

## CONVERGENCE OF POSITIVE LINEAR APPROXIMATION PROCESSES\*

Dedicated to Professor Tamotsu Tsuchikura on his sixtieth birthday

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**1. Introduction.** Let  $X$  be a compact Hausdorff space and let  $B(X)$  denote the Banach lattice of all real-valued bounded functions on  $X$  with the supremum norm  $\|\cdot\|$ .  $C(X)$  denotes the closed sublattice of  $B(X)$  consisting of all real-valued continuous functions on  $X$ . Let  $A$  be a linear subspace of  $B(X)$  and let  $\{T_{\alpha,\lambda}; \alpha \in D, \lambda \in A\}$  be a family of bounded linear operators of  $A$  into  $B(X)$ , where  $D$  is a directed set and  $A$  is an arbitrary index set. The family  $\{T_{\alpha,\lambda}\}$  is called an approximation process on  $A$  if for every  $f \in A$ ,

$$(1) \quad \lim_{\alpha} \|T_{\alpha,\lambda}(f) - f\| = 0 \quad \text{uniformly in } \lambda \in A$$

([23], cf. [21], [22]).

In this paper, we establish a theorem of Korovkin type with respect to the convergence behavior (1) for positive linear operators of  $C(X)$  into  $B(X)$  and give a quantitative version of this result under certain requirements.

Such problems are now classical for the usual convergence in  $C[a, b]$  with  $[a, b]$  being a finite closed interval of the real line  $R$ ; an excellent source for references and a systematic treatment of quantitative Korovkin theorems for positive linear operators in  $C[a, b]$  can be found in the book of DeVore [3]. Also, for the multi-dimensional case see Censor [2], and for an infinite dimensional case see the author [20].

Concerning the almost convergence ( $F$ -summability) introduced by Lorentz [12], in  $C[a, b]$  they were studied by King and Swetits [11] and by Mohapatra [17], whose results were recently extended by Swetits [27] to a general summability method considered by Bell [1] (cf. [15]), which includes  $F_A$ -summability of Lorentz [12],  $A_B$ -summability of Mazhar and Siddiqi [16] and order summability of Jurkat and Peyerimhoff [9, 10].

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In [23] a quantitative problem is discussed in the setting of compact convex subsets of a real locally convex Hausdorff vector space. Here this will be done in the setting of arbitrary compact metric spaces. Also, the direct theory of linear approximation processes of convolution operators and multiplier operators in an arbitrary Banach space setting is treated by the author [21] (cf. [22]).

The results obtained in this paper give an estimation of the rate of convergence of various summation processes of positive linear operators, which can be induced by the method of  $B$ -summability introduced by the author [22], which recovers that of Bell [1] (cf. [15]). Consequently, they extend results of the above authors and others to the setting of arbitrary compact metric spaces and more general summability methods, and yield a better estimation. Also, the most typical example of applications is given by the Bernstein-Lototsky-Schnabl functions on compact convex subsets of a real pre-Hilbert space (cf. [6], [23], [24]).

**2. A convergence theorem.** Throughout this paper, let  $\{T_{\alpha,\lambda}; \alpha \in D, \lambda \in A\}$  be a family of positive linear operators of  $C(X)$  into  $B(X)$  and  $1_x$  the unit function on  $X$  defined by  $1_x(x) = 1$  for all  $x \in X$ . Let  $T$  be a positive linear operator of  $C(X)$  into  $B(X)$  and  $\Psi$  a function in  $B(X^2)$ , where  $X^2 = X \times X$  denotes the product space of  $X$  and  $X$ , such that the function  $\Psi(\cdot, y)$  belongs to  $C(X)$  for each  $y \in X$ . Then we define

$$\mu(T, \Psi) = \sup \{ |T(\Psi(\cdot, y))(y)|; y \in X \}$$

and  $\mu_{\alpha,\lambda}(\Psi) = \mu(T_{\alpha,\lambda}, \Psi)$  for each  $\alpha \in D$  and  $\lambda \in A$ .

From now on let  $\Phi$  be a non-negative function in  $B(X^2)$  which satisfies the following properties:

- (2)  $\Phi(\cdot, y) \in C(X)$  for each  $y \in X$ ;
- (3)  $\inf \{ \Phi(x, y); (x, y) \in F \} > 0$  for every compact subset  $F$  of the complement of the diagonal set  $\Delta = \{(t, t); t \in X\}$  in  $X^2$ .

**REMARK 1.** If there exists a non-negative function  $G \in C(X^2)$  such that  $0 < G(x, y) \leq \Phi(x, y)$  for all  $(x, y) \in X^2$  with  $x \neq y$ , then (3) always holds. Hence, if  $\Phi$  is a non-negative function in  $C(X^2)$  satisfying  $\Phi(x, y) > 0$  for all  $(x, y) \in X^2$  with  $x \neq y$ , then (2) and (3) are fulfilled.

**LEMMA 1.** Let  $T$  be a positive linear operator of  $C(X)$  into  $B(X)$ . Then  $\mu(T, \Phi) = 0$  implies  $T(f) = fT(1_x)$  for all  $f \in C(X)$ . If, furthermore,  $\Phi(y, y) = 0$  for all  $y \in X$ , then the converse is also true.

**PROOF.** Let  $\Psi$  be a function in  $C(X^2)$  which vanishes in  $\Delta$  and let  $\varepsilon > 0$  be given. Then for each point  $(t, t) \in \Delta$ , there exists a neighbor-

hood  $V_i$  of  $(t, t)$  in  $X^2$  such that  $|\Psi(x, y)| < \varepsilon$  for all  $(x, y) \in V_i$ . Let  $F$  denote the complement of  $\bigcup \{V_i; t \in X\}$ . Then  $F$  is a compact subset of the complement of  $\Delta$ . Let

$$m = \inf \{\Phi(x, y); (x, y) \in F\} \quad \text{and} \quad M = \max \{|\Psi(x, y)|; (x, y) \in F\}.$$

By condition (3),  $m > 0$  and consequently, we obtain

$$|\Psi(x, y)| < \varepsilon + (M/m)\Phi(x, y)$$

for all  $(x, y) \in X^2$ . Thus, since  $T$  is positive and linear, it follows that

$$(4) \quad |T(\Psi(\cdot, y))(y)| \leq \varepsilon T(1_X)(y) + (M/m)T(\Phi(\cdot, y))(y)$$

for all  $y \in X$ . If  $\mu(T, \Phi) = 0$ , then (4) reduces to

$$|T(\Psi(\cdot, y))(y)| \leq \varepsilon T(1_X)(y),$$

which implies  $T(\Psi(\cdot, y))(y) = 0$ . Now let  $f \in C(X)$  and take  $\Psi(x, y) = f(x) - f(y)$ . Then for all  $y \in X$ , we have  $T(f - f(y)1_X)(y) = 0$ , which implies  $T(f) = fT(1_X)$ . Also, if  $\Phi(y, y) = 0$  for all  $y \in X$  and  $T(f) = fT(1_X)$  for every  $f \in C(X)$ , then  $T(\Phi(\cdot, y))(y) = \Phi(y, y)T(1_X)(y) = 0$ , and so  $\mu(T, \Phi) = 0$ . q.e.d.

