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# FINITE GROUPS OF POLYNOMIAL AUTOMORPHISMS IN $C^n$

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**Introduction.** Let $C^n$ be an $n$-dimensional complex Euclidean space. A biholomorphic transformation $g: C^n \to C^n$ of $C^n$ onto $C^n$ is called a polynomial automorphism if $g$ and the inverse $g^{-1}$ are given by $n$ polynomials in $n$ variables. We shall denote by $\mathrm{Aut}(C^n)$ the group of all polynomial automorphisms in $C^n$. Let $X$ be a projective algebraic compactification of $C^n$, let $\iota: C^n \to X$ be an inclusion and put $A = X - \iota(C^n)$. Then $A$ is a closed subvariety of $X$. For simplicity, we shall denote this compactification by $(C^n, \iota, X; A)$. Let us denote by $\mathrm{Aut}(X)$ the group of all birational and biregular automorphisms of $X$, and define a subgroup $\mathrm{Aut}(X; A)$ of $\mathrm{Aut}(X)$ by $\mathrm{Aut}(X; A) = \{\hat{g} \in \mathrm{Aut}(X); \hat{g}(A) = A\}$. Then we have the following theorem.

THEOREM 1. *Let $G$ be a finite subgroup of* $\mathrm{Aut}(C^n)$. *Then there exist a non-singular projective algebraic compactification* $(C^n, \iota, X; A)$ *and a finite subgroup $\hat{G}$ of* $\mathrm{Aut}(X; A)$ *such that $\iota^{-1} \circ \hat{G} \circ \iota = G$, namely* $\{\iota^{-1} \circ \hat{g} \circ \iota; \hat{g} \in \hat{G}\} = G$ *on* $C^n$.

Applying Theorem 1 and Morrow's classification of the minimal normal compactifications of $C^2$ [13], we shall give an elementary proof of the following theorem which was obtained by Gizatullin-Danilov [4], Miyanishi [12] and Kambayashi [10], independently (see also [3]).

THEOREM 2 ([4], [12], [10]). *Let $G$ be a finite subgroup of* $\mathrm{Aut}(C^2)$. *Then $G$ is conjugate in* $\mathrm{Aut}(C^2)$ *with a finite subgroup of $GL(2, C)$, namely, there exists a polynomial automorphism $\alpha \in \mathrm{Aut}(C^2)$ such that $\alpha \circ G \circ \alpha^{-1}$ is a finite subgroup of $GL(2, C)$.*

REMARK 1. For $n = 2$, Theorem 1 is a special case of the theorem of Gizatullin-Danilov [4, § 6]. For $n \geqq 3$, it seems to be effective in answering the following general question (see § 3).

QUESTION. Let $G$ be a finite subgroup of $\mathrm{Aut}(C^n)$. Then is $G$ conjugate in $\mathrm{Aut}(C^n)$ with a finite subgroup of $GL(n, C)$?

**1. Proof of Theorem 1.** Let $G$ be a finite subgroup of $\mathrm{Aut}(C^n)$ $(n \geqq 2)$. Let $C^n/G$ be the quotient space of $C^n$ by the group $G$, and

$\pi: C^n \to C^n/G$ the projection. Since $G$ is a finite group of polynomial automorphisms in $C^n$, by Cartan [2], $C^n/G$ is a normal affine algebraic variety of dimension $n$ and the projection $\pi$ is a proper finite regular mapping. Let $Y$ be the normalization of the algebraic closure of $C^n/G$ in some complex projective spece $P^N$, where $N > 0$ is a sufficiently large integer. Then $Y$ is a normal projective algebraic variety of dimension $n$. Let $\tau: C^n/G \to Y$ be the natural inclusion and put $B_0 = Y - \tau(C^n/G)$. The triple $R = (C^n, \pi, C^n/G)$ is a branched algebraic covering over $C^n/G$. Let $B_1$ be the algebraic closure in $Y$ of the branch locus in $C^n/G$ and put $B = B_0 \cup B_1$. Then $B$ is a closed subvariety of $Y$. Then the triple $\mathfrak{R}' = (C^n - \pi^{-1}(B), \pi, C^n/G - B)$ is an unbranched covering over $Y - B$ $(= C^n/G - B)$. By Stein [16, Satz 1], there exists a topologically branched finite covering $\mathfrak{R}_0 = (X_0, \pi_0, Y)$ over $Y$ with the following properties:

(i) the branch locus is contained in the set $B$,

(ii) $X_0$ contains $C^n$ as an open subset, and

(iii) $\pi_0 | C^n = \pi$.

