing $\|\xi_n(\omega)\|_2^2 = a_n$, the covariance function $\rho(u)$ is given by

$$\rho(u) = \sum_{n=-\infty}^{\infty} a_n e^{inu} . \tag{7.5}$$

$a_n \geqq 0$. $\rho(u)$ is indefinitely differentiable if and only if

$$\sum_{n=-\infty}^{\infty} |n|^k a_n < \infty , \quad \text{for all} \quad k = 0, 1, 2, \cdots . \tag{7.6}$$

We can show that if $C_n(\omega)$ is the Fourier coefficient of $X(t, \omega)$, then

$$\xi_n(\omega) = C_n(\omega) , \quad \text{a.s.} , \tag{7.7}$$

for all n. This is easily shown from

$$\begin{aligned} &E|C_n(\omega) - \xi_n(\omega)|^2 \\ &\quad = E\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} X(t, \omega)e^{-int}dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=-N}^{N} \xi_k(\omega)e^{i(k-n)t}dt\right|^2 \end{aligned}$$

for $N \geqq |n|$, which is seen to converge to zero as $N \to \infty$.

We also mention the well known theorem of Carleman that the class $C(m_n)$ is quasianalytic if and only if

$$\int_1^{\infty} \frac{\log T(x)}{x^2} dx = \infty , \tag{7.8}$$

where

$$T(x) = \sup_{k \geqq 0} (x^k / m_k) , \quad x \geqq 1 . \tag{7.9}$$

We need two lemmas.

LEMMA 7.1. *$C_2(l_n)$ is quasianalytic if and only if*

$$\int_1^{\infty} \frac{\log T_1(x)}{x^2} dx = \infty , \tag{7.10}$$

where

$$T_1(x) = \sup_{k \geqq 0} (x^k / l_n^{1/2}) , \quad x \geqq 1 . \tag{7.11}$$

This was given in Ivanova [9].

LEMMA 7.2. *$C_2(l_n)$ is quasianalytic if and only if $C(l_{n+1}^{1/2})$ is quasi-*

analytic.

PROOF. We note that (7.10) is equivalent to

$$\int_c^\infty \frac{\log T_1(x)}{x^2}dx = \infty , \tag{7.12}$$

for some $c \geqq 1$, since $T_1(x)$ is a nondecreasing function. Now choose c to be greater than $\max(1, (l_1/l_0)^{1/2})$. Then for $x \geqq c$,

$$\begin{aligned} T_1(x) &= \sup_{k\geqq 0}(x^k/l_k^{1/2}) = \max\left(\sup_{k\geqq 1}(x^k/l_k^{1/2}),\ l_0^{-1/2}\right) \\ &= \sup_{k\geqq 1}(x^k/l_k^{1/2}) = x\bar{T}(x) , \end{aligned}$$

where

$$\bar{T}(x) = \sup_{k\geqq 0}(x^k/l_{k+1}^{1/2})$$

from which we readily see that (7.12) is equivalent to

$$\int_c^\infty \frac{\log \bar{T}(x)}{x^2}dx = \infty . \tag{7.13}$$

This shows Lemma 7.2 in view of Carleman's theorem.

We now turn to the proof of Theorem 7.1. From the assumption that $\rho(u) \in C_2(l_n)$,

$$|\rho^{(2n)}(0)| = \sum_{k=-\infty}^{\infty} k^{2n}a_k \leqq AK^n l_n , \quad n = 0, 1, 2, \cdots . \tag{7.14}$$

We may obviously suppose $K > 1$.

$$\sum_{|k|\geqq m} a_k \leqq m^{-n} \sum_{|k|\geqq m} |k|^n a_k = o(m^{-n})$$

for large m, for all n (o depends on n) and hence from (3.6) with $\phi = 1$, $r = r' = 2$, we have that there is a set $\Omega_1 \subset \Omega$ with $P(\Omega_1) = 1$, such that

$$\sum_{n=-\infty}^{\infty} |k|^n |\xi_k(\omega)| < \infty , \tag{7.15}$$

for each $n = 0, 1, 2, \cdots$ for $\omega \in \Omega_1$.

Define

$$X_0(t, \omega) = \sum_{k=-\infty}^{\infty} \xi_k(\omega)e^{ikt} .$$

Then we see that $X_0(t, \omega)$ is a modification of $X(t, \omega)$, because the right hand side series converges in $L^2(\Omega)$ to $X(t, \omega)$ for each t. $X_0(t, \omega)$ is, for $\omega \in \Omega_1$, infinitely many times differentiable and

$$X_0^{(n)}(t, \omega) = \sum_{k=-\infty}^{\infty} (ik)^n \xi_k(\omega) e^{ikt} .$$

Hence, for $\omega \in \Omega_1$,

$$\sup_{t \in [-\pi, \pi]} | X_0^{(n)}(t, \omega) | \leqq \sum_{k=-\infty}^{\infty} |k|^n |\xi_k(\omega)| .$$

We, therefore, see

$$P\Big(\sup_{t \in [-\pi, \pi]} | X_0^{(n)}(t, \omega) | \geqq AK^n l_{n+1}^{1/2} \Big) \leqq P\Big(\sum_{k=-\infty}^{\infty} |k|^n |\xi_k(\omega)| \geqq AK^n l_{n+1}^{1/2} \Big)$$

which is, because of the Chebyshev inequality, not greater than

$$\begin{aligned} &A^{-2} K^{-2n} l_{n+1}^{-1} E\Big[\sum_{k \neq 0} |k|^n |\xi_k(\omega)| \Big]^2 \\ &\leqq A^{-2} K^{-2n} l_{n+1}^{-1} \sum_{k \neq 0} |k|^{-2} \cdot \sum_{k \neq 0} |k|^{2(n+1)} E |\xi_k(\omega)|^2 \\ &= C A^{-2} K^{-2n} l_{n+1}^{-1} \sum_{k=0} |k|^{2(n+1)} a_k \leqq C A^{-1} K^{-n+1} , \end{aligned}$$

by (7.14), where C is an absolute constant. We therefore have

$$\sum_{n=0}^{\infty} P\Big(\sup_{t \in [-\pi, \pi]} | X_0^{(n)}(t, \omega) | \geqq AK^n l_{n+1}^{1/2} \Big) < \infty .$$

Hence by the Borel-Cantelli lemma, the event inside the brace takes place only finite times for ω of some $\Omega_2 (\in \Omega_1)$ with $P(\Omega_2) = 1$. Hence for $\omega \in \Omega_2$ there is an $n_0(\omega)$ such that

$$\sup_{t \in [-\pi, \pi]} | X_0^{(n)}(t, \omega) | \leqq AK^n l_{n+1}^{1/2} , \tag{7.16}$$

for $n \geqq n_0(\omega)$. Writing

$$K(\omega) = \max \Big\{ K, \sup_{n \leqq n_0(\omega)} \Big[\sup_{t \in [-\pi, \pi]} | X_0^{(n)}(t, \omega) | / (A l_{n+1}^{1/2}) \Big]^{1/n} \Big\} ,$$

we have that (7.16) with $K = K(\omega)$ holds for $\omega \in \Omega_2$ and for all n. This together with Lemma 7.2 completes the proof of Theorem 7.1.

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