LEMMA 2. *If there exists an element  $\alpha_0 \in D$  such that*

$$(5) \quad \sup \{ \|T_{\alpha, \lambda}(1_X)\|; \alpha \geq \alpha_0, \alpha \in D, \lambda \in \Lambda \} < \infty$$

and if

$$(6) \quad \lim_{\alpha} \mu_{\alpha, \lambda}(\Phi) = 0 \quad \text{uniformly in } \lambda \in \Lambda,$$

then for every  $\Psi \in C(X^2)$  satisfying  $\Psi(y, y) = 0$  for all  $y \in X$ ,

$$(7) \quad \lim_{\alpha} \mu_{\alpha, \lambda}(\Psi) = 0 \quad \text{uniformly in } \lambda \in \Lambda.$$

PROOF. Let  $\varepsilon > 0$  be given. Let  $m$  and  $M$  be as in the proof of Lemma 1. Putting  $T = T_{\alpha, \lambda}$  in (4), and taking the norm, we have

$$\mu_{\alpha, \lambda}(\Psi) \leq \varepsilon \|T_{\alpha, \lambda}(1_X)\| + (M/m)\mu_{\alpha, \lambda}(\Phi),$$

which together with (5) and (6) implies (7). q.e.d.

REMARK 2. If  $\Phi(y, y) = 0$  for all  $y \in X$  and

$$\lim_{\alpha} \|T_{\alpha, \lambda}(\Phi(\cdot, y)) - \Phi(\cdot, y)\| = 0 \quad \text{uniformly in } \lambda \in \Lambda \text{ and } y \in X,$$

then (6) holds.

THEOREM 1. *If (6) holds and if there exists a strictly positive function  $g \in C(X)$  such that*

$$(8) \quad \lim_{\alpha} \|T_{\alpha, \lambda}(g) - g\| = 0 \quad \text{uniformly in } \lambda \in A,$$

then  $\{T_{\alpha, \lambda}\}$  is an approximation process on  $C(X)$ .

PROOF. There exists a constant  $C > 0$  such that  $g(x) \geq C$  for all  $x \in X$ . Thus for all  $\alpha \in D$  and all  $\lambda \in A$ , we have

$$\|T_{\alpha, \lambda}(1_X)\| \leq (1/C)\|T_{\alpha, \lambda}(g)\|,$$

which together with (8) gives (5). Now let  $f \in C(X)$  and define the function  $\Psi$  on  $X^2$  by

$$\Psi(x, y) = f(x) - (f(y)/g(y))g(x).$$

Then  $\Psi$  belongs to  $C(X^2)$  and  $\Psi(y, y) = 0$  for all  $y \in X$ . Therefore by Lemma 2, (7) implies

$$(9) \quad \lim_{\alpha} \|T_{\alpha, \lambda}(f) - (f/g)T_{\alpha, \lambda}(g)\| = 0 \quad \text{uniformly in } \lambda \in A.$$

Also, for all  $\alpha \in D$  and all  $\lambda \in A$  we have

$$(10) \quad \|T_{\alpha, \lambda}(f) - f\| \leq \|f/g\| \|T_{\alpha, \lambda}(g) - g\| + \|T_{\alpha, \lambda}(f) - (f/g)T_{\alpha, \lambda}(g)\|,$$

which establishes the desired result by (8) and (9). q.e.d.

COROLLARY 1. Under the hypotheses of Remark 2,  $\{T_{\alpha, \lambda}\}$  is an approximation process on  $C(X)$ .

Indeed, (8) is satisfied with  $g = \Phi(\cdot, y_1) + \Phi(\cdot, y_2)$ , where  $y_1$  and  $y_2$  are two distinct points of  $X$ .

COROLLARY 2. If (6) holds and if

$$(11) \quad \lim_{\alpha} \|T_{\alpha, \lambda}(1_X) - 1_X\| = 0 \quad \text{uniformly in } \lambda \in A,$$

then  $\{T_{\alpha, \lambda}\}$  is an approximation process on  $C(X)$ .

In view of these results and the classical Korovkin theory on the convergence of positive linear operators, we make the following definitions:

DEFINITION 1. A subset  $S$  of  $C(X)$  is called a Korovkin test system (or, briefly, KTS) in  $C(X)$  if for any family  $\{L_{\alpha, \lambda}; \alpha \in D, \lambda \in A\}$  of positive linear operators of  $C(X)$  into  $B(X)$ , the relation

$$\lim_{\alpha} \|L_{\alpha, \lambda}(g) - g\| = 0 \quad \text{uniformly in } \lambda \in A$$

for every  $g \in S$  implies the relation

$$\lim_{\alpha} \|L_{\alpha, \lambda}(f) - f\| = 0 \quad \text{uniformly in } \lambda \in A$$

for every  $f \in C(X)$ .

DEFINITION 2. A finite subset  $\{f_1, f_2, \dots, f_m\}$  of  $C(X)$  is called an extended Korovkin test system (or, briefly, EKTS) in  $C(X)$  if there exists a subset  $\{a_1, a_2, \dots, a_m\}$  of  $B(X)$  such that for all  $x, y \in X$ ,

$$\Phi(x, y) = \sum_{i=1}^m a_i(y)f_i(x) \geq 0, \quad \Phi(y, y) = 0$$

and (3) is satisfied.

We shall now mention some examples of  $\Phi(x, y)$ ,  $(x, y) \in X^2$ .

(1°) Let

$$(12) \quad \Phi(x, y) = \sum_{i=1}^m a_i(y)f_i(x),$$

where  $a_i$  is real-valued function on  $X$  and  $f_i \in C(X)$ , such that  $\Phi$  is a non-negative function in  $B(X^2)$  satisfying (3). Note that (2) always holds. If (6) holds, then the fact that for  $i = 1, 2, \dots, m$ ,

$$\lim_{\alpha} \|T_{\alpha, \lambda}(f_i) - f_i\| = 0 \quad \text{uniformly in } \lambda \in A$$

implies that for every  $f \in C(X)$ ,

$$\lim_{\alpha} \|T_{\alpha, \lambda}(f) - f\| = 0 \quad \text{uniformly in } \lambda \in A.$$