Further, such a covering $\mathfrak{R}_0$ is uniquely determined up to topological isomorphisms. Since $\pi_0$ is a proper finite mapping and $Y$ is compact, $X_0$ is also compact. Since $Y$ is a normal complex space, by the well-known theorem of Grauert-Remmert [6], we can introduce a normal complex structure on $X_0$ and the projection $\pi_0$ is holomorphic with respect to this complex structure. Since $Y$ is projective algebraic and $\pi_0$ is proper finite holomorphic, by Grauert-Remmert [5] (see also Remmert-Stein [15, Satz 8]), so is $X_0$. Thus, $\pi_0$ is a proper finite regular mapping. Let $\iota_0: C^n \to X_0$ be the natural inclusion and put $A_0 = X_0 - \iota_0(C^n)$. Then $A_0$ is a closed subvariety of $X_0$.

Let $g$ be an arbitrary element of $G$. Since $\pi_0 \circ g$ $(= \pi \circ g = \pi)$: $X_0 - A_0 \to Y$ is continued to the regular mapping $\pi_0: X_0 \to Y$ of $X_0$ into $Y$, by Stein [16, Hilfssatz 2], $g$ can be uniquely extended to a continuous mapping $g_0: X_0 \to X_0$. By the Riemann extension theorem, $g_0$ is a holomorphic (therefore regular) mapping of $X_0$ onto $X_0$. Similarly, the inverse $g^{-1}$ can be uniquely extended to a regular mapping $g_0^{-1}: X_0 \to X_0$ of $X_0$ onto $X_0$, and we have $g_0 \circ g_0^{-1} = \mathrm{id}_{X_0}$. Since $g(C^n) = C^n$, we have $g_0(A_0) = A_0$, namely, $g_0 \in \mathrm{Aut}\,(X_0; A_0)$, and further we have $\iota_0^{-1} \circ g_0 \circ \iota_0 = g$ on $C^n$. Thus we have the following:

PROPOSITION 1. *Let $G$ be a finite subgroup of* $\mathrm{Aut}\,(C^n)$. *Then there exist a (not necessarily non-singular) projective algebraic compactification $(C^n, \iota_0, X_0; A_0)$ and a finite subgroup $G_0$ of* $\mathrm{Aut}\,(X_0: A_0)$ *such that $\iota_0^{-1} \circ G_0 \circ \iota_0 = G$ on $C^n$.*

By Hironaka's equivariant resolution theorem [8, §7], there exists a non-singular model $\phi\colon X \to X_0$ of $X_0$ such that any automorphism $g_0 \in \mathrm{Aut}\,(X_0)$ can be uniquely extended to an automorphism $\hat{g} \in \mathrm{Aut}\,(X)$ and satisfies $\phi \circ \hat{g} = g_0 \circ \phi$.

From this theorem and the facts that the singularities of $X_0$ do not lie on $\boldsymbol{C}^n$ and that $g_0(\boldsymbol{C}^n) = \boldsymbol{C}^n$ for every $g_0 \in G_0$, there exists a finite subgroup $\hat{G}$ of $\mathrm{Aut}\,(X; A)$, where $A = \phi^{-1}(A_0)$, such that $\phi \circ \hat{G} = G_0 \circ \phi$, that is, for any $g_0 \in G_0$, there exists a unique element $\hat{g} \in \hat{G}$ such that $\phi \circ \hat{g} = g_0 \circ \phi$. Putting $\iota = \phi^{-1} \circ \iota_0\colon \boldsymbol{C}^n \to X$, the proof of Theorem 1 is completed.

**2. Proof of Theorem 2.** Let $G$ be a finite subgroup of $\mathrm{Aut}\,(\boldsymbol{C}^2)$. By Theorem 1, there exist a non-singular projective algebraic compactification $(\boldsymbol{C}^2, \iota, X; A)$ and a subgroup $\hat{G}$ of $\mathrm{Aut}\,(X; A)$ such that $\iota^{-1} \circ \hat{G} \circ \iota = G$. We put $A = \bigcup_{i=1}^{l} A_i$, where each $A_i$ is an irreducible algebraic curve. We need the following two elementary lemmas.

