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THE CANONICAL HALF-NORM, DUAL HALF-NORMS, AND MONOTONIC NORMS

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Abstract. Let $(\mathscr{B}, \mathscr{B}_+, \|\cdot\|)$ be an ordered Banach space and define the canonical half-norm

$$N(a) = \inf\{\|a + b\|;\ b \in \mathscr{B}_+\}\ .$$

We prove that $N(a) = \|a\|$ for $a \in \mathscr{B}_+$ if, and only if, the norm is (1-) monotonic on $\mathscr{B}$, and

$$N(a) = \inf\{\|b\|;\ b \in \mathscr{B}_+,\ b - a \in \mathscr{B}_+\}$$

if, and only if, the dual norm is (1-)monotonic on $\mathscr{B}^*$. Subsequently we examine the canonical half-norm in the dual and prove that it coincides with the dual of the canonical half-norm.

0. Introduction. Let $(\mathscr{B}, \|\cdot\|)$ be a Banach space ordered by a positive cone $\mathscr{B}_+$. The associated canonical half-norm N is defined by

$$N(a) = \inf\{\|a + b\|;\ b \in \mathscr{B}_+\}\ .$$

This half-norm has been useful in the analysis of positive semigroups [1] [2] [3] and it appears useful for the characterization of geometric properties of $(\mathscr{B}, \mathscr{B}_+, \|\cdot\|)$ [4] [5] [6]. If $\mathscr{B}$ is a Banach lattice, or the real part of a C^*-algebra then $N(a) = \|a_+\|$ where a_+ is the canonical positive component of $a \in \mathscr{B}$. In particular the half-norm and the norm coincide on $\mathscr{B}_+$. Moreover one has

$$N(a) = \inf\{\|b\|;\ b \in \mathscr{B}_+,\ b - a \in \mathscr{B}_+\}\ .$$

In this note we establish that these properties are general features of a Banach space whose norm and dual-norm are monotonic. Subsequently we examine the canonical half-norm in the dual $\mathscr{B}^*$ and prove that it is the dual, in an appropriate sense, of the canonical half-norm in $\mathscr{B}$.

Throughout this paper $\mathscr{B}_+$ is a norm-closed convex cone in $\mathscr{B}$ with the property

$$\mathscr{B}_+ \cap -\mathscr{B}_+ = \{0\}$$

and one sets $a \geqq b$ if $a - b \in \mathscr{B}_+$. Furthermore $\mathscr{B}_1$ denotes the unit ball, $\mathscr{B}^*$ the dual, $\mathscr{B}_+^*$ the dual cone, i.e.,

$$\mathscr{B}_+^* = \{f;\ f \in \mathscr{B}^*,\ f(a) \geqq 0 \quad \text{for all} \quad a \in \mathscr{B}_+\}\ ,$$

and $\mathscr{B}_1^*$ the unit ball of $\mathscr{B}^*$.

1. Monotonic norms. The norm of an ordered Banach space $(\mathscr{B}, \mathscr{B}_+, \|\cdot\|)$ is defined to be *α-monotonic* if

$$(*) \qquad 0 \leqq a \leqq b \quad \text{implies} \quad \|a\| \leqq \alpha \|b\|\ .$$

This condition is closely related to the concept of normality of $\mathscr{B}_+$ introduced by Krein [7].

The cone $\mathscr{B}_+$ is defined to be *β-normal* if

$$(**) \qquad a \leqq b \leqq c \quad \text{implies} \quad \|b\| \leqq \beta(\|a\| \vee \|c\|)\ .$$

Clearly $(**)$ implies $(*)$ with $\alpha = \beta$ but conversely $(*)$ implies $(**)$ with $\beta = 1 + 2\alpha$. Grosberg and Krein [8] established that normality of $\mathscr{B}_+$ is equivalent to a generation property of the dual cone $\mathscr{B}_+^*$.

The dual cone $\mathscr{B}_+^*$ is defined to be *β-generating* if each $f \in \mathscr{B}^*$ has a decomposition $f = f_+ - f_-$ with $f_\pm \in \mathscr{B}_+^*$ and

$$\beta \|f\| \geqq \|f_+\| + \|f_-\|\ .$$

The Grosberg-Krein theorem states that $\mathscr{B}_+$ is β-normal if, and only if, $\mathscr{B}_+^*$ is β-generating. A similar characterization of β-normality of $\mathscr{B}_+^*$ in terms of β'-generation of $\mathscr{B}_+$, where $\beta' > \beta$, was subsequently obtained by Ando [9] and Ellis [10]. (For further details see [11] [12].)