In fact, (8) is satisfied with  $g = \Phi(\cdot, y_1) + \Phi(\cdot, y_2)$ , where  $y_1$  and  $y_2$  are two distinct points of  $X$ , and so the statement follows from Theorem 1. It may be remarked that this extends the result of Lorentz [14; Chap. 1, Theorem 1] on the usual convergence to the more general convergence behavior (1) in a weaker condition (cf. [14; Footnote on p. 7]). Also, by Corollary 1, if  $\{f_1, f_2, \dots, f_m\}$  is an EKTS in  $C(X)$ , then it becomes a KTS in  $C(X)$ . For example, if  $X = X_r$  is a compact subset of  $R^r$ , then the set

$$K_r = \{1_x, e_1, e_2, \dots, e_r, e_1^2 + e_2^2 + \dots + e_r^2\}$$

is an EKTS in  $C(X_r)$ , where  $e_i$  denotes the  $i$ -th coordinate function on  $X_r$ , i.e.,  $e_i(x_1, x_2, \dots, x_r) = x_i$ . Also, if  $X = Y_r$  is the  $r$ -dimensional torus, then the set

$$S_r = \{1_x, c_1, c_2, \dots, c_r, s_1, s_2, \dots, s_r\}$$

is an EKTS in  $C(Y_r)$ , where  $c_i(x_1, x_2, \dots, x_r) = \cos x_i$  and  $s_i(x_1, x_2, \dots, x_r) = \sin x_i$ . Consequently,  $K_r$  and  $S_r$  are Korovkin test systems in  $C(X_r)$  and in  $C(Y_r)$ , respectively. This is well-known for the usual convergence and the almost convergence due to Lorentz [12]; see, for instance, [2], [3], [11], [14], [17]. Furthermore, these results can be extended to the following more general situation: Let  $\{g_1, g_2, \dots, g_r\}$  be a finite subset

of  $C(X)$ , which separates the points of  $X$ . Then the set

$$K = \{1_x, g_1, g_2, \dots, g_r, g_1^2 + g_2^2 + \dots + g_r^2\}$$

is an EKTS in  $C(X)$ . Thus  $K$  is a KTS in  $C(X)$  (cf. [23]). Indeed, with the help of the function

$$\Phi(x, y) = \sum_{i=1}^r (g_i(x) - g_i(y))^2, \quad (x, y) \in X^2,$$

we see that  $K$  is an EKTS in  $C(X)$ .

(2°) Let  $(X, d)$  be a compact metric space. Let

$$\Phi(x, y) = u(d(x, y)),$$

where  $u$  is a function of  $[0, \infty)$  into itself such that  $\Phi$  is a function in  $B(X^2)$  satisfying (2) and (3). It may be remarked that if  $u$  is a strictly increasing continuous function on  $[0, \infty)$  with  $u(0) = 0$ , then (2) and (3) are automatically satisfied. For example, the case where  $u$  is defined by  $u(t) = t^p$ ,  $p > 0$ , may be important (cf. (3°), (4°)).

(3°) Let  $X$  be a compact subset of a normed linear space with norm  $\|\cdot\|$ . Let

$$\Phi(x, y) = \|x - y\|^p, \quad p > 0.$$

(4°) Let  $(H, \langle \cdot, \cdot \rangle)$  be a real pre-Hilbert space and  $X$  a compact subset of  $H$ . Let

$$\Phi(x, y) = \langle x - y, x - y \rangle.$$

We define the functions

$$e: X \rightarrow [0, \infty) \quad \text{and} \quad \tau_{\alpha, \lambda}: X \rightarrow R$$

by  $e(y) = \langle y, y \rangle$  and  $\tau_{\alpha, \lambda}(y) = T_{\alpha, \lambda}(\langle \cdot, y \rangle)(y)$ , respectively. Note that if (11) holds and if

$$\lim_{\alpha} \|T_{\alpha, \lambda}(e) - e\| = 0 \quad \text{uniformly in } \lambda \in A,$$

and

$$(13) \quad \lim_{\alpha} \|\tau_{\alpha, \lambda} - e\| = 0 \quad \text{uniformly in } \lambda \in A,$$

then (6) holds, and so by Corollary 2,  $\{T_{\alpha, \lambda}\}$  is an approximation process on  $C(X)$ . For example, one takes  $H = R^r$  with the usual inner product

$$\langle x, y \rangle = \sum_{i=1}^r x_i y_i, \quad x = (x_1, x_2, \dots, x_r), \quad y = (y_1, y_2, \dots, y_r).$$

Then  $e = e_1^2 + e_2^2 + \dots + e_r^2$ , and if, for  $i = 1, 2, \dots, r$ ,

$$\lim_{\alpha} \|T_{\alpha, \lambda}(e_i) - e_i\| = 0 \quad \text{uniformly in } \lambda \in A,$$

then (13) holds. Consequently,  $K_r$  becomes again a KTS in  $C(X)$ .

**REMARK 3.** The results obtained in this section can be reformulated with respect to pointwise convergence and the following localization principle holds: Let  $y \in X$ . Suppose that

$$\lim_{\alpha} \mu_{\alpha, \lambda}(\Phi; y) = 0 \quad \text{uniformly in } \lambda \in A,$$

where

$$\mu_{\alpha, \lambda}(\Phi; y) = T_{\alpha, \lambda}(\Phi(\cdot, y))(y).$$

If  $f \in C(X)$  vanishes in a neighborhood of  $y$ , then

$$\lim_{\alpha} T_{\alpha, \lambda}(f)(y) = 0 \quad \text{uniformly in } \lambda \in A.$$

**3. A quantitative theorem.** In this section, it will be assumed that  $X$  is a compact metric space with metric  $d(x, y)$ . We give here a quantitative version of Theorem 1 with the rate of convergence, using the modulus of continuity of approximating functions  $f$ , which can be defined as the function

$$\omega(f, \delta) = \sup \{ |f(x) - f(y)|; x, y \in X, d(x, y) \leq \delta \}.$$

For each  $f \in B(X)$ ,  $\omega(f, \cdot)$  is a non-decreasing function on  $[0, \infty)$  with  $\omega(f, 0) = 0$ , and  $f \in C(X)$  if and only if  $\lim_{\delta \rightarrow 0+} \omega(f, \delta) = 0$ . Also, for each  $\delta \geq 0$ ,  $\omega(\cdot, \delta)$  is a seminorm on  $B(X)$ .

In order to achieve our purpose it is always supposed that the following condition holds:

(14) There exists a constant  $\eta > 0$  such that  $\omega(f, \xi\delta) \leq (1 + \eta\xi)\omega(f, \delta)$  for all  $f \in B(X)$  and all  $\xi, \delta > 0$ .

The following lemma gives sufficient conditions such that (14) holds for  $\eta = 1$ , which can be more convenient for later applications.

**LEMMA 3.** *The following statements hold:*

(i) *Suppose that  $d$  is convex, i.e., it has the property that if  $d(x, y) = a + b$ , where  $a, b > 0$ , then there exists a point  $z \in X$  such that  $d(x, z) = a$  and  $d(z, y) = b$ . Then (14) holds for  $\eta = 1$ .*

(ii) *Let  $X$  be a compact convex subset of a metric linear space  $Y$  with metric  $d(x, y)$ . Suppose that  $d$  is invariant, i.e.,  $d(x + z, z + y) = d(x, y)$  for all  $x, y, z \in Y$ , and that the function  $d(\cdot, 0)$  is starshaped, i.e.,  $d(\beta x, 0) \leq \beta d(x, 0)$  for all  $x \in Y$  and all  $\beta$  with  $0 \leq \beta \leq 1$ . Then (14) holds for  $\eta = 1$ .*

**PROOF.** (i) is proved by Gonska [4; Satz 6.2] and the proof of (ii)

is similar. Indeed, for every natural number  $n$  we have  $\omega(f, n\delta) \leq n\omega(f, \delta)$ , from which (ii) follows. q.e.d.