LEMMA 1. *Let $M$ be a two-dimensional complex manifold and $e = \{x_1, \cdots, x_k\}$ a set of finitely many points in $M$. Let $f\colon M \to M$ be a biholomorphic transformation with $f(e) = e$. Let $Q_e(M)$ be the quadratic transformation of $M$ at the set $e$, and $\phi\colon Q_e(M) \to M$ the projection. Put $\phi^{-1}(e) = E = \bigcup_{i=1}^{k} E_i$, where $E_i = \phi^{-1}(x_i)$ is an exceptional curve of the first kind. Then there exists a unique biholomorphic transformation $\hat{f}\colon Q_e(M) \to Q_e(M)$ with $\hat{f}(E) = E$ such that $\phi \circ \hat{f} = f \circ \phi$.*

LEMMA 2. *Let $\hat{M}$ be a two-dimensional complex manifold and $E = \bigcup_{i=1}^{k} E_i$ a disjoint union of exceptional curves of the first kind. Let $\hat{g}\colon \hat{M} \to \hat{M}$ be a biholomorphic transformation with $\hat{g}(E) = E$. Let $M = \hat{M}/E$ be the contraction of $E$, $\psi\colon \hat{M} \to M$ the projection and put $\psi(E) = e = \{x_1, \cdots, x_k\}$. Then there exists a unique biholomorphic transformation $g\colon M \to M$ with $g(e) = e$ such that $\psi \circ g = g \circ \psi$.*

The proof of Lemma 1 is contained in that of the Lemma of Hopf [9] and Lemma 2 follows from the Riemann extension theorem.

Since the singularities of the (reducible) curve $A$ is $\hat{G}$-invariant, blowing up such singularities and using Lemma 1, we may assume that each $A_i$ is non-singular and $A_i$'s cross each other normally if they intersect. Further, we may assume that $(\boldsymbol{C}^2, \iota, X; A)$ is a minimal normal compactification (see Morrow [13]). Indeed, taking account of Morrow's classification of the minimal normal compactifications of $\boldsymbol{C}^2$ (see also Figure), we see that the irreducible components $A_i$ $(1 \leqq i \leqq k)$ of $A$ with the following properties (i) and (ii) are $\hat{G}$-invariant.

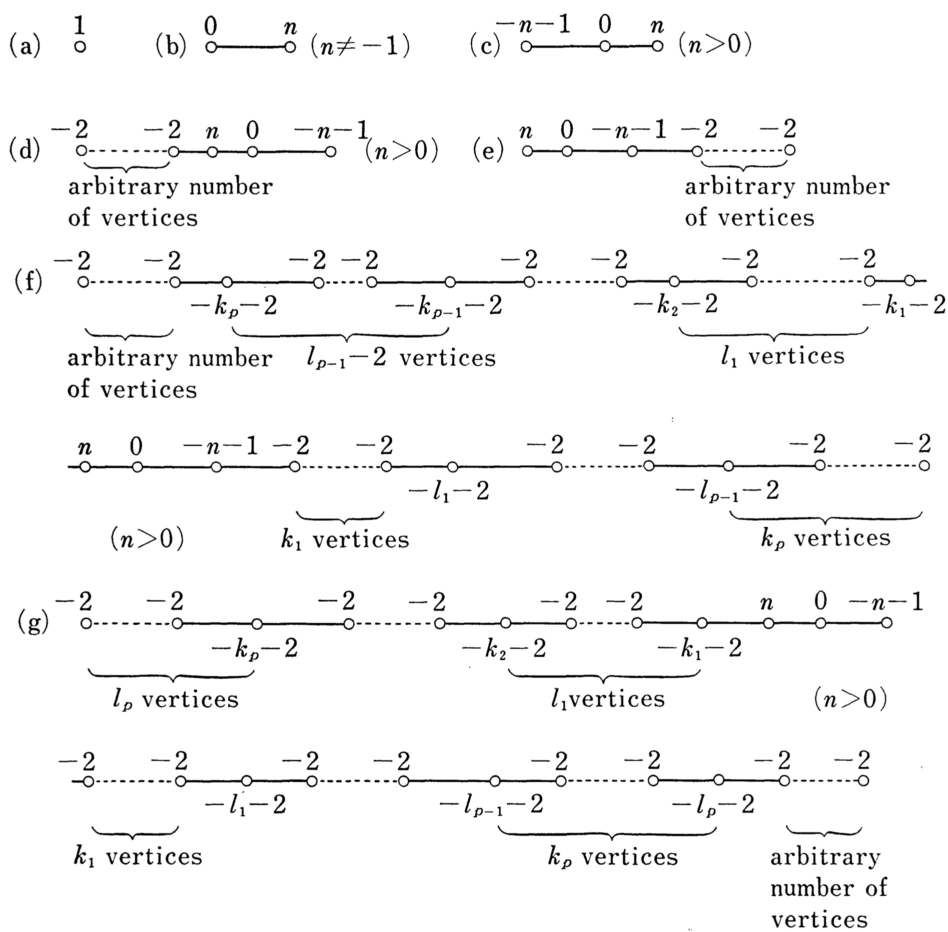
(i) $A_i$ is an exceptional curve of the first kind, and