Our first result is a one-sided version of the foregoing theorems.

THEOREM 1.1. *For each $\alpha \geqq 1$ the following conditions are equivalent:*

(1) *The norm is α-monotonic on $\mathscr{B}$,*

(2) *Each $f \in \mathscr{B}_1^*$ has a decomposition $f = f_+ - f_-$ with $f_+ \in \alpha\mathscr{B}_1^* \cap \mathscr{B}_+^*$ and $f_- \in \mathscr{B}_+^*$.*

Moreover the following conditions are equivalent:

(1*) *The norm is α-monotonic on $\mathscr{B}^*$,*

(2*) *For any $\alpha' > \alpha$ each $a \in \mathscr{B}$ has a decomposition $a = a_+ - a_-$ with $a_+ \in \alpha'\mathscr{B}_1 \cap \mathscr{B}_+$ and $a_- \in \mathscr{B}_+$.*

PROOF. The proof is by polar calculus [11] [12]. We begin by recalling the relevant results on polars.

If $\mathscr{A}$ is a subset of $\mathscr{B}$ the polar $\mathscr{A}^\circ$ of $\mathscr{A}$ is defined by

$$\mathscr{A}^\circ = \{f;\ f \in \mathscr{B}^*,\ f(a) \leqq 1 \quad \text{for} \quad a \in \mathscr{A}\}\ .$$

Hence if $\mathscr{A}_1$, $\mathscr{A}_2$, are norm (weakly) closed convex sets containing $\{0\}$ then

$$(\mathscr{A}_1 \cap \mathscr{A}_2)^\circ = \overline{co}(\mathscr{A}_1^\circ \cup \mathscr{A}_2^\circ)$$

where $\overline{co}$ denotes the weak*-closed convex hull (see, for example, [11] [12]). Moreover if $\mathscr{A}_1$ is a cone then

$$(\mathscr{A}_1 \cap \mathscr{A}_2)^\circ = \overline{co}(\mathscr{A}_1^\circ \cup \mathscr{A}_2^\circ) = (\overline{\mathscr{A}_1^\circ + \mathscr{A}_2^\circ})$$

where the bar denotes weak*-closure. Finally if $\mathscr{A}_2^\circ$ is weak*-compact then

$$(\overline{\mathscr{A}_1^\circ + \mathscr{A}_2^\circ}) = \mathscr{A}_1^\circ + \mathscr{A}_2^\circ$$

and hence

$$(\mathscr{A}_1 \cap \mathscr{A}_2)^\circ = \mathscr{A}_1^\circ + \mathscr{A}_2^\circ .$$

(1) $\Rightarrow$ (2). Condition (1) can be rephrased as

$$\mathscr{B}_+ \cap (\mathscr{B}_1 - \mathscr{B}_+) \subseteqq \alpha\mathscr{B}_1 .$$

Therefore if $\lambda > 1$ then

$$\mathscr{B}_+ \cap (\overline{\mathscr{B}_1 - \mathscr{B}_+}) \subseteqq \mathscr{B}_+ \cap \{\lambda\mathscr{B}_1 - \mathscr{B}_+\} \subseteqq \alpha\lambda\mathscr{B}_1 ,$$

by Corollary 3.3 of [12], Chapter 1. (Here the bar denotes norm closure.) Hence

$$\mathscr{B}_+ \cap (\overline{\mathscr{B}_1 - \mathscr{B}_+}) \subseteqq \alpha\mathscr{B}_1 .$$

But $\mathscr{B}_+$ is a cone and $\mathscr{B}_+^\circ = -\mathscr{B}_+^*$. Moreover $(\overline{\mathscr{B}_1 - \mathscr{B}_+})^\circ = \mathscr{B}_+^* \cap \mathscr{B}_1^*$ is weak*-closed. Hence by the above observations, applied with $\mathscr{A}_1 = \mathscr{B}_+$ and $\mathscr{A}_2 = (\overline{\mathscr{B}_1 - \mathscr{B}_+})$, one obtains

$$\mathscr{B}_1^* = \mathscr{B}_1^\circ \subseteqq \alpha(\mathscr{B}_+ \cap (\overline{\mathscr{B}_1 - \mathscr{B}_+}))^\circ = \alpha(\mathscr{B}_+^* \cap \mathscr{B}_1^* - \mathscr{B}_+^*) .$$

This is, however, a set-theoretic reformulation of Condition (2).