It may be remarked that if  $X$  is as in Part (ii) of Lemma 3 with  $d$  being invariant and if  $d(\beta x, 0) = \beta d(x, 0)$  for all  $x \in Y$  and all  $\beta$  with  $0 < \beta < 1$ , then  $d$  is convex. Gonska [4] obtained a quantitative theorem of Korovkin type in the setting of the metric convexity (cf. [19], [25; Sec. 8.8]). Also, Jiménez Pozo [7] introduced the concept of a coefficient of convex deformation in a metric space. This concept gives a characterization of the metric convexity and yields the condition (14), and is used to obtain a generalization of quantitative theorems of Korovkin type (see also [8]).

From now on let  $\Phi$  be a non-negative function in  $B(X^2)$  which satisfies (2) and the following condition:

- (15) There exist constants  $q \geq 1$  and  $\kappa > 0$  such that for all  $(x, y) \in X^2$  with  $x \neq y$ ,  $d^q(x, y) \leq \kappa \Phi(x, y)$ , where  $d^q(x, y) = (d(x, y))^q$  on  $X^2$ .

Condition (15) already implies (3) (see, Remark 1), and so the results obtained in Section 2 hold.

**LEMMA 4.** *Let  $L$  be a positive linear functional on  $C(X)$ . Let  $y \in X$  and  $f \in C(X)$ . Then we have*

$$(16) \quad |L(f) - f(y)L(1_x)| \leq \omega(f, \delta)\{L(1_x) + \delta^{-q}(\eta\kappa)L(\Phi(\cdot, y))\}$$

for every  $\delta > 0$ .

**PROOF.** Let  $x$  be an arbitrary point of  $X$ . If  $d(x, y) > \delta$ , then it follows from (14) and (15) that

$$(17) \quad |f(x) - f(y)| \leq \omega(f, \delta)\{1 + \eta(d(x, y)/\delta)\} \leq \omega(f, \delta)\{1 + \eta(d^q(x, y)/\delta^q)\} \\ \leq \omega(f, \delta)\{1 + \delta^{-q}(\eta\kappa)\Phi(x, y)\}.$$

Obviously, (17) holds whenever  $d(x, y) \leq \delta$ , and consequently, we have

$$|f - f(y)1_x| \leq \omega(f, \delta)\{1_x + \delta^{-q}(\eta\kappa)\Phi(\cdot, y)\}.$$

Applying  $L$  to both sides of this inequality and using the positivity and the linearity of  $L$ , we obtain (16). q.e.d.

As an immediate consequence of Lemma 4, we have the following.

**LEMMA 5.** *Let  $T$  be a positive linear operator of  $C(X)$  into  $B(X)$ . Let  $y \in X$  and  $f \in C(X)$ . Then we have*

$$|T(f)(y) - f(y)T(1_x)(y)| \leq \omega(f, \delta)\{T(1_x)(y) + \delta^{-q}(\eta\kappa)T(\Phi(\cdot, y))(y)\},$$

for every  $\delta > 0$ .



From now on we also suppose that for each  $\alpha \in D$ ,

$$(18) \quad \sup \{ \| T_{\alpha, \lambda}(1_X) \|; \lambda \in A \} < \infty .$$

For each  $\alpha \in D$  and  $f \in C(X)$ , let

$$\| \| T_{\alpha}(f) - f \| \| = \sup \{ \| T_{\alpha, \lambda}(f) - f \|; \lambda \in A \} ,$$

which is finite by (18). Note that  $\{T_{\alpha, \lambda}\}$  is an approximation process on  $C(X)$  if and only if

$$\lim_{\alpha} \| \| T_{\alpha}(f) - f \| \| = 0 \quad \text{for all } f \in C(X) .$$

We are now in a position to recast Theorem 1 in a quantitative form as follows.

**THEOREM 2.** *Let  $g$  be a strictly positive function in  $C(X)$  and let  $\varepsilon > 0$ . Then for all  $\alpha \in D$  and all  $f \in C(X)$ , we have*

$$(19) \quad \| \| T_{\alpha}(f) - f \| \| \leq \| f/g \| \| \| T_{\alpha}(g) - g \| \| + \| f/g \| C_{\alpha}(\varepsilon, q) \omega(g, (\eta\kappa)^{1/q} \varepsilon \mu_{\alpha}(\Phi, q)) + C_{\alpha}(\varepsilon, q) \omega(f, (\eta\kappa)^{1/q} \varepsilon \mu_{\alpha}(\Phi, q)) ,$$

where

$$C_{\alpha}(\varepsilon, q) = \sup \{ \| T_{\alpha, \lambda}(1_X) + \varepsilon^{-q} 1_X \|; \lambda \in A \}$$

and

$$\mu_{\alpha}(\Phi, q) = (\sup \{ \mu_{\alpha, \lambda}(\Phi); \lambda \in A \})^{1/q} .$$

In particular, if  $T_{\alpha, \lambda}(1_X) = 1_X$  for all  $\alpha \in D$  and all  $\lambda \in A$ , then (19) reduces to

$$\| \| T_{\alpha}(f) - f \| \| \leq \| f/g \| \| \| T_{\alpha}(g) - g \| \| + \| f/g \| (1 + \varepsilon^{-q}) \omega(g, (\eta\kappa)^{1/q} \varepsilon \mu_{\alpha}(\Phi, q)) + (1 + \varepsilon^{-q}) \omega(f, (\eta\kappa)^{1/q} \varepsilon \mu_{\alpha}(\Phi, q)) .$$

**PROOF.** By (10), we have

$$(20) \quad \| \| T_{\alpha}(f) - f \| \| \leq \| f/g \| \| \| T_{\alpha}(g) - g \| \| + K_{\alpha}(f, g) ,$$

where

$$K_{\alpha}(f, g) = \sup \{ \| T_{\alpha, \lambda}(f) - (f/g) T_{\alpha, \lambda}(g) \|; \lambda \in A \} .$$

Since

$$f(x) - (f(y)/g(y))g(x) = f(x) - f(y) + (f(y)/g(y))(g(y) - g(x))$$

for all  $x, y \in X$ , we have

$$\begin{aligned} T_{\alpha, \lambda}(f)(y) - (f(y)/g(y))T_{\alpha, \lambda}(g)(y) \\ = T_{\alpha, \lambda}(f)(y) - f(y)T_{\alpha, \lambda}(1_X)(y) + (f(y)/g(y))(g(y)T_{\alpha, \lambda}(1_X)(y) - T_{\alpha, \lambda}(g)(y)) . \end{aligned}$$