(ii) the number of irreducible components of $A$, different from $A_i$, which intersect $A_i$ is at most two.

Blowing down such irreducible components $A_i$ $(1 \leqq i \leqq k)$ to points, and using Lemma 2 at each step, the above assertion is finally proved. Thus we have the following:

PROPOSITION 2. *Let $G$ be a finite subgroup of* Aut $(\boldsymbol{C}^2)$. *Then there exist a minimal compactification* $(\boldsymbol{C}^2, \iota, X; A)$ *of* $\boldsymbol{C}^2$ *and a finite subgroup* $\hat{G}$ *of* Aut $(X; A)$ *such that* $\iota^{-1} \circ \hat{G} \circ \iota = G$ *on* $\boldsymbol{C}^2$.

REMARK 2. We can also prove Proposition 2 without using Hironaka's




FIGURE

equivariant resolution theorem. Indeed, by Proposition 1 and the uniqueness of the minimal resolution of singularities of a two-dimensional complex analytic space (cf. Laufer [11]), we can easily see that there exist a non-singular projective algebraic compactification $(\boldsymbol{C}^2, \iota_0, X_0; A_0)$ of $\boldsymbol{C}^2$ and a finite subgroup $\hat{G}$ of $\mathrm{Aut}\,(X_0; A_0)$ such that $\iota_0^{-1} \circ \hat{G} \circ \iota_0 = G$ on $\boldsymbol{C}^2$. Using Lemmas 1 and 2 repeatedly, we have finally Proposition 2.

Now, by Morrow [13], the types of the graph $\Gamma(A)$ of $A$ $(= \bigcup_{i=1}^{l} A_i)$ are the following, where each vertex of the graph represents a non-singular rational curve $A_i$, adjacent to which we write the self-intersection number $(A_i^2)$ of $A_i$. Two vertices are joined by a segment if and only if the two corresponding rational curves intersect each other (see Figure).

(CASE 1). The type of $\Gamma(A)$ is $(a)$. In this case, $X$ is a complex projective plane $\boldsymbol{P}^2$ and $A = X - \iota(\boldsymbol{C}^2)$ is a line $L$ in $\boldsymbol{P}^2$. More precisely, let $(X_i)_{0 \leqq i \leqq 2}$ be homogeneous coordinates in $\boldsymbol{P}^2$. Then $A = X - \iota(\boldsymbol{C}^2) = V(X_0)$.

(CASE 2). The type of $\Gamma(A)$ is $(b)$. In this case, $X$ is a rational ruled surface $\boldsymbol{F}_n$ with the minimal section $s_0$ whose self-intersection number is $(s_0^2) = -n$ $(n \geqq 0)$. Let $s_\infty$ be a section with $(s_\infty^2) = n$ and $l$ a fiber. Then we have $A = \boldsymbol{F}_n - \iota(\boldsymbol{C}^2) = s_\infty \cup l$.