To establish the converse implication we need to introduce polars of subsets of the dual. If $\mathscr{F} \subset \mathscr{B}^*$ then the polar $\mathscr{F}^\circ$ is defined by

$$\mathscr{F}^\circ = \{a; a \in \mathscr{B}, f(a) \leqq 1 \quad \text{for} \quad f \in \mathscr{F}\} .$$

(2) $\Rightarrow$ (1). Consider the above reformulation

$$\mathscr{B}_1^* \subseteqq \alpha(\mathscr{B}_+^* \cap \mathscr{B}_1^* - \mathscr{B}_+^*)$$

of Condition (2). Since $(\mathscr{B}_1^*)^\circ = \mathscr{B}_1$ the polar of this relation gives

$$(\mathscr{B}_+^* \cap \mathscr{B}_1^* - \mathscr{B}_+)^\circ \subseteqq \alpha\mathscr{B}_1 .$$

But it is readily checked that

$$\mathscr{B}_+ \cap (\mathscr{B}_1 - \mathscr{B}_+) \subseteqq (\mathscr{B}_+^* \cap \mathscr{B}_1^* - \mathscr{B}_+^*)^\circ$$

and hence

$$\mathscr{B}_+ \cap (\mathscr{B}_1 - \mathscr{B}_+) \subseteqq \alpha \mathscr{B}_1 .$$

This is, however, a reformulation of Condition (1).

$(1^*) \Leftrightarrow (2^*)$. Condition (1^*) can be rephrased as

$$\mathscr{B}_+^* \cap (\mathscr{B}_1^* - \mathscr{B}_+^*) \subseteqq \alpha \mathscr{B}_1^* .$$

But $\mathscr{B}_+^*$ and $(\mathscr{B}_1^* - \mathscr{B}_+^*)$ are both weak*-closed. Hence taking polars one finds that Condition (1^*) is equivalent to

$$\mathscr{B}_1 \subseteqq \alpha\, \overline{co}((\mathscr{B}_+ \cap \mathscr{B}_1) \cup (-\mathscr{B}_+)) = \alpha(\overline{(\mathscr{B}_+ \cap \mathscr{B}_1) - \mathscr{B}_+})$$

where the bar denotes norm (or weak) closure. Now since $\mathscr{B}_1$ is not norm compact one cannot use the previous argument to remove the closure sign. Nevertheless it follows from Corollary 3.3 of [12], Chapter 1, that

$$\mathscr{B}_1 \subseteqq \alpha'(\mathscr{B}_+ \cap \mathscr{B}_1 - \mathscr{B}_+)$$

for any $\alpha' > \alpha$. This is, however, a set-theoretic reformulation of Condition (2^*).

REMARK 1.2. Since Condition (1), for $\mathscr{B}$, is equivalent to Condition (2), for $\mathscr{B}^*$, which implies Condition (2^*), for $\mathscr{B}^*$, which in turn is equivalent to Condition (1^*), for the bidual $\mathscr{B}^{**}$, one concludes that α-monotonicity of the norm on $\mathscr{B}$ implies α-monotonicity of the norm on $\mathscr{B}^{**}$. Of course the converse is also true.

Next we examine the case of $\alpha = 1$ in more detail.

THEOREM 1.3. *The following conditions are equivalent:*

(1) *The norm is* 1*-monotonic on* $\mathscr{B}$,

(2) *Each* $f \in \mathscr{B}^*$ *has a decomposition* $f = f_+ - f_-$ *with* $f_\pm \in \mathscr{B}_+^*$ *such that* $\|f_+\| \leqq \|f\|$,

(3) *For each* $a \in \mathscr{B}_+$ *there is an* $f \in \mathscr{B}_+^*$ *with* $\|f\| = 1$ *and* $f(a) = \|a\|$.