Therefore, taking  $T = T_{\alpha,\lambda}$  in Lemma 5, we obtain

$$\begin{aligned} & |T_{\alpha,\lambda}(f)(y) - (f(y)/g(y))T_{\alpha,\lambda}(g)(y)| \\ & \leq |T_{\alpha,\lambda}(f)(y) - f(y)T_{\alpha,\lambda}(1_X)(y)| + |f(y)/g(y)| |T_{\alpha,\lambda}(g)(y) - g(y)T_{\alpha,\lambda}(1_X)(y)| \\ & \leq \omega(f, \delta)\{T_{\alpha,\lambda}(1_X)(y) + \delta^{-q}(\eta\kappa)T_{\alpha,\lambda}(\Phi(\cdot, y))(y)\} \\ & \quad + |f(y)/g(y)|\omega(g, \delta)\{T_{\alpha,\lambda}(1_X)(y) + \delta^{-q}(\eta\kappa)T_{\alpha,\lambda}(\Phi(\cdot, y))(y)\} \\ & \leq \omega(f, \delta)\{T_{\alpha,\lambda}(1_X)(y) + (\eta\kappa)(\mu_\alpha(\Phi, q)/\delta)^q\} \\ & \quad + \|f/g\|\omega(g, \delta)\{T_{\alpha,\lambda}(1_X)(y) + (\eta\kappa)(\mu_\alpha(\Phi, q)/\delta)^q\}. \end{aligned}$$

If  $\mu_\alpha(\Phi, q) > 0$ , then take  $\delta = (\eta\kappa)^{1/q}\varepsilon\mu_\alpha(\Phi, q)$  in this inequality. Then we have

$$\begin{aligned} & |T_{\alpha,\lambda}(f)(y) - (f(y)/g(y))T_{\alpha,\lambda}(g)(y)| \\ & \leq \omega(f, (\eta\kappa)^{1/q}\varepsilon\mu_\alpha(\Phi, q))\{T_{\alpha,\lambda}(1_X)(y) + \varepsilon^{-q}\} \\ & \quad + \|f/g\|\omega(g, (\eta\kappa)^{1/q}\varepsilon\mu_\alpha(\Phi, q))\{T_{\alpha,\lambda}(1_X)(y) + \varepsilon^{-q}\}, \end{aligned}$$

and so

$$\begin{aligned} \|T_{\alpha,\lambda}(f) - (f/g)T_{\alpha,\lambda}(g)\| & \leq \omega(f, (\eta\kappa)^{1/q}\varepsilon\mu_\alpha(\Phi, q))\|T_{\alpha,\lambda}(1_X) + \varepsilon^{-q}1_X\| \\ & \quad + \|f/g\|\omega(g, (\eta\kappa)^{1/q}\varepsilon\mu_\alpha(\Phi, q))\|T_{\alpha,\lambda}(1_X) + \varepsilon^{-q}1_X\|. \end{aligned}$$

Thus we conclude

$K_\alpha(f, g) \leq C_\alpha(\varepsilon, q)\omega(f, (\eta\kappa)^{1/q}\varepsilon\mu_\alpha(\Phi, q)) + \|f/g\|C_\alpha(\varepsilon, q)\omega(g, (\eta\kappa)^{1/q}\varepsilon\mu_\alpha(\Phi, q))$ , which together with (20) establishes (19). If  $\mu_\alpha(\Phi, q) = 0$ , then taking  $T = T_{\alpha,\lambda}$  in Lemma 1, we have that  $T_{\alpha,\lambda}(h) = hT_{\alpha,\lambda}(1_X)$  whenever  $h$  belongs to  $C(X)$ . Hence (10) reduces to

$$\|T_{\alpha,\lambda}(f) - f\| \leq \|f/g\|\|T_{\alpha,\lambda}(g) - g\|,$$

and so

$$\| \|T_\alpha(f) - f\| \| \leq \|f/g\| \| \|T_\alpha(g) - g\| \|.$$

This also implies (19). q.e.d.

**COROLLARY 3.** *Let  $\varepsilon > 0$ . Then for all  $\alpha \in D$  and all  $f \in C(X)$ , we have*

$$(21) \quad \| \|T_\alpha(f) - f\| \| \leq \|f\| \| \|T_\alpha(1_X) - 1_X\| \| + C_\alpha(\varepsilon, q)\omega(f, (\eta\kappa)^{1/q}\varepsilon\mu_\alpha(\Phi, q)).$$

*In particular, if  $T_{\alpha,\lambda}(1_X) = 1_X$  for all  $\alpha \in D$  and all  $\lambda \in \Lambda$ , then (21) reduces to*

$$\| \|T_\alpha(f) - f\| \| \leq (1 + \varepsilon^{-q})\omega(f, (\eta\kappa)^{1/q}\varepsilon\mu_\alpha(\Phi, q)).$$

**REMARK 4.** Theorem 2 and Corollary 3 are applicable to  $\Phi$  considered in Example (1°). In particular, for an EKTS  $\{f_1, f_2, \dots, f_m\}$  in  $C(X)$  with

$\Phi$  taking the form of (12) and satisfying (15), one can estimate the rate of convergence of  $||| T_\alpha(f) - f |||$ ,  $f \in C(X)$ , in terms of those of  $||| T_\alpha(f_i) - f_i |||$ ,  $i = 1, 2, \dots, m$ , since

$$(\mu_\alpha(\Phi, q))^q \leq \sum_{i=1}^m \| \alpha_i \| ||| T_\alpha(f_i) - f_i ||| .$$

In view of Examples (2°) and (3°), we have the following.

**THEOREM 3.** *Let  $g$  be a strictly positive function in  $C(X)$ . Let  $\varepsilon > 0$  and  $p \geq 1$ . Then the conclusion of Theorem 2 holds for  $\Phi = d^p$ ,  $q = p$  and  $\kappa = 1$ .*

**COROLLARY 4.** *Let  $\varepsilon > 0$  and  $p \geq 1$ . Then the conclusion of Corollary 3 holds for  $\Phi = d^p$ ,  $q = p$  and  $\kappa = 1$ .*

Concerning an estimation of the rate of convergence of  $\{T_{\alpha,\lambda}\}$  considered in Example (4°) one can assert:

**THEOREM 4.** *Let  $X$  be a compact convex subset of a real pre-Hilbert space with inner product  $\langle \cdot, \cdot \rangle$ . Let  $\varepsilon > 0$ . Then for all  $\alpha \in D$  and all  $f \in C(X)$ , we have*

$$(22) \quad ||| T_\alpha(f) - f ||| \leq \| f \| ||| T_\alpha(1_X) - 1_X ||| + C_\alpha(\varepsilon)\omega(f, \varepsilon\mu_\alpha) ,$$

where

$$C_\alpha(\varepsilon) = \sup \{ ||| T_{\alpha,\lambda}(1_X) + \varepsilon^{-2}1_X ||| ; \lambda \in A \}$$

and

$$\mu_\alpha = (\sup \{ \mu_{\alpha,\lambda}(d^2); \lambda \in A \})^{1/2}$$

with metric  $d(x, y) = \langle x - y, x - y \rangle^{1/2}$ . In particular, if  $T_{\alpha,\lambda}(1_X) = 1_X$  for all  $\alpha \in D$  and all  $\lambda \in A$ , then (22) reduces to

$$||| T_\alpha(f) - f ||| \leq (1 + \varepsilon^{-2})\omega(f, \varepsilon\mu_\alpha) .$$