(CASE 3). The type of $\Gamma(A)$ is one of $(c) \sim (g)$. Let $A_0$ (resp. $A_1$, $A_2$) be the irreducible component of $A$ with $(A_0^2) = 0$ (resp. $(A_1^2) = n$, $(A_2^2) = -n-1$). Since the self-intersection number is invariant under an automorphism of $X$, we have $\hat{g}(A_i) = A_i$ $(i = 0, 1)$ for every $\hat{g}$ of $\mathrm{Aut}\,(X; A)$. Since $A_0$ and $A_1$ are $\hat{g}$-invariant, so is $A_2$. Blowing up the intersection point of $A_0$ and $A_1$, and blowing down the proper transform of $A_0$ to a point, we have a new minimal normal compactification $(\boldsymbol{C}^2, \iota_1, X_1; B)$ of $\boldsymbol{C}^2$. It is easily seen that the type of the graph $\Gamma(B)$ of $B$ is the same as that of $\Gamma(A)$ with $n$ replaced by $n-1$, provided $n \geqq 2$. If $n = 1$, the type changes as follow:

$$(c) \to (b)\,, \qquad (d) \to (e)\,, \qquad (e) \to (b) \quad \text{and} \quad (f) \leftrightarrow (g)\,.$$

Thus repeating this process finitely many times and using Lemmas 1 and 2 at each step, we see finally that every element of $\mathrm{Aut}\,(X; A)$ induces a unique element of $\mathrm{Aut}\,(\boldsymbol{F}_n, s_\infty \cup l)$. More precisely, let $\psi: X \to \boldsymbol{F}_n$ be the birational mapping obtained by the above process. By the construction, the restriction $\psi|\iota(\boldsymbol{C}^2)$ of $\psi$ to $\iota(\boldsymbol{C}^2) = X - A$ is a one-to-one regular mapping and the mapping $\psi \circ \iota: \boldsymbol{C}^2 \to \boldsymbol{F}_n$ gives an inclusion. We put $s_\infty \cup l = \boldsymbol{F}_n - \psi \circ \iota(\boldsymbol{C}^2)$. Then there exists a finite subgroup $\hat{G}$ of $\mathrm{Aut}\,(\boldsymbol{F}_n, s_\infty \cup l)$ such that $(\psi \circ \iota)^{-1} \circ \hat{G} \circ (\psi \circ \iota) = G$. Thus we have the following:

PROPOSITION 3. *Let $G$ be a finite subgroup of* $\mathrm{Aut}\,(\boldsymbol{C}^2)$. *Then the* *following two cases arise*:

(1) *There exists a finite subgroup* $\hat{G}$ *of* $\mathrm{Aut}(\boldsymbol{P}^2, L)$ *such that* $\iota^{-1}\circ\hat{G}\circ\iota = G$, *where* $\iota\colon \boldsymbol{C}^2\to\boldsymbol{P}^2$ *is an inclusion and* $L=\boldsymbol{P}^2-\iota(\boldsymbol{C}^2)$ *is a line.*

(2) *There exists a finite subgroup* $\tilde{G}$ *of* $\mathrm{Aut}(\boldsymbol{F}_n, s_\infty\cup l)$ *such that* $\tau^{-1}\circ\tilde{G}\circ\tau = G$, *where* $\tau\colon\boldsymbol{C}^2\to\boldsymbol{F}_n$ *is an inclusion,* $s_\infty$ *is a section with the self-intersection number* $(s_\infty^2)=n$ $(n\geqq 0)$ *and* $l$ *is a fiber of* $\boldsymbol{F}_n$.

Now, since $\iota^{-1}\circ\hat{G}\circ\iota=G$ (resp. $\tau^{-1}\circ\tilde{G}\circ\tau=G$), we have $\iota\circ G\circ\iota^{-1}=\hat{G}|\boldsymbol{C}^2$ (resp. $\tau\circ G\circ\tau^{-1}=\tilde{G}|\boldsymbol{C}^2$), where $\hat{G}|\boldsymbol{C}^2$ (resp. $\tilde{G}|\boldsymbol{C}^2$) means the restriction of the group $\hat{G}$ (resp. $\tilde{G}$) to $\iota(\boldsymbol{C}^2)$ (resp. $\tau(\boldsymbol{C}^2)$). For simplicity, we identify $\iota(\boldsymbol{C}^2)$ and $\tau(\boldsymbol{C}^2)$ with $\boldsymbol{C}^2$. On the other hand, $\mathrm{Aut}(\boldsymbol{P}^2)$ and $\mathrm{Aut}(\boldsymbol{F}_n)$ are well-known, and we can write down every element of $\mathrm{Aut}(\boldsymbol{P}^2, L)$ or $\mathrm{Aut}(\boldsymbol{F}_n, s_\infty\cup l)$ (see [4]). In fact, choosing suitable coordinates $x$ and $y$ in $\boldsymbol{C}^2$, we find that for every element $\hat{g}$ of $\mathrm{Aut}(\boldsymbol{P}^2, L)$ (resp. $\tilde{g}$ of $\mathrm{Aut}(\boldsymbol{F}_n, s_\infty\cup l)$) the restriction $\hat{g}|\boldsymbol{C}^2$ (resp. $\tilde{g}|\boldsymbol{C}^2$) has the following form:

$$\begin{cases} x' = ax+by+\lambda \\ y' = cx+dy+\mu\,, \quad \text{where} \quad ad-bc\neq 0 \text{ and } \lambda, \mu\in\boldsymbol{C}\end{cases}$$

$$\left(\text{resp.}\ \begin{cases} x' = ax+\lambda \\ y' = dy+\nu(x)\,, \quad \text{where} \quad ad\neq 0 \text{ and } \nu(x)\in\boldsymbol{C}[x]\end{cases}\right).$$

Since $\iota$ (resp. $\tau$) is a regular mapping of $\boldsymbol{C}^2$ into $\boldsymbol{P}^2$ (resp. $\boldsymbol{F}_n$), $\iota$ and $\tau$ can be regarded as elements of $\mathrm{Aut}(\boldsymbol{C}^2)$. Consequently we have the following:

PROPOSITION 4. *Let $G$ be a finite subgroup of* $\mathrm{Aut}(\boldsymbol{C}^2)$. *Then there exists a polynomial automorphism $\beta$ in $\boldsymbol{C}^2$ such that for every $g$ of $G$, we have*

$$\beta\circ g\circ\beta^{-1}\colon \begin{cases} x' = a_g x + b_g y + \lambda_g \\ y' = c_g x + d_g y + \mu_g \end{cases}$$

*or*

$$\beta\circ g\circ\beta^{-1}\colon \begin{cases} x' = l_g x + \lambda_g' \\ y' = m_g y + \nu_g(x)\,, \end{cases}$$

*where* $a_g, b_g, c_g, d_g, l_g, m_g, \lambda_g, \lambda_g', \mu_g\in\boldsymbol{C}$, $a_gd_g-b_gc_g\neq 0$, $l_gm_g\neq 0$ *and* $\nu_g(x)\in\boldsymbol{C}[x]$.

Finally, put

$$\gamma_1 = 1/|G|\cdot\sum_{g\in G}\begin{pmatrix} a_g & b_g \\ c_g & d_g\end{pmatrix}^{-1}\circ(\beta\circ g\circ\beta^{-1})$$

or

$$\gamma_2 = 1/|G|\cdot\sum_{g\in G}\begin{pmatrix} l_g & 0 \\ 0 & m_g\end{pmatrix}^{-1}\circ(\beta\circ g\circ\beta^{-1})\,.$$

We can easily see that

$$\gamma_1\colon \begin{cases} x' = x + 1/|G|\cdot\sum_{g\in G}(\lambda_g d_g - b_g\mu_g)/(a_g d_g - b_g c_g) \\ y' = y + 1/|G|\cdot\sum_{g\in G}(a_g\mu_g - \lambda_g c_g)/(a_g d_g - b_g c_g) \end{cases}$$

and

$$\gamma_2\colon \begin{cases} x' = x + 1/|G|\cdot\sum_{g\in G}\gamma'_g/l_g \\ y' = y + 1/|G|\cdot\sum_{g\in G}\nu_g(x)/m_g\,. \end{cases}$$

Thus $\gamma_1$ and $\gamma_2$ are polynomial automorphisms in $C^2$. For any element $h$ of $G$, we have

$$\begin{aligned} &\gamma_1\circ(\beta\circ h\circ\beta^{-1}) \\ &= 1/|G|\cdot\sum_{g\in G}\begin{pmatrix} a_g & b_g \\ c_g & d_g\end{pmatrix}^{-1}\circ(\beta\circ g\circ\beta^{-1})\circ(\beta\circ h\circ\beta^{-1}) \\ &= 1/|G|\cdot\sum_{g\in G}\begin{pmatrix} a_h & b_h \\ c_h & d_h\end{pmatrix}\circ\left\{\begin{pmatrix} a_g & b_g \\ c_g & d_g\end{pmatrix}\circ\begin{pmatrix} a_h & b_h \\ c_h & d_h\end{pmatrix}\right\}^{-1}\circ(\beta\circ g\circ h\circ\beta^{-1}) \\ &= \begin{pmatrix} a_h & b_h \\ c_h & d_h\end{pmatrix}\circ 1/|G|\cdot\sum_{g\circ h\in G}\left\{\begin{pmatrix} a_g & b_g \\ c_g & d_g\end{pmatrix}\circ\begin{pmatrix} a_h & b_h \\ c_h & d_h\end{pmatrix}\right\}^{-1}\circ\beta\circ(g\circ h)\circ\beta^{-1} \\ &= \begin{pmatrix} a_h & b_h \\ c_h & d_h\end{pmatrix}\circ\gamma_1\,. \end{aligned}$$