PROOF. $(1) \Rightarrow (2)$. This follows from Theorem 1.1 with $\alpha = 1$.

$(2) \Rightarrow (3)$. Given $a \in \mathscr{B}_+$ the Hahn-Banach theorem establishes the existence of an $f \in \mathscr{B}_1^*$ with $f(a) = \|a\|$. But if $f = f_+ - f_-$ is the decomposition of Condition (2) then

$$\|a\| = f(a) \leqq f_+(a) \leqq \|f_+\|\, \|a\| \leqq \|a\| .$$

Therefore $\|f_+\| = \|f\| = 1$ and $f_+(a) = \|a\|$.

$(3) \Rightarrow (1)$. Choose f to satisfy Condition (3) then $0 \leqq a \leqq b$ implies

$$\|a\| = f(a) \leqq f(b) \leqq \|b\| .$$

THEOREM 1.4. *The following conditions are equivalent:*

(1) *The norm is 1-monotonic on* $\mathscr{B}^*$,

(2) *Given* $\varepsilon > 0$ *each* $a \in \mathscr{B}$ *has a decomposition* $a = a_+ - a_-$ *with* $a_\pm \in \mathscr{B}_+$ *and* $\|a_+\| \leqq (1 + \varepsilon)\|a\|$,

(3) *Given* $\varepsilon > 0$ *and* $f \in \mathscr{B}_+^*$ *there is an* $a \in \mathscr{B}_+$ *with* $\|a\| \leqq 1$ *and* $f(a) = (1 - \varepsilon)\|f\|$.

PROOF. (1) $\Leftrightarrow$ (2). This equivalence follows from Theorem 1.1 with $\alpha = 1$.

(2) $\Rightarrow$ (3). This follows from the argument used to prove the similar implication in Theorem 1.3 together with the fact that $\mathscr{B}_1$ is weakly dense in the unit ball of the bidual $\mathscr{B}^{**}$.

(3)$\Rightarrow$(1). This follows by the argument used to prove the similar implication in Theorem 1.3.

Finally we remark that 1-monotonicity of the norm can be re-expressed as an hereditary property. Recall that a subset $\mathscr{A} \subseteq \mathscr{B}_+$ is defined to be hereditary if $0 \leqq a \leqq b$ and $b \in \mathscr{A}$ always implies $a \in \mathscr{A}$. Thus 1-monotonicity of $\|\cdot\|$ on $\mathscr{B}_+$ is equivalent to hereditarity of $\mathscr{B}_+ \cap \mathscr{B}_1$.

2. The Canonical half-norm. The canonical half-norm N was defined in the introduction and the principal aim of this section is to evaluate N when the norm and dual-norm are 1-monotonic. First, however, we demonstrate that N can be characterized in a variety of other fashions, by maximality, by duality, or order-theoretically.

Generally a half-norm on $\mathscr{B}$ is a function N' with the properties

$$\begin{aligned}
&0 \leqq N'(a) \leqq k\|a\| \quad \text{for some} \quad k > 0\,,\\
&N'(a + b) \leqq N'(a) + N'(b)\,,\\
&N'(\lambda a) = \lambda N'(a) \quad \text{for all} \quad \lambda \geqq 0\,,\\
&N'(a) \vee N'(-a) = 0 \quad \text{if, and only if,} \quad a = 0\,.
\end{aligned}$$

For each $k > 0$ we denote the corresponding set of half-norms by $\mathscr{N}_k$ and let $\mathscr{N}_k(\mathscr{B}_+)$ denote the $N' \in \mathscr{N}_k$ which are associated with $\mathscr{B}_+$, i.e., which satisfy

$$\mathscr{B}_+ = \{a;\, N'(-a) = 0\}\,.$$

THEOREM 2.1. *The canonical half-norm* N *satisfies the following*:

$$\begin{aligned}
N(a) &= \sup\{N'(a);\, N' \in \mathscr{N}_1(\mathscr{B}_+)\} = \sup\{f(a);\, f \in \mathscr{B}_+^* \cap \mathscr{B}_1^*\}\\
&= \inf\{\lambda \geqq 0;\, a \leqq \lambda u,\, u \in \mathscr{B}_1\}\,.
\end{aligned}$$

PROOF. The third characterization of N was given in [5] and is included because it is useful for establishing the first characterization.