**PROOF.** Taking  $p = 2$ , this follows from (ii) of Lemma 3 and Corollary 4. q.e.d.

**REMARK 5.** Let  $e$  and  $\tau_{\alpha,\lambda}$  be as in Example (4°). Then we have an estimation of  $\mu_\alpha$ :

$$(23) \quad \mu_\alpha^2 \leq \| e \| ||| T_\alpha(1_X) - 1_X ||| + ||| T_\alpha(e) - e ||| + 2\tau_\alpha(e) ,$$

where

$$\tau_\alpha(e) = \sup \{ ||| \tau_{\alpha,\lambda} - e ||| ; \lambda \in A \} .$$

In particular, if  $T_{\alpha,\lambda}(1_X) = 1_X$  and  $T_{\alpha,\lambda}(\langle \cdot, y \rangle) = \langle \cdot, y \rangle$  for all  $\alpha \in D$ ,  $\lambda \in A$  and all  $y \in X$ , then (23) reduces to

$$\mu_\alpha^2 = ||| T_\alpha(e) - e ||| .$$

4.  $\mathcal{A}$ -summation processes of positive linear operators. Let  $N$  denote the set of all non-negative integers. Let  $\mathcal{A} = \{A^{(\lambda)}; \lambda \in A\}$  be a family of infinite matrices  $A^{(\lambda)} = (a_{nm}^{(\lambda)})_{n,m \in N}$  of real numbers. A sequence  $\{L_m\}_{m \in N}$  of bounded linear operators of  $C(X)$  into  $B(X)$  is called an  $\mathcal{A}$ -summation process on  $C(X)$  if  $\{L_m(f)\}$  is  $\mathcal{A}$ -summable to  $f$  for every  $f \in C(X)$ , i.e.,

$$(24) \quad \lim_{n \rightarrow \infty} \left\| \sum_{m=0}^{\infty} a_{nm}^{(\lambda)} L_m(f) - f \right\| = 0 \quad \text{uniformly in } \lambda \in A,$$

where it is assumed that the series in (24) converges for each  $n, \lambda$  and  $f$  ([23], cf. [22]).

We shall now mention some examples of  $A^{(\lambda)} = (a_{nm}^{(\lambda)})_{n,m \in N}$ .

(5°) Given a matrix  $A$ , if  $A^{(\lambda)} = A$  for all  $\lambda \in A$ , then  $\mathcal{A}$ -summability is just the usual matrix summability by  $A$ .

(6°) Let  $Q = \{q^{(\lambda)}; \lambda \in A\}$  be a family of sequences  $q^{(\lambda)} = \{q_m^{(\lambda)}\}_{m \in N}$  of non-negative real numbers such that

$$Q_n^{(\lambda)} = q_0^{(\lambda)} + q_1^{(\lambda)} + \cdots + q_n^{(\lambda)} > 0$$

for all  $n \in N$  and all  $\lambda \in A$ . Let

$$\begin{aligned} a_{nm}^{(\lambda)} &= q_{n-m}^{(\lambda)} / Q_n^{(\lambda)} \quad \text{for } 0 \leq m \leq n \\ &= 0 \quad \text{for } m > n. \end{aligned}$$

Then  $\mathcal{A}$ -summability is called a  $(N, Q)$ -summability. Clearly, if for a sequence  $\{q_m\}_{m \in N}$  of non-negative real numbers with  $q_0 > 0$ , one takes  $q_m^{(\lambda)} = q_m$  for all  $m \in N$  and all  $\lambda \in A$ , then  $(N, Q)$ -summability reduces to the Nörlund summability. Also, a typical example of this type is the following: Let  $A$  be a subset of  $[0, \infty)$  and  $\beta > 0$ . Let  $q_m^{(\lambda)} = A_m^{(\lambda+\beta-1)}$ , where  $A_m^{(\tau)} = \binom{m+\tau}{m}$ ,  $\tau > -1$ . In particular, if  $A = \{0\}$ , then this method reduces to the Cesàro  $(C, \beta)$ -summability of order  $\beta$ .

(7°) Let  $A$  be subset of  $(0, \infty)$  and  $\beta > -1$ . Let

$$\begin{aligned} a_{nm}^{(\lambda)} &= A_{n-m}^{(\lambda-1)} A_m^{(\beta)} / A_n^{(\beta+\lambda)} \quad \text{for } 0 \leq m \leq n \\ &= 0 \quad \text{for } m > n. \end{aligned}$$

(8°) Let  $A$  be a subset of  $[0, 1]$ , and let

$$\begin{aligned} a_{nm}^{(\lambda)} &= \binom{n}{m} \lambda^m (1-\lambda)^{n-m} \quad \text{for } 0 \leq m \leq n \\ &= 0 \quad \text{for } m > n. \end{aligned}$$

(9°) Let  $A$  be a subset of  $[0, \infty)$ , and let

$$a_{nm}^{(\lambda)} = \exp(-n\lambda) (n\lambda)^m / m!.$$

(10°) Let  $A$  be a subset of  $[0, 1)$ , and let

$$a_{nm}^{(\lambda)} = \binom{n+m}{m} \lambda^m (1-\lambda)^{n+1}.$$

(11°) Let  $A$  be a subset of  $[0, \infty)$ , and let

$$a_{nm}^{(\lambda)} = \binom{n+m-1}{m} \lambda^m (1+\lambda)^{-n-m}.$$

(12°) If one takes  $A = N$ , then  $\mathcal{A}$ -summability reduces to that by Bell [1] (cf. [15]). This method includes  $F$ -summability (almost convergence method) and  $F_A$ -summability of Lorentz [12],  $A_B$ -summability of Mazhar and Siddiqi [16] and order summability of Jurkat and Peyerimhoff [9, 10].

It may be remarked that all the matrices given in Examples (6°)–(11°) satisfy that  $a_{nm}^{(\lambda)} \geq 0$  for all  $n, m, \lambda$ , and  $\sum_{m=0}^{\infty} a_{nm}^{(\lambda)} = 1$  for each  $n$  and  $\lambda$ . Also, concerning detailed statements for  $\mathcal{A}$ -summability methods in arbitrary Banach spaces one may consult [22; Sec. 4].

Let  $\mathcal{A} = \{(a_{nm}^{(\lambda)})_{n,m \in N}; \lambda \in A\}$  be a family of infinite matrices of non-negative real numbers and  $\{L_m\}_{m \in N}$  a sequence of positive linear operators of  $C(X)$  into  $B(X)$  such that for each  $n \in N$  and each  $\lambda \in A$ ,

$$(25) \quad \sum_{m=0}^{\infty} a_{nm}^{(\lambda)} \|L_m(1_X)\| < \infty.$$

Furthermore, for each  $n \in N, \lambda \in A$  and  $f \in C(X)$ , let

$$(26) \quad T_{n,\lambda}(f) = \sum_{m=0}^{\infty} a_{nm}^{(\lambda)} L_m(f),$$

which is well-defined by (25), and belongs to  $B(X)$ .