Similarly, we have

$$\gamma_2\circ(\beta\circ h\circ\beta^{-1}) = \begin{pmatrix} l_g & 0 \\ 0 & m_g\end{pmatrix}\circ\gamma_2\,.$$

Therefore, for every element $g$ of $G$, we have

$$\gamma_1\circ(\beta\circ g\circ\beta^{-1})\circ\gamma_1 = \begin{pmatrix} a_g & b_g \\ c_g & d_g\end{pmatrix}\in GL(2, C)$$

or

$$\gamma_2\circ(\beta\circ g\circ\beta^{-1})\circ\gamma_2 = \begin{pmatrix} l_g & 0 \\ 0 & m_g\end{pmatrix}\in GL(2, C)\,.$$

We have only to let $\alpha = \gamma_1\circ\beta$ or $\alpha = \gamma_2\circ\beta$. Thus the proof of Theorem 2 is completed.

**3. Example.** Let $G$ be a finite subgroup of Aut$(\boldsymbol{C}^3)$. By Theorem 1, there exists a non-singular projective algebraic compactification $(\boldsymbol{C}^3, \iota, X; A)$ and a finite subgroup $\hat{G}$ of Aut$(X; A)$ such that $\iota^{-1}\circ\hat{G}\circ\iota = G$. Here, if we can choose the complex projective space $\boldsymbol{P}^3$ or a non-singular quadric hypersurface $\boldsymbol{Q}^3$ in $\boldsymbol{P}^4$ as such a compactification $X$, there exists an element $\alpha$ of $\mathrm{Aut}(\boldsymbol{C}^3)$ such that $\alpha \circ G \circ \alpha^{-1}$ is a finite subgroup of $GL(3, \boldsymbol{C})$. Indeed, if $X = \boldsymbol{P}^3$, then it is obvious. Suppose that $X = \boldsymbol{Q}^3 \hookrightarrow \boldsymbol{P}^4$. Let $(X_i)_{0\leqq i\leqq 4}$ (resp. $(Y_i)_{1\leqq i\leqq 4}$) be the homogeneous coordinates of $\boldsymbol{P}^4$ (resp. $\boldsymbol{P}^3$). We may assume that

$$X \cong V(X_0X_1 + X_2^2 + X_3^2 + X_1^2)\,,$$
$$A \cong V(X_0) \cap X \cong V(Y_2^2 + Y_3^2 + Y_4^2) \hookrightarrow \boldsymbol{P}^3\,.$$

In fact, we shall first consider the following standard sequence:

$$\to H_c^i(\boldsymbol{C}^3, \boldsymbol{Z}) \to H^i(X, \boldsymbol{Z}) \to H^i(A, \boldsymbol{Z}) \to H_c^{i+1}(\boldsymbol{C}^3, \boldsymbol{Z}) \to .$$

Since $H_i^c(\boldsymbol{C}^3, \boldsymbol{Z}) = 0$ for $1 \leqq i \leqq 4$, we have

$$H^i(X, \boldsymbol{Z}) \cong H^i(A, \boldsymbol{Z}) \quad \text{for} \quad 1 \leqq i \leqq 4\,.$$