Clearly $N \in \mathscr{N}_1(\mathscr{B}_+)$ and hence for the first equality it suffices to prove that $N \geqq N'$ for all $N' \in \mathscr{N}_1(\mathscr{B}_+)$. But given $\varepsilon > 0$ and $a \in \mathscr{B}$ there is a $u \in \mathscr{B}_1$ such that

$$a \leqq N(a)(1 + \varepsilon)u$$

because of the third characterization of N. Therefore

$$N'(a) \leqq N(a)(1 + \varepsilon)N'(u) \leqq N(a)(1 + \varepsilon)$$

because $N' \in \mathscr{N}_1(\mathscr{B}_+)$. Taking the limit $\varepsilon \to 0$ one obtains $N' \leqq N$.

The second characterization of N follows directly from two lemmas established in [6] which can be rephrased as follows.

LEMMA 2.2. *The following conditions are equivalent*:

(1) $f \in \mathscr{B}_+^* \cap \mathscr{B}_1^*$,

(2) *f is a linear functional over $\mathscr{B}$ satisfying*

$$f(a) \leqq N(a)\,, \quad a \in \mathscr{B}.$$

Moreover for each $a \in \mathscr{B}$ there is an $f \in \mathscr{B}_+^* \cap \mathscr{B}_1^*$ such that

$$f(a) = N(a)\,.$$

Next we examine the evaluation of N on positive elements.

THEOREM 2.3. *The following conditions are equivalent*:

(1) *The norm is* 1-*monotonic on* $\mathscr{B}$,

(2) $N(a) = \|a\|$ *for all* $a \in \mathscr{B}_+$.

PROOF. (1) $\Rightarrow$ (2). If $a, b \geqq 0$ then $\|a + b\| \geqq \|a\|$. Hence

$$\|a\| \leqq \inf\{\|a + b\|;\, b \in \mathscr{B}_+\} = N(a) \leqq \|a\|\,.$$

(2) $\Rightarrow$ (1). Given $a \in \mathscr{B}_+$ it follows from Lemma 2.2 that there exists an $f \in \mathscr{B}_+^* \cap \mathscr{B}_1^*$ such that

$$f(a) = N(a) = \|a\|\,.$$

But this is equivalent to Condition (1) by Theorem 1.3.

If the dual norm is 1-monotonic one has a further partial evaluation of N.

THEOREM 2.4. *The following conditions are equivalent*:

(1) *The norm is* 1-*monotonic on* $\mathscr{B}^*$,

(2) $N(a) = \inf\{\|b\|;\, b \geqq 0, b \geqq a\}$.

PROOF. Define N_+ by

$$N_+(a) = \inf\{\|b\|;\, b \geqq 0, b \geqq a\}\,.$$

It follows straightforwardly that

$$N_+(a) = \inf\{\lambda \geqq 0;\ a \leqq \lambda u,\ u \in \mathscr{B}_+ \cap \mathscr{B}_1\}\ .$$

Therefore it follows from Theorem 8 of [4] that Condition (2) is equivalent to Condition (2) of Theorem 1.4. Consequently the theorem is a corollary of Theorem 1.4.

REMARK 2.5. The property $N = N_+$ can be characterized in several other ways. In fact the conditions of Theorem 2.4 are also equivalent to the following:

(3) $N_+(a) \leqq \|a\|$, $a \in \mathscr{B}$,

(4)(4_+) *For each* $a \in \mathscr{B}$ *there is an* $f \in \mathscr{B}^*(f \in \mathscr{B}_+^*)$ *with* $\|f\| \leqq 1$ *and* $f(a) = N_+(a)$.

To prove this we first remark that by Lemma 2.2 one can choose an $f \in \mathscr{B}_+^* \cap \mathscr{B}_1^*$ with $f(a) = N(a)$. Thus if $N = N_+$ then f satisfies Condition (4_+) and one concludes that (2) $\Rightarrow$ (4_+). But (4_+) $\Rightarrow$ (4) and if f satisfies Condition (4) then

$$N_+(a) = f(a) \leqq \|f\|\ \|a\| \leqq \|a\|\ ,$$

i.e., (4) $\Rightarrow$ (3). Finally $a \leqq a + b$ for $b \geqq 0$ and hence Condition (3) implies that

$$N_+(a) \leqq N_+(a + b) \leqq \|a + b\|\ .$$

Therefore $N_+ \leqq N$. But in general $N \leqq N_+$ and hence (3) $\Rightarrow$ (2).