Consequently, under the above setting all the results obtained in the preceding sections are applicable to the family  $\{T_{n,\lambda}\}$ , with  $D = N$ . Thus our results extend the results of Censor [2], King and Swetits [11], Mohapatra [17], Mond [18] and Swetits [27] to the setting of arbitrary compact metric spaces and more general  $\mathcal{A}$ -summability methods. Moreover, the following example will show that our estimations can be sharper than theirs:

(13°) Let  $X = [a, b]$  be a finite closed interval in  $R$  with the usual metric  $d(x, y) = |x - y|$ , and let  $p \geq 1$ . Let  $\{L_n\}_{n \geq 1}$  be a sequence of positive linear operators of  $C(X)$  into  $B(X)$  such that  $L_n(1_X) = 1_X$  for all  $n \geq 1$  and

$$(27) \quad A_p = \sup \{n^p \mu(L_n, d^{2p}); n \geq 1\} < \infty.$$

Then it follows from Lemma 3 and Corollary 4 that for all  $n \geq 1$ , and all  $f \in C(X)$ ,

$$\begin{aligned} \|L_n(f) - f\| &\leq \inf \{(1 + \varepsilon^{-2p})\omega(f, \varepsilon(\mu(L_n, d^{2p}))^{1/2p}); \varepsilon > 0\} \\ &\leq \inf \{(1 + \varepsilon^{-2p})\omega(f, \varepsilon(A_p/n^p)^{1/2p}); \varepsilon > 0\} \\ &\leq (1 + A_p)\omega(f, n^{-1/2}). \end{aligned}$$

For example, take  $X = [0, 1]$ , and let  $L_n, n \geq 1$ , be the Bernstein operators on  $C(X)$ , i.e.,

$$L_n(f)(x) = \sum_{i=0}^n f(i/n) \binom{n}{i} x^i (1-x)^{n-i}.$$

Then  $L_n(1_X) = 1_X$  for all  $n \geq 1$ , and (27) is satisfied (see [13; p. 14 ff]). We have  $A_1 = 1/4$ , and so

$$(28) \quad \|L_n(f) - f\| \leq (5/4)\omega(f, n^{-1/2})$$

for all  $n \geq 1$  and all  $f \in C(X)$ . This is the well-known result of Lorentz [13; Theorem 1.6.1], which can also be an immediate consequence of Mond [18]. Also, we have  $A_2 = 3/16$ , and so (28) can be sharpened further as

$$\|L_n(f) - f\| \leq (19/16)\omega(f, n^{-1/2}).$$

It may be remarked that, by the result of Sikkema [26],

$$\inf \{A_p; p \geq 1\} \geq 0.0898873 \dots$$

The following result can be an immediate consequence of Theorem 4, and is more convenient for later applications.

**COROLLARY 5.** *Let  $X$  and  $\varepsilon$  be as in Theorem 4. Let  $\mathcal{A} = \{(a_{nm}^{(\lambda)})_{n,m \in N}; \lambda \in A\}$  be a family of infinite matrices of non-negative real numbers such that  $\sum_{m=0}^{\infty} a_{nm}^{(\lambda)} = 1$  for all  $n \in N$  and all  $\lambda \in A$ . Let  $\{L_m\}_{m \in N}$  be a sequence of positive linear operators such that  $L_n(1_X) = 1_X$  for all  $n \in N$ . Then the conclusion of Theorem 4 holds for  $D = N, T_{\alpha, \lambda} = T_{n, \lambda}$ , which is defined by (26).*

**5. Bernstein-Lototsky-Schnabl operators.** Let  $S$  be a linear subspace of  $C(X)$  containing  $1_X$  and  $T$  a Markov operator on  $C(X)$ , i.e., a positive linear operator of  $C(X)$  into itself with  $T(1_X) = 1_X$ . Given a point  $x \in X$ , a Radon probability measure  $\nu_x$  on  $X$  is called a  $T(S)$ -representing measure for  $x$  if

$$T(f)(x) = \int_x f d\nu_x$$

for all  $f \in S$  (cf. [5]).

From now on let  $X$  be a compact convex subset of a real pre-Hilbert space with inner product  $\langle \cdot, \cdot \rangle$ , and let  $A(X)$  denote the space of all real-valued continuous affine functions on  $X$ . Let  $V = \{V_n\}_{n \geq 1}$  be a sequence of Markov operators on  $C(X)$ ,  $\mathcal{U}^{(V)} = \{\nu_{x,n}; n \geq 1, x \in X\}$  a family of Radon probability measures on  $X$  such that  $\nu_{x,n}$  is a  $V_n(A(X))$ -representing measure for  $x$ ,  $P = (p_{nj})_{n,j \geq 1}$  an infinite lower triangular stochastic matrix,  $\mathcal{Y} = \{y_x; x \in X\}$  a family of points of  $X$ , and  $\rho = \{\rho_n\}_{n \geq 1}$  a sequence of functions mapping  $X$  into  $[0, 1]$ . Then we define

$$\nu_{x,n,\rho}^{(V,\mathcal{Y})} = \rho_n(x)\nu_{x,n} + (1 - \rho_n(x))\varepsilon_{y_x} \circ V_n,$$

where  $\varepsilon_t$  denotes the Dirac measure at  $t$ , and

$$\pi_{n,P}: X^n \rightarrow X \text{ by } (x_1, x_2, \dots, x_n) \rightarrow \sum_{j=1}^n p_{nj}x_j.$$

Given a function  $f \in C(X)$ , the  $n$ -th Bernstein-Lototsky-Schnabl function  $f$  on  $X$  with respect to  $\mathcal{U}^{(V)}$ ,  $P$ ,  $\mathcal{Y}$  and  $\rho$  is defined by

$$B_n(f)(x) = B_{n,P,\rho}^{(\mathcal{U}^{(V)},\mathcal{Y})}(f)(x) = \int_{X^n} f \circ \pi_{n,P} d \bigotimes_{1 \leq j \leq n} \nu_{x_j,\rho}^{(V,\mathcal{Y})}$$

([23], cf. [6], [24]).

LEMMA 6. Suppose that  $V_n(\langle \cdot, y \rangle) = \langle \cdot, y \rangle$  for all  $n \geq 1$  and all  $y \in X$ . Then the following statements hold:

(i) If  $f$  belongs to  $A(X)$ , then for all  $n \geq 1$  and all  $x \in X$ , we have

$$B_n(f)(x) = \sum_{j=1}^n p_{nj}\rho_j(x)V_j(f)(x) + \sum_{j=1}^n p_{nj}(1 - \rho_j(x))V_j(f)(y_x).$$

In particular, for all  $n \geq 1$  and all  $x, y \in X$ , we have

$$B_n(\langle \cdot, y \rangle)(x) = \sum_{j=1}^n p_{nj}\rho_j(x)\langle x, y \rangle + \sum_{j=1}^n p_{nj}(1 - \rho_j(x))\langle y_x, y \rangle.$$

(ii) If  $y_x = x$  for all  $x \in X$ , then for all  $n \geq 1$  and all  $x \in X$ , we have

$$(29) \quad B_n(e)(x) = \sum_{j=1}^n p_{nj}^2\{\rho_j(x)\nu_{x,j}(e) + (1 - \rho_j(x))V_j(e)(x)\} + (1 - \sum_{j=1}^n p_{nj}^2)e(x),$$

where  $e(x) = \langle x, x \rangle$  for all  $x \in X$ .