By the Lefschetz hyperplane section theorem, we have $H^2(X, \boldsymbol{Z}) \cong H^2(\boldsymbol{P}^4, \boldsymbol{Z}) \cong \boldsymbol{Z}$. We can see that the line bundle $[A]$ is ample on $X$, and the first Chern class $\boldsymbol{C}_1([A])$ of $[A]$ generates the cohomology ring $H^2(X, \boldsymbol{Z})$ $(\cong \boldsymbol{Z})$. By the adjunction formula, we have $K_X \cong [A]^{-3}$ (cf. Brenton-Morrow [1]). Since $A$ is a hyperplane section and $H^2(A, \boldsymbol{Z}) \cong \boldsymbol{Z}$, $A$ is an irreducible quadric hypersurface in $V(X_0) \cong \boldsymbol{P}^3$ with an isolated singularity. By elementary arguments, we see that the minimal resolution of $A$ is the rational ruled surface $\boldsymbol{F}_2$. Thus we may assume that $A$ is isomorphic to the variety $V(Y_2^2 + Y_3^2 + Y_4^2) \hookrightarrow \boldsymbol{P}^3$, and that $X$ is isomorphic to the variety $V(X_0X_1 + X_2^2 + X_3^2 + X_4^2)$ (see Griffiths-Harris [7]). It is easy to verify that such a $(X, A)$ is a non-singular compactification of $\boldsymbol{C}^3$.

Now, we put $x = (1:0:0:0) \in X$. Then $x$ is a singular point of $A$. Let $p_1\colon Q_x(X) \to X$ be the quadratic transformation of $X$ at the point $x$ with $p_1^{-1}(x) = E \cong \boldsymbol{P}^2$. We define the projection $p_2\colon Q_x(X) \to \boldsymbol{P}^3$ of $Q_x(X)$ onto $\boldsymbol{P}^3$ by

$$p_2^{-1}(y) = \begin{cases} \text{(i)} & \text{the point with} \quad X_0 = -\sum_{i=2}^{4} y_i^2/y_1\,, \quad X_i = y_i \quad (1 \leqq i \leqq 4) \\ & \text{if} \quad y_1 \neq 0\,, \\ \text{(ii)} & \text{the point with} \quad X_0 = 1\,, \quad X_i = 0 \quad (1 \leqq i \leqq 4) \\ & \text{if} \quad y_1 = 0 \quad \text{and} \quad \sum_{i=2}^{4} y_i^2 \neq 0\,, \\ \text{(iii)} & \text{any of the line of points with} \quad X_0 = t\,, \quad x_i = sy_i \\ & (1 \leqq i \leqq 4) \quad \text{if} \quad y_1 = \sum_{i=2}^{4} y_i^2 = 0\,. \end{cases}$$

Thus we have the following diagram

$$\begin{tikzcd} & Q_x(X) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[rr, dotted, no head] & & P^3 \end{tikzcd} .$$

Let $\bar{A}$ be the proper transform of $A$ in $Q_x(X)$. Then we have $p_2(p_1^{-1}(A)) = V(Y_1) \hookrightarrow \boldsymbol{P}^3$ and $p_2(\bar{A})$ is a conic $\gamma: \{Y_1 = Y_2^2 + Y_3^2 + Y_4^2 = 0\} \hookrightarrow V(Y_1)$ (see Mumford [14]).

Let $g$ be an arbitrary element of $\mathrm{Aut}(X; A)$. Then $g(x) = x$, since the point $x$ is the only singular point of $A$. Therefore, for the same reason as in Lemma 1, there exists a unique automorphism $\hat{g}$ of $\mathrm{Aut}(Q_x(X); p_1^{-1}(A))$ such that $p_1 \circ \hat{g} = g \circ p_1$. Further by the Riemann extension theorem, there exists a unique automorphism $\tilde{g}$ of $\mathrm{Aut}(\boldsymbol{P}^3; V(Y_1))$ such that $p_2 \circ \tilde{g} = \hat{g} \circ p_2$. We put $\alpha = p_2 \circ p_1^{-1}$. Then $\alpha$ is a one-to-one regular mapping of $\boldsymbol{C}^3$ into $\boldsymbol{P}^3$ with $V(Y_1) = \boldsymbol{P}^3 - \alpha(\boldsymbol{C}^3)$ and $\alpha \circ g = \tilde{g} \circ \alpha$, namely, $\alpha \circ g \circ \alpha^{-1} = \tilde{g} | \boldsymbol{C}^3$. Since $\tilde{g} \in \mathrm{Aut}(\boldsymbol{P}^3; V(Y_1))$, $\tilde{g} | \boldsymbol{C}^3$ is a linear transformation. Therefore $G$ is conjugate in $\mathrm{Aut}(\boldsymbol{C}^3)$ with a finite subgroup of $GL(3, \boldsymbol{C})$.

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