3. Dual half-norms. Next we consider the canonical half-norm N in the dual $\mathscr{B}^*$ and identify it as the dual of the canonical half-norm in $\mathscr{B}$. There are, however, two natural definitions of the dual half-norm which coincide if, and only if, the norm is 1-monotonic on $\mathscr{B}$. Before demonstrating this we examine the implications of Section 2 for N.

First remark that if $\mathscr{B} = \overline{\mathscr{B}_+ - \mathscr{B}_+}$, where the bar denotes norm closure, then the dual cone $\mathscr{B}_+^*$ is proper, i.e.,

$$\mathscr{B}_+^* \cap -\mathscr{B}_+^* = \{0\}\ .$$

Hence the results of Section 2 can be applied to $\mathscr{B}_+^*$ and the associated canonical half-norm N.

THEOREM 3.1. *The following conditions are equivalent*:

(1) *The norm is* 1-*monotonic on* $\mathscr{B}^*$,

(2) $N(f) = \|f\|$ *for all* $f \in \mathscr{B}_+^*$.

Moreover the following are equivalent:

(1_*) *The norm is* 1-*monotonic on* $\mathscr{B}$,

(2_*) $N(f) = \inf\{\|g\|;\ g \geqq 0,\ g \geqq f\}$.

PROOF. $(1) \Leftrightarrow (2)$. This follows from Theorem 2.3 applied to $(\mathscr{B}^*, \mathscr{B}_+^*, \|\cdot\|)$.

$(1_*) \Leftrightarrow (2_*)$. Condition (2_*) is equivalent to 1-monotonicity of the norm on the bidual $\mathscr{B}^{**}$, by Theorem 2.4, but this is equivalent to 1-monotonicity of the norm on $\mathscr{B}$, by Remark 1.2.

Next we consider dual, or conjugate, half-norms. In analogy with the dual norm there are two natural definitions. These are given by N^c and N^* where

$$N^c(f) = \sup\{f(a);\, a \geqq 0,\, N(a) \leqq 1\}$$
$$N^*(f) = \sup\{f(a);\, a \geqq 0,\, \|a\| \leqq 1\}\,.$$

Note that since $N(a) \leqq \|a\|$ one has $N^* \leqq N^c$.

THEOREM 3.2. *The following conditions are equivalent:*

(1) *The norm is* 1-*monotonic on* $\mathscr{B}$,

(2) $N^* = N^c$.

PROOF. $(1) \Rightarrow (2)$. It follows from Theorem 2.3 that Condition (1) is equivalent to $N(a) = \|a\|$ for $a \geqq 0$. Therefore Condition (1) implies that $N^* = N^c$ by definition.

$(2) \Rightarrow (1)$. Given $a \geqq 0$ choose f such that $f(a) = \|f\|\ \|a\|$. Therefore

$$N^*(f) = \|f\| = f(a)/\|a\| = N^c(f)$$

by Condition (2). But this implies that

$$f(a)/\|a\| \geqq f(b)/N(b)$$

for all $b \geqq 0$. Setting $b = a$ one then deduces that $N(a) \geqq \|a\|$. But one also has $N(a) \leqq \|a\|$. Hence $N(a) = \|a\|$ for $a \geqq 0$ and Condition (1) follows from Theorem 2.3.

REMARK 3.3. If $N' \in \mathscr{N}_1(\mathscr{B}_+)$ then $N \geqq N'$ by Theorem 2.1. Hence defining N'^c by

$$N'^c(f) = \sup\{f(a);\, a \geqq 0,\, N'(a) \leqq 1\}$$

one deduces that $N^c \leqq N'^c$, i.e., N^c is the minimal half-norm conjugate to a half-norm in $\mathscr{N}_1(\mathscr{B}_+)$.

Next we prove that $N^* = N$, the canonical half-norm associated with $\mathscr{B}_+^*$. The proof again uses polar calculus.

We are indebted to Professor T. Ando for pointing out the following identities and their significance for the proof of Theorem 3.5.