(iii) If  $\rho_n = 1_X$  for all  $n \geq 1$ , then (29) reduces to

$$B_n(e)(x) = \sum_{j=1}^n p_{nj}^2\nu_{x,j}(e) + (1 - \sum_{j=1}^n p_{nj}^2)e(x).$$

This follows by immediate computations.

Let  $\mathcal{A} = \{(a_{nm}^{(\lambda)})_{n,m \in N}; \lambda \in A\}$  be a family of infinite matrices of non-negative real numbers such that  $\sum_{m=0}^\infty a_{nm}^{(\lambda)} < \infty$  for each  $n$  and  $\lambda$ . For each  $n \in N$ ,  $\lambda \in A$  and  $f \in C(X)$ , let

$$(30) \quad U_{n,\lambda}(f) = a_{n0}^{(\lambda)}f + \sum_{m=1}^\infty a_{nm}^{(\lambda)}B_m(f),$$

which belongs to  $B(X)$  since each  $B_m$  is a positive linear operator of  $C(X)$  into  $B(X)$  with  $B_m(1_X) = 1_X$ .

It follows that in view of Corollary 2 and Example (4°) if

$$\lim_{n \rightarrow \infty} \|U_{n,\lambda}(1_X) - 1_X\| = 0 \quad \text{uniformly in } \lambda \in A$$

and

$$\lim_{n \rightarrow \infty} \mu(U_{n,\lambda}, d^2) = 0 \quad \text{uniformly in } \lambda \in A,$$

where  $d(x, y) = \langle x - y, x - y \rangle^{1/2}$ , then

$$\lim_{n \rightarrow \infty} \|U_{n,\lambda}(f) - f\| = 0 \quad \text{uniformly in } \lambda \in A$$

for every  $f \in C(X)$  (cf. [23; Theorem 3]). In particular, if  $\lim_{n \rightarrow \infty} \mu(B_n, d^2) = 0$ , then we have  $\lim_{n \rightarrow \infty} \|B_n(f) - f\| = 0$  for all  $f \in C(X)$ .

Concerning the rate of convergence we have the following.

**THEOREM 5.** *Let  $\mathcal{A} = \{(a_{nm}^{(\lambda)})_{n,m \in N}; \lambda \in A\}$  be a family of infinite matrices of non-negative real numbers such that  $\sum_{m=0}^{\infty} a_{nm}^{(\lambda)} = 1$  for each  $n$  and  $\lambda$ . Let  $U_{n,\lambda}$  be as in (30), and for each  $n \in N$  and  $f \in C(X)$  let*

$$\| \|U_n(f) - f\| \| = \sup \{ \|U_{n,\lambda}(f) - f\|; \lambda \in A \}.$$

Let  $E_n(x) = \nu_{x,n}(e)$  for every  $n \geq 1$  and  $x \in X$ . Suppose that  $V_n(\langle \cdot, y \rangle) = \langle \cdot, y \rangle$  for all  $n \geq 1, y \in X$ . Let  $\varepsilon > 0$ . Then the following statements hold:

(i) *If  $y_x = x$  for all  $x \in X$ , then for all  $n \in N$  and all  $f \in C(X)$  we have*

$$(31) \quad \| \|U_n(f) - f\| \| \leq (1 + \varepsilon^{-2})\omega(f, \varepsilon\delta_n),$$

where

$$\delta_n = \left( \sup \left\{ \sum_{m=1}^{\infty} a_{nm}^{(\lambda)} \xi_m; \lambda \in A \right\} \right)^{1/2}$$

and

$$(32) \quad \xi_m = \left\| \sum_{j \geq 1} p_{mj}^2 (\rho_j E_j + (1_X - \rho_j) V_j(e) - e) \right\|.$$

(ii) *If  $\rho_n = 1_X$  for all  $n \geq 1$ , then for all  $n \in N$  and all  $f \in C(X)$  (31) also holds with*

$$(33) \quad \xi_m = \left\| \sum_{j \geq 1} p_{mj}^2 (E_j - e) \right\|.$$

**PROOF.** Suppose that  $y_x = x$  for all  $x \in X$ . Then, by (i) and (ii) of Lemma 6, we conclude that

$$\mu(U_{n,\lambda}, d^2) \leq \sum_{m=1}^{\infty} a_{nm}^{(\lambda)} \xi_m$$



for all  $n \in N$  and all  $\lambda \in A$ . Thus (31) follows from Corollary 5. The proof of Part (ii) is similar. q.e.d.

**COROLLARY 6.** *Let  $E_n$  and  $V$  be as in Theorem 5, and let  $\varepsilon > 0$ . Then the following statements hold:*

(i) *If  $y_x = x$  for all  $x \in X$ , then for all  $n \geq 1$  and all  $f \in C(X)$  we have*

$$(34) \quad \|B_n(f) - f\| \leq (1 + \varepsilon^{-2})\omega(f, \varepsilon\xi_n^{1/2}),$$

where  $\xi_n$  is defined by (32).

(ii) *If  $\rho_n = 1_X$  for all  $n \geq 1$ , then for all  $n \geq 1$  and all  $f \in C(X)$  (34) holds with  $\xi_n$  being given by (33).*

**REMARK 6.** Let the hypotheses of Theorem 5 be fulfilled. Then for every  $n \geq 1$ , we have a simple estimation for  $\delta_n$  and  $\xi_n^{1/2}$ :

$$\delta_n \leq \|e\|^{1/2} \left( \sup \left\{ \sum_{m=1}^{\infty} a_{nm}^{(\lambda)} \sum_{j \geq 1} p_{mj}^2; \lambda \in A \right\} \right)^{1/2};$$

$$\xi_n^{1/2} \leq \|e\|^{1/2} \left( \sum_{j \geq 1} p_{nj}^2 \right)^{1/2}.$$

Hence, if  $\{\sum_{j \geq 1} p_{mj}^2\}_{m \in N}$ , where  $p_{0j} = 0$  for all  $j \geq 1$ , is  $\mathcal{A}$ -summable to zero, then  $\lim_{n \rightarrow \infty} \delta_n = 0$ , and so  $\{U_{n,\lambda}\}$  is an approximation process on  $C(X)$  by Theorem 5. Note that if  $\mathcal{A}$  is regular (see, [22; Definition 5]) and  $\lim_{n \rightarrow \infty} \sum_{j \geq 1} p_{nj}^2 = 0$ , then  $\{\sum_{j \geq 1} p_{mj}^2\}_{m \in N}$  is  $\mathcal{A}$ -summable to zero by [22; Proposition 5]. Also, if  $\lim_{n \rightarrow \infty} \sum_{j \geq 1} p_{nj}^2 = 0$ , then  $\lim_{n \rightarrow \infty} \xi_n = 0$ , and so for every  $f \in C(X)$ , we have  $\lim_{n \rightarrow \infty} \|B_n(f) - f\| = 0$  by Corollary 6. This implies that the result of Grossman [6] can be sharpened with the rate of convergence (cf. [23; Theorem 4]).

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