THEOREM 3.4. *The following identities*

$$(\mathscr{B}_1 \cap \mathscr{B}_+)^\circ = \mathscr{B}_1^* - \mathscr{B}_+^*\,, \quad (\mathscr{B}_1 \cap \mathscr{B}_+)^{\circ\circ} = \mathscr{B}_1^{**} \cap \mathscr{B}_+^{**}\,,$$

are valid, where the bipolar is now taken in the bidual $\mathscr{B}^{**}$.

PROOF. In Section 1 we used the identity

$$(\mathscr{B}_1 \cap \mathscr{B}_+)^\circ = \overline{co}(\mathscr{B}_1^* \cup (-\mathscr{B}_+^*))$$

where $\overline{co}$ denotes the weak*-closed convex hull. Now consider $\mathscr{B}_1^* - \mathscr{B}_+^*$. This set is convex and weak*-closed, because $\mathscr{B}_1^*$ is weak*-compact and $\mathscr{B}_+^*$ is weak*-closed. Furthermore

$$co(\mathscr{B}_1 \cup (-\mathscr{B}_+^*)) \subseteqq \mathscr{B}_1^* - \mathscr{B}_+^* \subseteqq (\mathscr{B}_1 \cap \mathscr{B}_+)^\circ .$$

Hence we have the identity

$$(\mathscr{B}_1 \cap \mathscr{B}_+)^\circ = \mathscr{B}_1^* - \mathscr{B}_+^* .$$

Now it can be easily verified that

$$\mathscr{B}_1^{**} \cap \mathscr{B}_+^{**} \subseteqq (\mathscr{B}_1^* - \mathscr{B}_+^*)^\circ = (\mathscr{B}_1 \cap \mathscr{B}_+)^{\circ\circ} .$$

The converse inclusion $(\mathscr{B}_1 \cap \mathscr{B}_+)^{\circ\circ} \subseteqq \mathscr{B}_1^{**} \cap \mathscr{B}_+^{**}$ is, however, obvious.

THEOREM 3.5. *The dual half-norm N^* and the canonical half-norm N on the dual coincide, i.e.,*

$$N^*(f) = N(f) , \quad f \in \mathscr{B}^* .$$

PROOF. From Theorem 2.1 one has

$$N(f) = \sup\{f(a);\ a \in \mathscr{B}_1^{**} \cap \mathscr{B}_+^{**}\} \geqq \sup\{f(a);\ a \in \mathscr{B}_1 \cap \mathscr{B}_+\} = N^*(f) .$$

But equality occurs because $\mathscr{B}_1 \cap \mathscr{B}_+$ is weakly dense in $\mathscr{B}_1^{**} \cap \mathscr{B}_+^{**}$ by Theorem 3.4, and the bipolar theorem.

Finally we give another version of Theorem 1.3 which uses the canonical half-norm N on $\mathscr{B}^*$. For this purpose, we need two lemmas; one is a double Hahn-Banach theorem and the other an inequality for N.

LEMMA 3.6. *Let $\mathscr{B}$ be a vector space and q, r subadditive, positively homogeneous on $\mathscr{B}$. Then, if*

$$q(x) + r(-x) \geqq 0 \quad \textit{for all} \quad x \in \mathscr{B},$$

there is a linear functional g on $\mathscr{B}$ such that

$$g(x) \leqq q(x) \quad \textit{and} \quad g(x) \leqq r(x) \quad \textit{for all} \quad x \in \mathscr{B}.$$

PROOF. In the product $\mathscr{B} \times \mathscr{B}$, we consider the subset

$$M = \{(x, -x);\ x \in \mathscr{B}\}$$

and let G be a linear functional on M which is identically zero. Then,

$$p(x, y) = q(x) + r(y) \quad \text{for} \quad (x, y) \in \mathscr{B} \times \mathscr{B}$$

defines a subadditive and positively homogeneous function p which satisfies $G \leqq p$ on M. We denote by the same G an extension of G to $\mathscr{B} \times \mathscr{B}$ retaining the relation $G \leqq p$ and set

$$g_1(x) = G(x, 0) \quad \text{and} \quad g_2(x) = G(0, x) .$$

Then

$$g_1(x) - g_2(x) = G(x, -x) = 0 \quad \text{for all} \quad x \in \mathscr{B}$$

and $g = g_1 = g_2$ is the required functional.

LEMMA 3.7. *For any $f \in \mathscr{B}^*$ there exists a $\gamma > 0$ such that*

$$f(a) \leqq N(f)\|a\| + \gamma N(-a) \quad \textit{for all} \quad a \in \mathscr{B}.$$

PROOF. We first note that

$$N(f) = \inf\{\|f + g\|; g \in \mathscr{B}_+^*, \|g\| \leqq 3\|f\|\} .$$

In fact if $N_1(f)$ denotes the right hand side and we choose $\varepsilon > 0$ such that $N(f) + \varepsilon \leqq \|f\|$ if $N(f) < \|f\|$ and $\varepsilon < \|f\|$ if $N(f) = \|f\|$ then we can choose $g \in \mathscr{B}_+^*$ such that

$$\|f + g\| - N(f) < \varepsilon$$

and hence

$$\|g\| \leqq \|f + g\| + \|f\| \leqq \|f\| + N(f) + \varepsilon \leqq 3\|f\| .$$

It follows that $N(f) \leqq N_1(f) \leqq N(f) + \varepsilon$ and therefore $N(f) = N_1(f)$.

Now to prove our inequality, we take $g \in \mathscr{B}_+^*$ such that $\|g\| \leqq 3\|f\|$. Then it follows that

$$\begin{aligned} f(a) &\leqq \|f + g\| \, \|a\| + g(-a) \leqq \|f + g\| \, \|a\| + \|g\| N(-a) \\ &\leqq \|f + g\| \, \|a\| + 3\|f\| N(-a) \end{aligned}$$

where the second inequality follows from Lemma 2.2 and the fact that $g \in \mathscr{B}_+^*$. Therefore, we have the inequality with $\gamma = 3\|f\|$.

In the following theorem, we denote by N both the canonical half-norm associated with $\mathscr{B}_+$ and that associated with $\mathscr{B}_+^*$.

THEOREM 3.8. *The following conditions are equivalent*:

(1) *The norm is 1-monotonic on $\mathscr{B}$,*

(2) $\|a\| \leqq N(a) + 2N(-a)$ *for all $a \in \mathscr{B}$,*

(3) *Each $f \in \mathscr{B}^*$ has a decomposition $f = f_+ - f_-$ with $f_\pm \in \mathscr{B}_+^*$ such that $\|f_+\| = N(f)$.*

PROOF. (1) $\Rightarrow$ (2). By the definition of canonical half-norms, there exist $b_n \geqq 0$ and $c_n \geqq 0$ such that

$$\|a + b_n\| < N(a) + 1/n \quad \text{and} \quad \|-a + c_n\| < N(-a) + 1/n\ .$$

Therefore,

$$\|b_n + c_n\| < N(a) + N(-a) + 2/n$$

and

$$\|a\| \leqq \|a - c_n\| + \|c_n\| \leqq \|a - c_n\| + \|b_n + c_n\| \leqq N(a) + 2N(-a) + 3/n\ .$$

Hence we obtain the required inequality.

$(2) \Rightarrow (3)$. It follows from Lemma 3.7 that

$$f(a) \leqq N(f)(N(a) + 2N(-a)) + \gamma N(-a) \leqq N(f)N(a) + \gamma' N(-a)\ ,$$

where $\gamma' = 2N(f) + \gamma$. We now apply Lemma 3.6 with

$$q(a) = N(f)N(a) \quad \text{and} \quad r(a) = f(a) + \gamma' N(a)\ .$$

Then we obtain a linear functional g on $\mathscr{B}$ such that

$$g(a) \leqq N(f)N(a) \quad \text{and} \quad g(a) - f(a) \leqq \gamma' N(a)$$

for all $a \in \mathscr{B}$. The first relation implies that $\|g\| \leqq N(f)$ and $g \geqq 0$, and the second relation shows that $g \geqq f$. Then, since

$$N(f) \leqq N(g) \leqq \|g\| \leqq N(f)\ ,$$

we have $\|g\| = N(f)$ and $f_+ = g$ and $f_- = g - f$ satisfy the required property.

$(3) \Rightarrow (1)$. Condition (3) implies Condition (2) in Theorem 3.1.

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